SOLUTIONS

EE 4361
Exam 2
April 15, 2004

Please Print.

Last Name **First Name** **4-digit ID**

Instructions

1. Examination Duration: 1 hour 15 minutes
2. ONE 8.5" x 11"sheet, with both sides of hand-written material, is allowed. No solved problems of any kind are allowed. Turn in the help sheet.
3. One calculator is allowed.
4. Answer in the space/sheets provided.
5. **Answer any 3 of the first 5 questions. Answer question 6 (compulsory), a 40-point problem. You can answer an extra problem for a bonus credit of 5 points. But you should clearly designate the bonus problem.**
6. **DO NOT RELY ON PARTIAL CREDIT, WHICH IS SOLELY AT THE DISCRETION OF THE INSTRUCTOR.**
7. Any copying or cheating will result in appropriate action as per university regulations.

1. **(20 points)**

Determine the Discrete Fourier Series coefficients of the periodic sequence

$$x(n) = 2\cos[(2\pi/3)\,n] + 2\cos[(2\pi/5)\,n] \quad — (1)$$

The fundamental period of $x(n)$ is <u>15 samples</u>.

The harmonics are:

$$s_k(n) = e^{j\frac{2\pi}{15}k}, \quad k = 0, 1, 2 \ldots 14.$$

(1) can be written using Euler's rules as:

$$x(n) = e^{j\frac{2\pi}{3}n} + e^{-j\frac{2\pi}{3}n} + e^{j\frac{2\pi}{5}n} + e^{-j\frac{2\pi}{5}n}$$

$$= e^{j\frac{2\pi \cdot 5}{15}n} + e^{j\frac{2\pi}{15}(15-5)\cdot n} + e^{j\frac{2\pi 3}{15}n} + e^{j\frac{2\pi(15-3)}{15}n}$$

$$= e^{j\frac{2\pi 5}{15}n} + e^{j\frac{2\pi \cdot 10}{15}n} + e^{j\frac{2\pi \cdot 3}{15}n} + e^{j\frac{2\pi \cdot 12}{15}n} \quad — (2)$$

We used the fact that $s_k(n)$'s are periodic with $N = 15$.

Using Fourier Series Synthesis formula, we can write:

$$x(n) = \sum_{k=0}^{14} c_k \cdot e^{j\frac{2\pi k}{15}\cdot n} \quad — (3)$$

Comparing (2) and (3), we have:

$$\boxed{\begin{array}{l} c_3 = c_5 = c_{10} = c_{12} = 1. \\ \text{All other } c_k\text{'s are zero.} \end{array}}$$

2. **(20 points)**

Given the finite sequence

x(n) {-3, -1, 2, 3, -2, 1, 3}
^

with a DTFT of X(ω), compute the following quantities without explicitly computing X(ω). {You may use the definitions, however.}:

(1) $X(\pi)$, (2) $X(0)$, (3) $\int_{-\pi}^{\pi} X(\omega)\, d\omega$ and (4) $\int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega$

(1) $$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\omega}$$

$$X(\pi) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\pi n} = \sum_{n=-\infty}^{\infty} x(n).(-1)^n.$$

For our case:

$$X(\pi) = (-1)^0.(-3) + (-1)^1(-1) + (-1)^2(2) + (-1)^3.(3) + (-1)^4.(-2) + (-1)^5(1) + (-1)^6.(3)$$

$$= -3 + 1 + 2 - 3 - 2 - 1 + 3$$

$$= \boxed{-3}$$

(2) $$X(0) = \sum_{n=-\infty}^{\infty} x(n) = -3 - 1 + 2 + 3 - 2 + 1 + 3 = \boxed{3}$$

(3) $$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

$$\Longrightarrow x(0) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} X(\omega) e^{j\omega \cdot 0} d\omega$$

$$\Longrightarrow \int_{-\pi}^{\pi} X(\omega) d\omega = 2\pi . x(0) = 2\pi.(-3) = \boxed{-6\pi}$$

(4) $$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$ (Parseval's Theorem)

$$\Longrightarrow \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= 2\pi(9 + 1 + 4 + 9 + 4 + 1 + 9) = \boxed{74\pi}$$

3. **(20 points)**

Design a digital resonator (i.e. find H(z)) with the following characteristics:

Resonance Frequency = $\pi/2$ radians/sample
3-dB Bandwidth $\approx$ 0.1 rad/sample
Frequency response of zero at $\omega = 0$ and $\omega = \pi$ radians/sample

Bandwidth:

$$\Delta\omega = 2(1-r) = 0.1 \implies \boxed{r = 0.95}$$

Resonance: $\omega_r = \cos^{-1}\left(\frac{1+r^2}{2r} \cos(\omega_0)\right) = \pi/2$

or $\left(\frac{1+r^2}{2r}\right)\cos\omega_0 = \cos\pi/2 = 0 \implies \boxed{\omega_0 = \pi/2}$

Freq. resp at zero & π:

$H(0) = 0 \implies$ zero location at $\omega = 0$

$H(\pi) = 0 \implies$ zero location at $\omega = \pi$

$$H(z) = G \cdot \frac{(1-z^{-1})(1+z^{-1})}{1-(2r\cos\omega_0)z^{-1} + r^2 z^{-2}}$$

$$= G \cdot \frac{(1-z^{-2})}{1-(0)\cdot z^{-1} + (0.95)^2 z^{-2}}$$

$$= \boxed{G \cdot \frac{(1-z^{-2})}{(1+0.9025 z^{-2})}}$$

The gain G can be set according to a desired mag. response at any specified frequency.

4. **(20 points)**

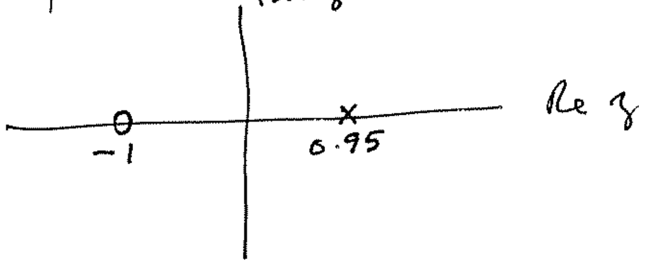
A digital filter H(z) is characterized by the following properties:

(1) It is low pass and has one pole and one zero.
(2) The pole is at a distance of 0.95 from the origin of the z-plane.
(3) $|H(0)| = 1$
(4) $H(\pi) = 0$

Compute the output of the system if the input is

$$x(n) = 2\cos[(\pi/2)n + \pi/4]$$

The information given [1, 2 & 4] leads to the following pole-zero diagram



$$H(z) = G \cdot \frac{(1+z^{-1})}{(1-0.95z^{-1})}.$$

$$H(0) = H(\omega)\Big|_{\omega=0} = \frac{G\,(1+e^{-j\omega})}{(1-0.95\,e^{-j\omega})}\Bigg|_{\omega=0} = G\left(\frac{2}{0.05}\right)$$

$$= 40G.$$

From #3, i.e. $|H(0)| = 40$, and above result

$$\boxed{G = \frac{1}{40}}$$

Thus $\boxed{H(z) = \frac{1}{40}\cdot\frac{(1+z^{-1})}{(1-0.95z^{-1})}}$

$$H(\pi/2) = H(\omega)\Big|_{\omega=\pi/2} = \frac{1}{40}\,\frac{(1+e^{-j\pi/2})}{(1-0.95\,e^{-j\pi/2})} = \frac{1}{40}\cdot\frac{(1-j)}{(1+0.95j)}$$

$$= \frac{1}{40}\cdot\frac{\sqrt{2}\;e^{-j\pi/4}}{(1.3793)\;e^{+j0.7598}} \cong 0.0256\,e^{-j(\pi/4+0.7598)}$$

$$y(n) = |H(\pi/2)|\cdot 2\cdot\cos\left(\frac{\pi}{2}n + \frac{\pi}{4} + \theta_H(n)\right)$$

$$\cong \boxed{0.0513\,\cos\left(\frac{\pi}{2}n - 0.7598\right)}.$$

5. (20 points)

Let

$$H(z)=\frac{(1-1.5z^{-1}-z^{-2})(1+0.9z^{-1})}{(1-0.9z^{-1})(1+j0.7z^{-1})(1-j0.7z^{-1})} \quad \text{—(1)}$$

be the transfer function of a <u>causal</u> digital filter.

(A) Is the inverse system $H_i(z)$ stable? Is it causal?
(B) Express H(z) as the product of a minimum phase and all pass system. (Clearly indicate which one is min. phase and which one is all pass.)

$$H(z)=\frac{(1-2z^{-1})(1+\frac{1}{2}z^{-1})(1+0.9z^{-1})}{(1+j0.7z^{-1})(1-j0.7z^{-1})(1-0.9z^{-1})}, \quad \text{ROC: } |z|>0.9 \text{ (Causal)}$$

A. $$H_i(z)=\frac{(1+j0.7z^{-1})(1-j0.7z^{-1})(1-0.9z^{-1})}{(1-2z^{-1})(1+\frac{1}{2}z^{-1})(1+0.9z^{-1})}.$$

The ROC of $H_i(z)$ should overlap with that of $H(z)$.
The choices are: (i) $|z|>2$, causal (ii) $|z|<\frac{1}{2}$, anticausal
and (iii) $0.9<|z|<2$, two-sided.
The ROC choices (i) and (iii) will overlap with that of $H(z)$.
If (i), i.e. $|z|>2$ is chosen, $H_i(z)$ is <u>causal</u>, <u>but unstable</u>
If (iii), i.e. $0.9<|z|<2$ is chosen, $H_i(z)$ is <u>two-sided</u> (<u>not causal</u>), <u>but stable</u> as it contains the unit circle.

B. $H(z)$ has a zero at $z=2$.

$$H(z)=\underbrace{\left[G\cdot\frac{(1+0.9z^{-1})\;(1-\frac{1}{2}z^{-1})(1+\frac{1}{2}z^{-1})}{(1-0.9z^{-1})\;(1+j0.7z^{-1})(1-j0.7z^{-1})}\right]}_{\text{Min. Phase}}\underbrace{\left[\frac{(1-2z^{-1})}{(1-\frac{1}{2}z^{-1})}\right]}_{\text{All pass}} \quad \text{—(2)}$$

All pass system has a gain of -2. For $H(z)$ above to have the same characteristic as (1), we need to set $\boxed{G=-\frac{1}{2}}$.

6. (40 points)
The periodic signal

$$x(t) = \cos 2\pi.50\,t + \cos 2\pi.100\,t$$

is sampled at a rate of 4 times the highest frequency present in x(t), and input to a digital FIR notch filter to suppress the 50 Hz component while maintaining a magnitude response of unity at the (digital) frequency of $\omega = \pi/2$ radians/sample. The output of the digital FIR notch filter is fed to a D/C (discrete to continuous time) converter (with the same rate of sampling of x(t)). Determine the output signal y(t).

$F_s = 400$ Hz.

$$x(n) = \cos\left(2\pi \cdot \frac{50}{400} n\right) + \cos\left(2\pi \frac{100}{400} n\right)$$

$$= \cos\left(\frac{\pi}{4} n\right) + \cos\left(\frac{\pi}{2} n\right) \quad — (1)$$

corresponds to 50 Hz in CT domain.

$$H(z) = G\left(1 - 2\cos\omega_0 z^{-1} + z^{-2}\right) \quad — (2)$$

is the notch filter transfer function. ω_0 is the notch frequency of the notch filter. G can be chosen to meet any mag. response spec. at any specified frequency.

For us, $\omega_0 = \frac{\pi}{4} \Rightarrow H(z) = G\left(1 - 2\cos\frac{\pi}{4} \cdot z^{-1} + z^{-2}\right)$

$$H(z) = \boxed{G\left(1 - \sqrt{2} z^{-1} + z^{-2}\right)} \quad — (3)$$

$$H(\omega) = H(z)\Big|_{z = e^{j\omega}} = G\left(1 - \sqrt{2} e^{-j\omega} + e^{-j2\omega}\right)$$

$$H\left(\frac{\pi}{2}\right) = G\cdot\left(1 - \sqrt{2}\cdot e^{-j\pi/2} + e^{-j2\cdot\pi/2}\right) = G\,(\sqrt{2}j) \quad — (4)$$

$$= G\cdot\sqrt{2}\cdot e^{j\pi/2} \quad — (5)$$

But by spec, we require $\left|H\left(\frac{\pi}{2}\right)\right| = 1$

$$\Rightarrow G\cdot\sqrt{2} = 1 \quad \text{or} \quad G = \boxed{\frac{1}{\sqrt{2}}} \quad — (6)$$

$$y(n) = \underbrace{\left|H\left(\frac{\pi}{4}\right)\right|}_{\text{This equals 0 by design}} \cos\left(\frac{\pi}{4} n + \theta_H(\pi/4)\right) + \left|H\left(\frac{\pi}{2}\right)\right| \cdot \cos\left(\frac{\pi}{2} n + \theta_H\left(\frac{\pi}{2}\right)\right) \quad — (7)$$

$$y(n) = 1 \cdot \cos\left(\frac{\pi}{2} n + \frac{\pi}{2}\right) \quad — (7)$$

$$\boxed{y(t) = \cos\left(2\pi\, 100 t + \frac{\pi}{2}\right)}$$