SOLUTION

EE 6360
Exam 1
February 27, 2007

Please Print.

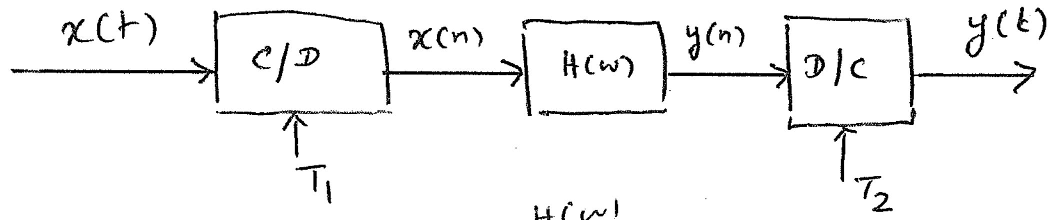
Last Name **First Name**

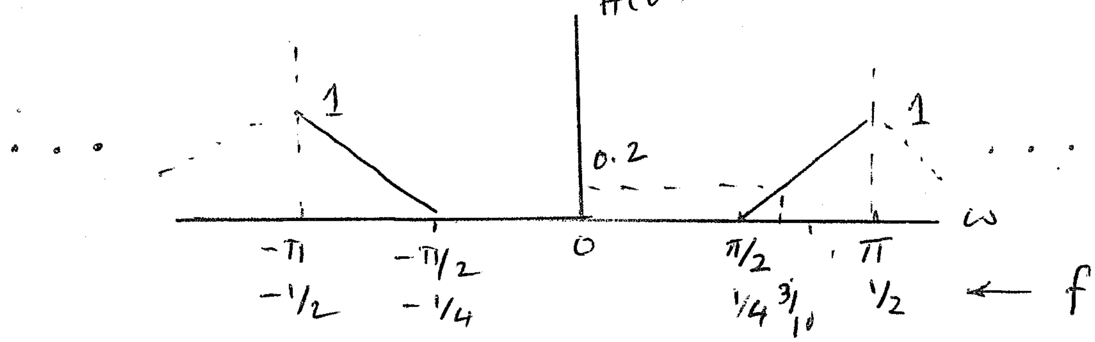
Instructions

1. Examination Duration: 1 hour 15 minutes. You may have an additional 10 minutes if you come in earlier.
2. ONE 8.5" x 11"sheet, with both sides of hand-written material, is allowed. No solved problems of any kind are allowed.
3. One calculator is allowed.
4. Answer in the space/sheets provided. **Highlight your answers.**
5. **Answer all 5 questions. There is a bonus question at the end for 5 points. <u>Show all steps/logic. DO NOT RELY ON PARTIAL CREDIT, WHICH IS SOLELY AT THE DISCRETION OF THE INSTRUCTOR.</u>**
6. Any copying or cheating will result in appropriate action as per university regulations.

1. (Two independent parts)

(A) A continuous time signal x(t) is processed as in the diagram shown below. Determine the output y (t), when the input is
$x(t) = \cos(2\pi\, 10t) + \cos(2\pi\, 20t) + \cos(2\pi\, 30t)$, and $T_1 = 10$ ms, and $T_2 = 20$ ms.





$T_1 = 10\,\text{msec} \Rightarrow F_s = 100\,\text{Hz};\ T_2 = 20\,\text{msec} \Rightarrow F_r = 50\,\text{Hz}.$

$x(n) = \cos(2\pi\cdot\frac{1}{10} n) + \cos(2\pi\cdot\frac{2}{10} n) + \cos(2\pi\frac{3}{10} n)$

The filter $H(w)$ will not pass the first two component [because $H(2\pi\cdot\frac{1}{10})$ and $H(2\pi\cdot\frac{2}{10})$ are zero], and will pass the third component with a gain $\boxed{G = 0.2}$.

Therefore,

$y(n) = 0.2 \cos(2\pi\cdot\frac{3}{10} n)$

The reconstructed signal is:

$y(t) = 0.2 \cos(2\pi\frac{3}{10}\cdot F_r t)$

$= \frac{1}{5} \cos(2\pi\cdot\frac{3}{\cancel{10}}\cdot \cancel{50}^{5} t)$

$\Rightarrow \boxed{y(t) = \frac{1}{5} \cos(2\pi\, 15 t)}$

(B) The signal $x(t) = \cos(2\pi\, 10t + \pi/4)$ is C/D converted at the rate of 15 Hz to obtain the DT signal x(n), which is D/C converted at the rate of F_r Hz to yield the CT signal $y(t) = \cos(2\pi\, 10t - \pi/4)$. Determine the D/C rate, F_r Hz.

$$x(n) = \cos\left(2\pi \cdot \frac{10}{F_s} n + \pi/4\right) = \cos\left(2\pi \cdot \frac{\cancel{10}^{\,2}}{\cancel{15}_{\,3}} n + \pi/4\right)$$

$$= \cos\left(2\pi \cdot \left(1 - \tfrac{1}{3}\right) n + \pi/4\right) \quad \left[\text{Putting the "f" in principal range}\right]$$

$$= \cos\left(2\pi \tfrac{1}{3} n - \pi/4\right)$$

Reconstructed signal y(t) from x(n) is:

$$y(t) = \cos\left(2\pi \tfrac{1}{3} F_r t - \pi/4\right)$$

For this to equal the given expression, we require that

$$\boxed{F_r = 30\,Hz}$$

2. A causal, LTI system is described by the following LCCDE:

$$\boxed{y(n) - 0.25\, y(n-1) = x(n)} \quad n \geq 0 \text{ with } y(-1) = 1, \text{ and } x(n) = u(n) \qquad (1)$$

Determine the transient and steady state responses. Identify the zero input and zero state responses. Identify the homogeneous and particular solutions.

Taking 1-sided z-transform of the LCCDE in (1), we get:

$$Y(z) - \frac{1}{4}\left[z^{-1}Y(z) + y(-1)\right] = X(z)$$

$$\Rightarrow Y(z) = \underbrace{\frac{X(z)}{(1 - \frac{1}{4}z^{-1})}}_{Y_{zs}(z)} + \underbrace{\frac{\frac{1}{4}\, y(-1)}{(1 - \frac{1}{4}z^{-1})}}_{Y_{ZI}(z)} \qquad (2)$$

$$= \frac{1}{(1-z^{-1})(1-\frac{1}{4}z^{-1})} + \frac{\frac{1}{4}\, y(-1)}{(1 - \frac{1}{4}z^{-1})}$$

$$= \underbrace{\frac{A}{(1-z^{-1})}}_{\substack{Y_{s.s}(z) \\ Y_p(z)}} + \underbrace{\frac{B}{(1-\frac{1}{4}z^{-1})} + \frac{\frac{1}{4}\, y(-1)}{(1-\frac{1}{4}z^{-1})}}_{\substack{Y_{tr}(z) \\ Y_h(z)}} \qquad (3)$$

Zero Input Response:

$$y_{zi}(n) = \frac{1}{4}\, y(-1)\left(\frac{1}{4}\right)^n u(n) = \boxed{\left(\frac{1}{4}\right)\cdot\left(\frac{1}{4}\right)^n u(n)} \qquad (4)$$

ZERO INPUT RESP

Zero state Response

$$Y_{zs}(z) = \frac{A}{(1-z^{-1})} + \frac{B}{(1-\frac{1}{4}z^{-1})} = \frac{1}{(1-z^{-1})(1-\frac{1}{4}z^{-1})}$$

$$A = Y_{zs}(z)\cdot(1-z^{-1})\Big|_{z^{-1}=1} = \frac{1}{(1-\frac{1}{4}\cdot 1)} = \boxed{\frac{4}{3}}$$

$$B = Y_{zs}(z)\,(1-\tfrac{1}{4}z^{-1})\Big|_{z^{-1}=4} = \frac{1}{(1-4)} = \boxed{-\frac{1}{3}}$$

$$\Rightarrow \boxed{y_{zs}(n) = \frac{4}{3}u(n) + \left(-\frac{1}{3}\right)\left(\frac{1}{4}\right)^n u(n)} \quad (5)$$

ZERO STATE RESPONSE

Transient & Steady State Response

From (3), (4) & (5), we identify these responses:

Transient Response:

$$\boxed{y_{tr}(n) = \left(\tfrac{1}{4}-\tfrac{1}{3}\right)\left(\frac{1}{4}\right)^n u(n) = \left(-\frac{1}{12}\right)\left(\frac{1}{4}\right)^n u(n)}$$

Steady-State Response:

$$\boxed{y_{ss}(n) = \frac{4}{3}u(n)}$$

Homogeneous & Particular Solutions.

$$\boxed{y_h(n) = y_{tr}(n) \quad \text{and} \quad y_p(n) = y_{ss}(n)}$$

Note: You can arrive at the answer in several ways.

3. The input to an LTI system is

$$x(n) = \{\underset{\wedge}{1}\ \ -1\ \ 1\}$$

and the cross correlation function between the input x(n) and output y(n) is given by

$$r_{xy}(n) = \{2\ \ -3\ \ 4\ \ -\underset{\wedge}{1}\ \ 0\ \ 1\}.$$

Determine the autocorrelation function of y(n).

Range of various quantities:

$x(n) \rightarrow [0,2] \qquad r_{xy}(n) = x(n) * y(-n) \rightarrow [-3,2]$

$\Rightarrow y(-n) \rightarrow [-3,0] \Rightarrow y(n) \rightarrow [0,3]$

$\Rightarrow h(n) \rightarrow [0,1].$

Let $h(n) = [\underset{\uparrow}{\alpha}\ \ \beta].$

$y(n) = \alpha \cdot x(n) + \beta\, x(n-1)$

$$\left.\begin{array}{cccc} \alpha & -\alpha & \alpha & \\ & -\beta & -\beta & -\beta \\ \hline [\underset{\uparrow}{\alpha} & (\beta-\alpha) & (\alpha-\beta) & \beta\ ] \end{array}\right\} \Rightarrow \boxed{y(n) = \{\underset{\uparrow}{\alpha}\ \ (\beta-\alpha)\ \ (\alpha-\beta)\ \ \beta\}}$$

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-l) = \sum_{n=0}^{2} x(n)\, y(n-l).$$

$$r_{xy}(2) = x(0)\, \cancel{y(-2)}^{\,0} + x(1)\, \cancel{y(-1)}^{\,0} + \overset{1}{x(2)}\, \overset{\alpha}{y(0)} = \boxed{\alpha}$$

But $r_{xy}(2) = 1 \Rightarrow \boxed{\alpha = 1}$

$$r_{xy}(1) = x(0)\, \cancel{y(-1)}^{\,0} + x(1)\, y(0) + x(2)\, y(1)$$

$$= \alpha - (\beta - \alpha) = 2\alpha - \beta = 0 \Rightarrow \beta = 2\alpha \Rightarrow \boxed{\beta = 2}$$

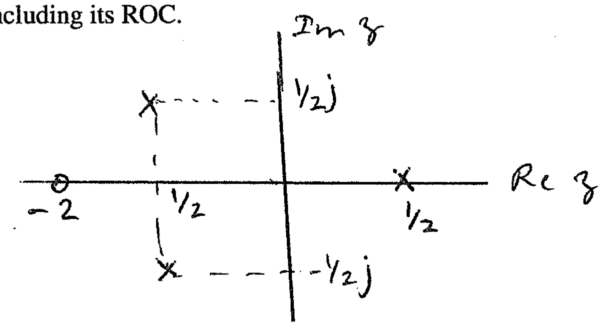
Thus, $\boxed{y(n) = \{\underset{\uparrow}{1}\ \ 1\ \ -1\ \ 2\}}$, $\quad h(n) = \{\underset{\uparrow}{1}, 2\}$

$$R_{yy}(z) = Y(z)\, Y(z^{-1}) = [1 + z^{-1} - z^{-2} + 2z^{-3}]\,[1 + z - z^{2} + 2z^{3}]$$

$$= 2z^{3} + z^{2} - 2z + 7 - 2z^{-1} + z^{-2} + 2z^{-3}$$

$$\Rightarrow \boxed{r_{yy}(l) = \{2\ \ 1\ \ -2\ \ \underset{\uparrow}{7}\ \ -2\ \ 1\ \ 2\}}$$

4. The pole-zero plot of a causal signal x(n) is shown below. (A) Determine the pole-zero plot of the signal $x_1(n) = x(-n+2)$, including its ROC. (B) Determine the pole-zero plot of the signal $x_2(n) = (1/2)^n x(n)$, including its ROC.



(A) $x_1(n) = x(-(n-2)) \Longrightarrow X_1(z) = z^{-2} X(z^{-1})$.

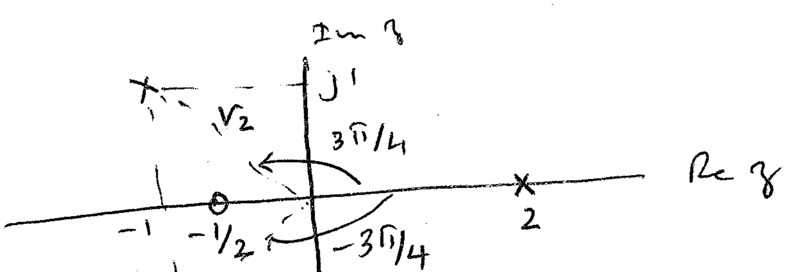
$$X(z) = \frac{(z+2)}{(z-1/2)(z-re^{j\theta})(z+re^{j\theta})} \quad \text{— (1)} \quad \boxed{ROC: |z| > \frac{1}{\sqrt{2}}}$$

where $r = \frac{1}{\sqrt{2}}$ and, $\theta = \frac{3\pi}{4}$.

$$X_1(z) = z^{-2} X(z^{-1}) = \frac{z^{-2}(z^{-1}+2)}{(z^{-1}-1/2)(z^{-1}-re^{j\theta})(z^{-1}-re^{-j\theta})}$$

$$= \frac{z^{-3}(1+2z)}{(z^{-1}-1/2)(z^{-1}-re^{j\theta})(z^{-1}-re^{-j\theta})}$$

$$= \frac{(1+2z)}{(1-\frac{1}{2}z)(1-re^{j\theta}z)(1-re^{-j\theta}z)}$$



Two zeros at $z = \infty$..

ROC: $\boxed{|z| < \sqrt{2}}$

(why?)

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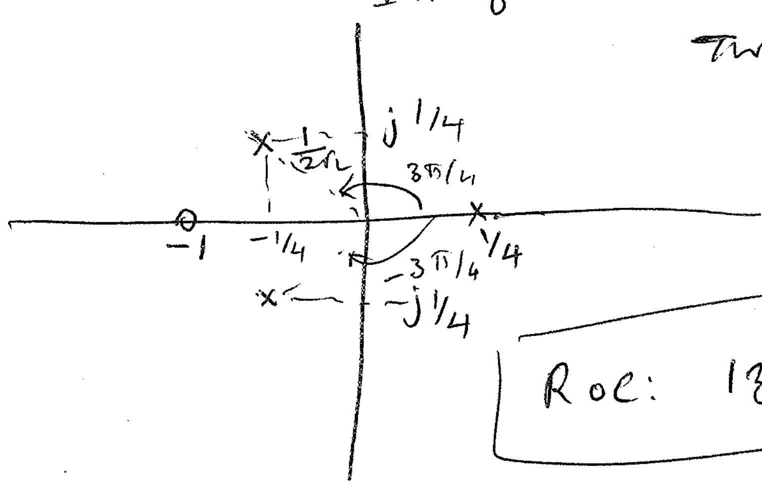
$x_2(n) = \left(\frac{1}{2}\right)^n x(n)$

$X_2(z) = X\left(\frac{z}{\lambda}\right), \quad \lambda = 1/2.$

Thus the poles & zeros of $X(z)$ will be the poles & zeros of $X_2(z)$ with the relationship $\frac{z}{\lambda} = p_0$ or z_0 (pole or zero of $x(n)$)

	Pole or zero of $X(z)$	$X_2(z)$
zeros:	-2	$(-2)\cdot\frac{1}{2} = -1$
	∞	∞
poles	$1/2$	$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$
	$re^{j\theta}$	$\frac{1}{2}re^{j\theta}$
	$re^{-j\theta}$	$\frac{1}{2}re^{-j\theta}$

where $r = \frac{1}{\sqrt{2}}$ $\theta = \frac{3\pi}{4}$.



Two zeros at ∞

$\boxed{\text{ROC: } |z| > \frac{1}{2\sqrt{2}}}$

5. [Note: You are not allowed to use z-transform for this problem.] The impulse response of a relaxed causal LTI system is given by

$$h(n) = (1/2)^n u(n)$$

and the input is given by $x(n) = u(-n)$. Determine the output of the system $y(n)$ for all possible values of n.

$$y(n) = x(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} u(-k)\left(\tfrac{1}{2}\right)^{n-k} u(n-k)$$

$$= \left(\tfrac{1}{2}\right)^n \sum_{k=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^{-k} u(-k)\, u(n-k) \quad — ①$$

$$u(-k) = \begin{cases} 1, & k \le 0 \\ 0, & \text{otherwise} \end{cases} \qquad u(n-k) = \begin{cases} 1, & k \le n \\ 0, & \text{otherwise} \end{cases} \quad — ②$$

we have to consider $n \le 0$ and $n > 0$.

(i) $n \le 0$, $\Rightarrow -\infty \le k \le n$

$$y(n) = \left(\tfrac{1}{2}\right)^n \sum_{k=-\infty}^{n} \left(\tfrac{1}{2}\right)^{-k} = \left(\tfrac{1}{2}\right)^n \sum_{k=-n}^{\infty} \left(\tfrac{1}{2}\right)^k$$

$$= \left(\tfrac{1}{2}\right)^n \left\{ \left[\sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k\right] - \left[\sum_{k=0}^{-n-1} \left(\tfrac{1}{2}\right)^k\right] \right\}$$

$$= \left(\tfrac{1}{2}\right)^n \left\{ \left(\frac{1}{1-1/2}\right) - \frac{\left[1-\left(\tfrac{1}{2}\right)^{-n}\right]}{(1-1/2)} \right\}$$

$$= \left(\tfrac{1}{2}\right)^n \left\{ 2 - 2\left(1-\left(\tfrac{1}{2}\right)^{-n}\right) \right\}$$

$$= \boxed{2}, \quad n \le 0.$$

$$\boxed{y(n) = 2, \quad n \le 0} \quad — ③$$

(ii) $n > 0 \Rightarrow -\infty \leq k \leq 0$

$$y(n) = \left(\tfrac{1}{2}\right)^n \sum_{k=-\infty}^{0} \left(\tfrac{1}{2}\right)^{-k}$$

$$= \left(\tfrac{1}{2}\right)^n \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^{k}$$

$$= 2(1/2)^n, \quad n > 0$$

$$\boxed{y(n) = 2(1/2)^n, \quad n > 0} \quad \text{— (4)}$$

Combining (3) & (4), we can write

$$\boxed{y(n) = 2\left[u(-n-1) + \left(\tfrac{1}{2}\right)^n u(n)\right]}$$

You can easily verify the above result by solving the problem using z-transform.

BONUS QUESTION (5 Points)

Indicate the ROC and stability of the following sequences.

1.. $x(n) = u(n+2) - u(n-2)$

ROC: $0 < |z| < \infty$. stable (finite sequence)

2. $x(n) = u(-n) - u(-n-1)$ $= \delta(n)$

ROC: Entire z-plane. stable

3. $x(n) = (1/2)^{(n+1)} u(n)$ $= \frac{1}{2}\left(\frac{1}{2}\right)^n u(n)$ → causal

ROC: $|z| > \frac{1}{2}$. stable, ROC includes unit ck.

4. $x(n) = u(n) - u(n-1)$ $= \delta(n)$

ROC: Entire z-plane. stable

5. $x(n) = (1/2)^n u(n-1)$ $= \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u(n-1)$ causal

ROC: $|z| > \frac{1}{2}$, stable, ROC includes unit ck.