SOLUTION

EE 6360-001
Exam 1
October 6, 2007

Please Print.

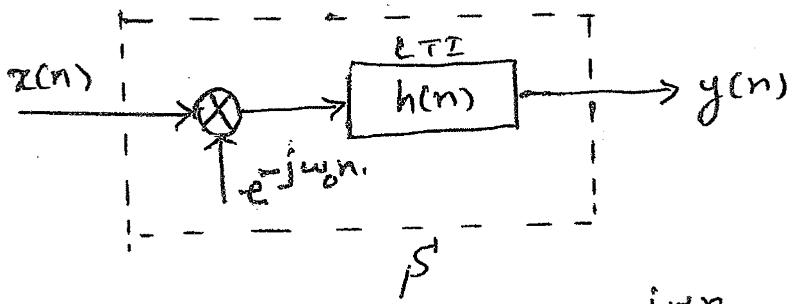
Last Name **First Name**

Instructions

1. Examination Duration: 2 hours.
2. ONE 8.5" x 11"sheet, with both sides of hand-written material, is allowed. No solved problems of any kind are allowed.
3. One calculator is allowed.
4. Answer in the space/sheets provided. **Highlight your answers.**
5. **Answer all questions. Part A has 6 questions, equally weighted for 60 points; Part B has 10 questions, equally weighted for 40 points. Show all steps/logic, unless the question instructs you otherwise. DO NOT RELY ON PARTIAL CREDIT, WHICH IS SOLELY AT THE DISCRETION OF THE INSTRUCTOR.**
6. Any copying or cheating will result in appropriate action as per university regulations.

Part A:

1. Shown below is a system *S* with input x (n) and output y (n). The LTI system has a stable impulse response h (n). The system is at rest. (a) Is the system *S* linear? (b) Is the system *S* time invariant? (c) Is the system *S* BIBO stable? You must <u>prove or disprove</u> the answers to your questions to receive credit.



$$y(n) = h(n) * [x(n)e^{-j\omega_0 n}].$$

$$= \sum_{k=-\infty}^{\infty} x(k)e^{-j\omega_0 k}h(n-k) \quad \text{—①}$$

(A) <u>Linearity:</u>

$$x_1(n) \longrightarrow y_1(n) = \sum_{k=-\infty}^{\infty} x_1(k)e^{-j\omega_0 k}h(n-k)$$

$$x_2(n) \longrightarrow y_2(n) = \sum_{k=-\infty}^{\infty} x_2(k)e^{-j\omega_0 k}h(n-k)$$

$$x_3(n) = \alpha x_1(n) + \beta x_2(n) \longrightarrow y_3(n) = \sum_{k=-\infty}^{\infty} [\alpha x_1(k) + \beta x_2(k)]e^{-j\omega_0 k}h(n-k)$$

$$= \alpha y_1(n) + \beta y_2(n)$$

$$\boxed{S \text{ is LINEAR}}$$

(b) <u>Time Invariance</u>

Let $x_1(n) = x(n-n_0)$

$$\Rightarrow y_1(n) = \sum_{k=-\infty}^{\infty} x_1(k)e^{-j\omega_0 k}h(n-k) \quad \text{—②}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k-n_0)e$$

From ① & shifting by n_0 samples, we have

$$y(n-n_0) = \sum_{k=-\infty}^{\infty} x(k)e^{-j\omega_0 k}h(n-n_0-k).$$

$$\Rightarrow y(n-n_0) = \sum_{m=-\infty}^{\infty} x(m-n_0)\, e^{-j\omega_0 (m-n_0)} \cdot h(n-m) \quad — ③$$

[we used the change of variable, $m = k + n_0$]

Note that

$$y(n-n_0) \neq y_1(n)$$

$$\Rightarrow \boxed{S \text{ is TIME VARYING}}$$

(c) BIBO stability

Let $\max_n |x(n)| = B_x$.

From ①,

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} x(k)\, e^{-j\omega_0 k} \cdot h(n-k) \right|$$

$$\leq \sum_{k=-\infty}^{\infty} |x(k)\, e^{-j\omega_0 k}\, h(n-k)| \quad — ④$$

$$\leq B_x \sum_{k=-\infty}^{\infty} |h(n-k)|$$

But we know that $h(n)$ is stable and is absolute summable. Therefore

$$|y(n)| < \infty \quad \text{for all } n$$

$$\Rightarrow \boxed{S \text{ is BIBO Stable}}$$

2. Determine the autocorrelation of the finite sequence x (n) = 1, $0 \le n \le (N-1)$, and x (n) = 0 otherwise.

$$r_{xx}(l) = x(l) * x(-l)$$
$$= Z^{-1}\{X(z) \cdot X(z^{-1})\}$$

$$X(z) = 1 + z^{-1} + z^{-2} + \cdots + z^{-(N-1)}, \quad |z| > 0 \quad (1)$$

$$X(z^{-1}) = z^{(N-1)} + \cdots + z + 1, \quad |z| < \infty \quad (2)$$

$$Z[r_{xx}(z)] = X(z)\, X(z^{-1}), \quad 0 < |z| < \infty$$

using (2) & (1) we have: Multiplying term by term

N-terms

$$1 + z^{-1} + z^{-2} + \cdots + z^{-(N-1)}$$
$$z + 1 + z^{-1} + \cdots + z^{-(N-2)}$$
$$z^2 + z + 1 + z^{-1} + \cdots + z^{-(N-3)}$$
$$\vdots$$
$$z^{(N-1)} + z^{(N-2)} + \cdots + z + 1$$

Add

$$z^{(N-1)} + \cdots + (N-1)z + N + (N-1)z^{-1} + \cdots + z^{-(N-1)}$$

$$\Rightarrow \boxed{r_{xx}(l) = \begin{cases} N - |l|, & -(N-1) \le l \le (N-1) \\ 0, & \text{otherwise} \end{cases}}$$

3. A causal, LTI system is described by the following LCCDE:

$$y(n) - \tfrac{1}{4}\, y(n-1) = x(n), \quad n \geq 0 \text{ with } y(-1) = 1, \text{ and } x(n) = (\tfrac{1}{2})^n u(n) \qquad (1)$$

Determine the complete response y(n), n ≥ 0. Identify the zero input and zero state responses. Identify the homogeneous and particular solutions.

Taking 1-sided z-transform of the LCCDE, we have:

$$Y(z) - \frac{1}{4}\left[z^{-1}Y(z) + y(-1)\right] = X(z)$$

$$\Rightarrow Y(z) = \underbrace{\frac{X(z)}{(1 - \frac{1}{4}z^{-1})}}_{Y_{zs}(z)} + \underbrace{\frac{\frac{1}{4}y(-1)}{(1 - \frac{1}{4}z^{-1})}}_{Y_{zI}(z)}$$

Zero State Response

$$Y_{zs}(z) = \frac{X(z)}{1 - \frac{1}{4}z^{-1}} = \frac{1}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})}$$

$$= \frac{A}{(1 - \frac{1}{4}z^{-1})} + \frac{B}{(1 - \frac{1}{2}z^{-1})}$$

$$A = Y_{zs}(z) \cdot (1 - \tfrac{1}{4}z^{-1})\Big|_{z^{-1}=4} = \frac{1}{1 - \frac{1}{2}z^{-1}}\Big|_{z^{-1}=4} = \boxed{-1}$$

$$B = Y_{zs}(z)(1 - \tfrac{1}{2}z^{-1})\Big|_{z^{-1}=2} = \frac{1}{1 - \frac{1}{4}z^{-1}}\Big|_{z^{-1}=2} = \boxed{2}$$

$$\Rightarrow \boxed{y_{zs}(n) = \left[-(\tfrac{1}{4})^n + (\tfrac{1}{2})^n\right] u(n)}$$

Zero Input Response:

$$Y_{ZI}(z) = \frac{1/4\, y(-1)}{(1 - 1/4\, z^{-1})} = \frac{1/4}{(1 - 1/4\, z^{-1})}$$

$$\boxed{y_{zi}(n) = \frac{1}{4} \cdot (1/4)^n\, u(n)}$$

Total Solution:

$$y(n) = y_{zs}(n) + y_{zi}(n)$$

$$\boxed{y(n) = \left[-3/4 \cdot (1/4)^n + 2(1/2)^n\right] u(n).}$$

Homogeneous Soln:

$$y_h(n) = -3/4\,(1/4)^n\, u(n)$$

Particular Soln.

$$y_p(n) = 2(1/2)^n\, u(n)$$

4. The input to an LTI system is given by x(n) = u(n) – u(n-3), and the impulse response of the LTI system is given by h(n) = u(n) – u(n-6). Determine and sketch the output, y (n).

$$X(z) = 1 + z^{-1} + z^{-2} \quad , |z| > 0$$

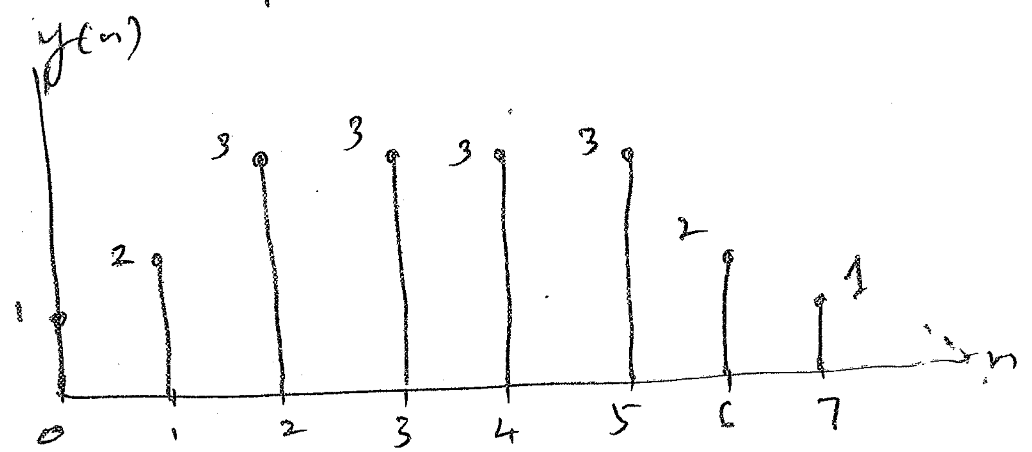
$$H(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} \quad , |z| > 0.$$

$$Y(z) = \cancel{X}(z)\, H(z) \quad , \quad |z| > 0.$$

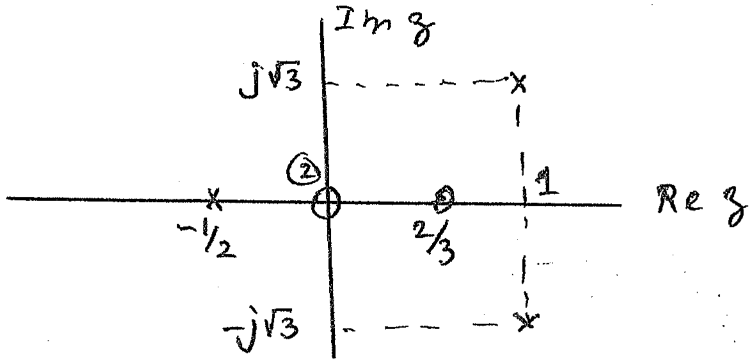
$$\begin{aligned} = \; & 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} \\ & + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6} \\ & + z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6} + z^{-7} \end{aligned}$$

$$Y(z) = 1 + 2z^{-1} + 3z^{-2} + 3z^{-3} + 3z^{-4} + 3z^{-5} + 2z^{-6} + z^{-7}$$

$$\rightarrow \quad y(n) = \{\underset{\uparrow}{1}, 2, 3, 3, 3, 3, 2, 1\}$$



5. The pole-zero plot of an LTI system is shown below. The gain of the system is 9/8. Determine <u>all possible impulse response sequences</u> associated with the information given. Comment on the stability of each sequence, giving appropriate reason. [Note: The zero at the origin is of multiplicity 2, i.e. it is a double zero at the origin.]



$$H(z) = 9/8 \; \frac{z^2 (z - 2/3)}{(z+1/2)(z^2 - 2r\cos\omega_0 z + r^2)}$$

where $r = 2$, $\cos\omega_0 = 1/2 \longrightarrow \boxed{\omega_0 = \pi/3 \text{ rad}}$

or $$H(z) = \frac{9/8\,(1 - 2/3\, z^{-1})}{(1 + 1/2\, z^{-1})(1 - 2z^{-1} + 4z^{-2})}$$

$$= \frac{A}{(1 + 1/2\, z^{-1})} + \frac{Bz^{-1} + C}{(1 - 2z^{-1} + 4z^{-2})} \quad \text{—(1)}$$

$$\boxed{A} = H(z)(1 + 1/2\, z^{-1})\Big|_{z^{-1} = -2} = \frac{9/8\,(1 - 2/3(-2))}{1 - 2(-2) + 4(-2)^2} = \frac{9/8 \times 7/3}{21} = \boxed{\frac{1}{8}}$$

$$\frac{1}{8}(1 - 2z^{-1} + 4z^{-2}) + Bz^{-1} + \frac{1}{2}Bz^{-2} + C + \frac{1}{2}Cz^{-1} = 9/8 - 3/4\, z^{-1}$$

$\Rightarrow \boxed{B = -1}$ $\boxed{C = 1}$

$$H(z) = \frac{1/8}{(1 + 1/2\, z^{-1})} + \frac{(1 - z^{-1})}{(1 - 2z^{-1} + 4z^{-2})}$$

ROC 1: $|z| > 2$, causal $h(n)$, unstable (unit circle not in ROC)

$$\boxed{h(n) = \left[\frac{1}{8}\left(-\frac{1}{2}\right)^n + (2)^n \cos\left(\frac{\pi}{3} n\right)\right] u(n)}$$

ROC 2: $|z| < \frac{1}{2}$, $h(n)$ anticausal, unstable
(unit cle not in ROC)

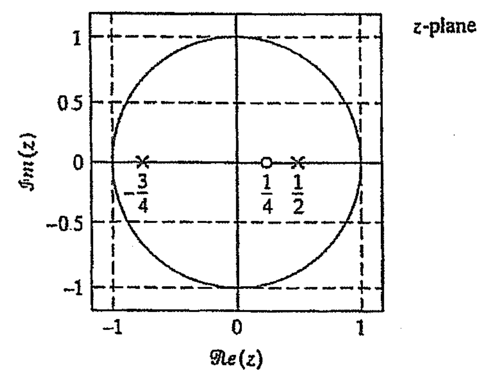
$$\boxed{h(n) = \left[-\frac{1}{8}\left(-\frac{1}{2}\right)^n - (2)^n \cos(\pi/3\, n)\right] u(-n-1)}$$

ROC 3: $\frac{1}{2} < |z| < 2$, $h(n)$ two-sided, stable
(ROC contains unit circle)

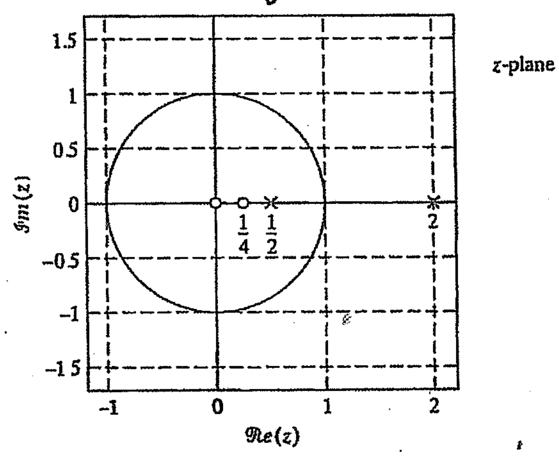
$$\boxed{h(n) = \frac{1}{8}(-1/2)^n u(n) - 2^n \cos(\pi/3\, n) u(-n-1)}$$

6. The signal y (n) is the output of an LTI system with impulse response h (n) for a given stable, input sequence x(n). Throughout the problem, assume that y (n) is stable. The pole-zero configurations of X (z) and Y (z) are shown below. (a) What is the ROC of Y (z)? (b) Is y (n) right-sided, left-sided or two-sided? (c) What is the ROC of X (z)? (d) Is x (n) a causal sequence? (e) Draw the pole-zero plot of H(z) and specify its ROC. (f) Is h (n) causal, anti-causal or two-sided? [Note: Label your answers.]

$X(z)$



$Y(z)$



Stability implies that ROC includes unit circle.

(a) ROC of $Y(z)$: $\boxed{\frac{1}{2} < |z| < 2}$

(b) $y(n)$ is $\boxed{\text{two-sided.}}$

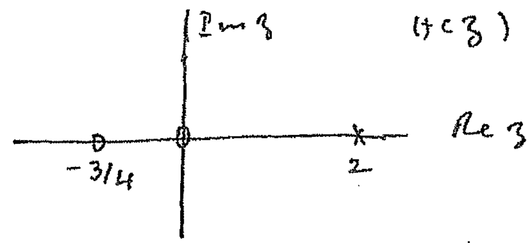
(c) ROC of $X(z)$: $\boxed{|z| > \frac{3}{4}}$

(d) $x(n)$ is $\boxed{\text{Causal}}$.

(e) $$H(z) = \frac{Y(z)}{X(z)} = \left[\frac{\cancel{(1-\frac{1}{4}z^{-1})}\,z}{\cancel{(1-\frac{1}{2}z^{-1})}(1-2z^{-1})}\right]\left[\frac{(1+\frac{3}{4}z^{-1})\cancel{(1-\frac{1}{2}z^{-1})}}{\cancel{(1-\frac{1}{4}z^{-1})}}\right]$$

$$= \frac{(1+\frac{3}{4}z^{-1})\,z}{(1-2z^{-1})}$$

Valid ROC : $\boxed{|z| < 2}$



(f) $h(n)$ is $\boxed{\text{anti Causal}}$

Part B: SHORT QUESTIONS (<u>Write only the answers</u>. Do any scratch work needed on separate sheet(s), which should not be submitted. <u>Credit is given only for correct answers</u>.) (40 points)

1. Which of the following three DT sequences are (i) stable? (ii) causal?

   (a) $x(n) = (-1)^n u(-n)$, (b) $= x(n) = (-1)^n u(n)$, (c) $x(n) = (-1/2)^n u(n) + (-1/2)^n u(-n)$

(i) $\boxed{\text{NONE}}$ (ii) $\boxed{\text{(b) only}}$

2. The CT signal $x(t) = \sin(2\pi\, 10\, t + \pi/4)$ is sampled at the rate of 15 samples per second to obtain the DT signal x(n), which is reconstructed at the rate of 30 samples per second to a obtain y(t). Write the expression for y (t). Assume ideal C/D and D/C converters.

$$\boxed{y(t) = -\sin(2\pi 5t - \pi/4)}$$

3. What is the DT sequence x (n) that corresponds to $X(z) = (1+2z)(1+2z^{-1})$?

$$\boxed{x(n) = \{2 \ \underset{\uparrow}{5} \ 2\}}$$

4. Which of the following DT signals are periodic? For periodic signals, write the corresponding fundamental periods.

   (a) $x(n) = \cos(1/3\ n)$, (b) $x(n) = \cos(3\pi n/7)$,

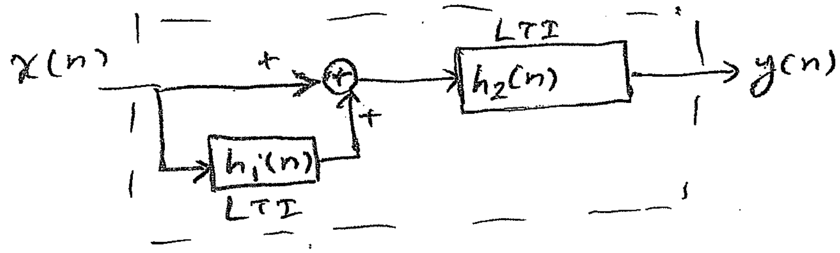
   (c) $x(n) = \sum_{k=-\infty}^{\infty} w(n-4k)$, with $w(n) = \delta(n) - \delta(n-5)$

(a) → not periodic

(b) → Periodic. Fund. Period = 14 samples

(c) → Periodic. Fund. Period = 4 samples

5. What is the impulse response, h (n) of the overall causal system shown below?



$h_1(n) = \frac{1}{2}\delta(n-1)$

$h_2(n) = \left(\frac{1}{2}\right)^n u(n)$

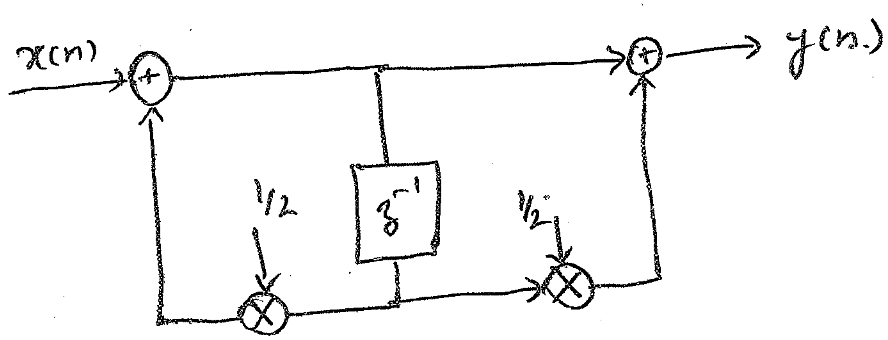
$h(n) = h_2(n) + \frac{1}{2} h_2(n-1)$

$$\Rightarrow \boxed{h(n) = \left(\frac{1}{2}\right)^n u(n) + \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u(n-1) = \delta(n) + \left(\frac{1}{2}\right)^{n-1} u(n-1).}$$

6. What is the LCCDE that describes the input-output relationship for the system shown in the previous question?

$y(n) - \frac{1}{2} y(n-1) = x(n) + \frac{1}{2} x(n-1).$

7. Draw a Direct Form II implementation of the system in #5 above.



8. A band pass signal extends from 88 kHz to 100 kHz. What is the minimum sampling in kHz in order to reconstruct the signal from the sampled values?

$$\boxed{F_{s_{min}} = 25 \text{ kHz}}$$

9. A signal is uniformly sampled at 1 kHz and using uniform binary quantization the data rate is 10 KBPS. The resulting SQNR is 50 dB. (a) Customer A desires an SQNR of at least 60 dB. What will be the smallest data rate to satisfy this requirement? What will be the new SQNR? (b) Customer B desires only a minimum SQNR of 40 dB. What will be the data rate to satisfy this requirement? What will be the new SQNR?

(a) $\boxed{12 \text{ KBPS}}$, $\boxed{62 \text{ dB}}$

(b) $\boxed{9 \text{ KBPS}}$, $\boxed{44 \text{ dB}}$

10. For each of the following cases, write the ROC of the z-transform (if it exists).

(a) $x_1(n) = (1/2)^n\, u(n+3)$, (b) $x_2(n) = (1/2)^n\, u(-n+3)$, (c) $x_3(n) = x_1(n) - x_2(n)$

(a) ROC: $\frac{1}{2} < |z| < \infty$

(b) ROC: $0 < |z| < 1/2$

(c) Z- transform does not exist.