SoLUTION

EE 6360-501
Exam 1
October 13, 2009

Please Print.

Last Name **First Name**

Instructions

1. Examination Duration: 1 hour 15 minutes.
2. ONE 8.5" x 11"sheet, with both sides of hand-written material, is allowed. No solved problems of any kind are allowed.
3. One calculator is allowed.
4. Answer in the space/sheets provided. **Highlight your answers.**
5. **Answer all questions. Part A has 7 questions for 40 points; Part B has 3 questions, equally weighted for 60 points. There is a bonus question at the end for 10 points. Show all steps/logic, unless the question instructs you otherwise. DO NOT RELY ON PARTIAL CREDIT, WHICH IS SOLELY AT THE DISCRETION OF THE INSTRUCTOR.**
6. Any copying or cheating will result in appropriate action as per university regulations.

Part A: SHORT QUESTIONS (<u>Write only the answers</u>. Do any scratch work needed on separate sheet(s), which should not be submitted. <u>Credit is given only for correct answers</u>.) The first 5 questions, 5 points each, #6, 7 points and #7, 8 points. (40 points)

1. Let x (n) be a z-transformable sequence, and N a positive integer. Let y(n) = x(n/N) for n divisible by N, and zero otherwise. Express Y (z) in terms of X (z).

$$Y(z) = X(z^N)$$

2. The impulse response of an LTI has the following constraint: $(2)^n < h(n) < (3)^n$, $n \le -1$, and h(n) =0 otherwise. Is this system BIBO stable? State clearly your reason for the answer.

- BIBO stable.
- Reason: $\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{-1} |h(n)| = S_H$

$\sum_{n=-\infty}^{-1} 2^n < S_H < \sum_{n=-\infty}^{-1} 3^n$. Both bounds are finite. $\Rightarrow$ BIBO stable

3. The continuous time signal x (t) = cos (2 π 1000 t) is sampled with a sampling period of T seconds to obtain a discrete time x(n) = cos (πn/4). (i) What is the value of T consistent with this information, and (ii) Is the choice of T unique? If not, specify another choice of T consistent with the information given.

- $(k + 1/8) = 1000T$, where $k = 0, 1, 2, \cdots$ (why?)
- $T = \frac{1}{8000}$ sec $= 125\,\mu$sec.
- Another choice: $T = 9/8000$ sec.

4. The sequence x (n) = $(½)^n$ u (n) is periodically extended to repeat every 3 samples. Write one period of the resulting periodic signal.

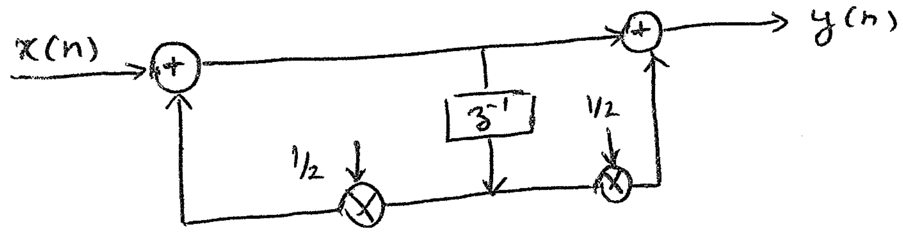
$$[8/7,\ 4/7,\ 2/7]$$

5. A periodic signal, x (n) created by repeating the prototype signal xp (n) = {1 2 3} every 4 samples, is input to an LTI system with the impulse response h (n) = {1 -1 1}. What is the output signal, y (n)?

$$y_p(n) = \sum_{l=-\infty}^{\infty} y(n - l4)$$

where $y(n) = [4 \quad 1 \quad 2 \quad -1]$

6. The Direct Form II block diagram implementation of a relaxed causal LTI system is shown below. Write the LCCDE that describes the system. (7 points)



$$y(n) - \tfrac{1}{2} y(n-1) = x(n) + \tfrac{1}{2} x(n-1)$$

7. Identify all possible ROCs that correspond to the transfer function of an LTI system. For each ROC, identify whether the sequence is (i) stable, (ii) causal, anti-causal or two-sided. (8 points)

$$H(z) = \frac{(z-3)^4}{(z-\frac{1}{2})(z+2)^2}$$

Poles at $z = \frac{1}{2}$, $z = -2$ and $z = \infty$. (why?)

ROC	stability	Causal/anti-Causal/2-sided
1. $\lvert z \rvert < \frac{1}{2}$	unstable	anti-Causal
2. $\frac{1}{2} < \lvert z \rvert < 2$	stable	2-sided
3. $2 < \lvert z \rvert < \infty$	unstable	2-sided. (Right sided).

Part B: (60 points)

1. The impulse response of an LTI system is given by h (n) = (½)^n u(n). Determine the output of the system, y(n), for the input

$$x(n) = [1/2^{1/2}]^n \sin(\pi n/4)\, u(n).$$

$$H(z) = \frac{1}{(1-\frac{1}{2}z^{-1})}, \; |z| > \frac{1}{2}, \quad X(z) = \frac{\frac{1}{2}z^{-1}}{(1-z^{-1}+\frac{1}{2}z^{-2})}, \; |z| > \frac{1}{\sqrt{2}}$$

$$Y(z) = H(z)X(z)$$

$$\Longrightarrow$$

$$Y(z) = \frac{\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1}+\frac{1}{2}z^{-2})}, \quad |z| > \frac{1}{\sqrt{2}}.$$

$$Y(z) = \frac{\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1}+\frac{1}{2}z^{-2})} = \frac{A}{(1-\frac{1}{2}z^{-1})} + \frac{Bz^{-1}+C}{(1-z^{-1}+\frac{1}{2}z^{-2})}$$

$$A = Y(z)(1-\tfrac{1}{2}z^{-1})\Big|_{z^{-1}=2} = \boxed{1}$$

At $z^{-1}=0$, $Y(z) = 0 = A + C \Longrightarrow \boxed{C=-1}$

At $z^{-1}=1$, $Y(z) = \frac{1}{2} = \frac{A}{2} + \frac{B+C}{2} \Longrightarrow \boxed{B=1}$

Thus,

$$Y(z) = \frac{1}{(1-\frac{1}{2}z^{-1})} + \frac{-(1-z^{-1})}{(1-z^{-1}+\frac{1}{2}z^{-2})}$$

$$= \frac{1}{(1-\frac{1}{2}z^{-1})} - \frac{(1-\frac{1}{2}z^{-1})}{(1-z^{-1}+\frac{1}{2}z^{-2})} + \frac{\frac{1}{2}z^{-1}}{(1-z^{-1}+\frac{1}{2}z^{-2})},$$

$$|z| > \frac{1}{\sqrt{2}}.$$

$\Longrightarrow$

$$\boxed{y(n) = \left\{ \left(\tfrac{1}{2}\right)^n - \left(\tfrac{1}{\sqrt{2}}\right)^n \cos\left(\tfrac{\pi}{4}n\right) + \left(\tfrac{1}{\sqrt{2}}\right)^n \sin\left(\tfrac{\pi}{4}n\right) \right\} u(n)}$$

2. The input to an LTI system is

$$x(n) = \{\underset{\wedge}{1}\ \ -1\ \ 1\}$$

and the cross correlation function between the input x(n) and output y(n) is given by

$$r_{xy}(n) = \{2\ \ -3\ \ 4\ \ \underset{\wedge}{-1}\ \ 0\ \ 1\}.$$

Determine the autocorrelation function of y (n).

Range of various quantities:

$x(n) \to [0,2]$ $\quad r_{xy}(n) = x(n) * y(-n) \to [-3,2]$

$\Rightarrow y(-n) \to [-3,0] \Rightarrow y(n) \to [0,3]$

$\Rightarrow h(n) \to [0,1]$.

Let $h(n) = [\underset{\uparrow}{\alpha}\ \ \beta]$.

$y(n) = \alpha \cdot x(n) + \beta\, x(n-1)$

$$\left.\begin{array}{cccc} \alpha & -\alpha & \alpha & \\ & -\beta & -\beta & -\beta \\ \hline [\underset{\uparrow}{\alpha} & (\beta-\alpha) & (\alpha-\beta) & \beta\ ] \end{array}\right\} \Rightarrow \boxed{y(n) = \{\underset{\uparrow}{\alpha}\ (\beta-\alpha)\ (\alpha-\beta)\ \beta}$$

$$r_{xy}(\ell) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-\ell) = \sum_{n=0}^{2} x(n)\, y(n-\ell).$$

$$r_{xy}(2) = x(0)\cancel{y(-2)}^{\,0} + x(1)\cancel{y(-1)}^{\,0} + \cancel{x(2)}^{\,1}\cancel{y(0)}^{\,\alpha} = \boxed{\alpha}$$

But $r_{xy}(2) = 1 \Rightarrow \boxed{\alpha = 1}$

$$r_{xy}(1) = x(0)\cancel{y(-1)}^{\,0} + x(1)y(0) + x(2)y(1)$$
$$= \alpha - (\beta-\alpha) = 2\alpha - \beta = 0 \Rightarrow \beta = 2\alpha \Rightarrow \boxed{\beta = 2}$$

Thus, $\boxed{y(n) = \{\underset{\uparrow}{1}\ \ 1\ \ -1\ \ 2\}}$, $\quad h(n) = \{\underset{\uparrow}{1}, 2\}$

$$R_{yy}(z) = Y(z)\,Y(z^{-1}) = [1 + z^{-1} - z^{-2} + 2z^{-3}][1 + z - z^{2} + 2z^{3}]$$
$$= 2z^{3} + z^{2} - 2z + 7 - 2z^{-1} + z^{-2} + 2z^{-3}$$

$$\Rightarrow \boxed{r_{yy}(\ell) = \{2\ \ 1\ \ -2\ \ \underset{\uparrow}{7}\ \ -2\ \ 1\ \ 2\}}$$

Alternate method:

$$R_{xy}(z) = X(z)\, Y(z^{-1})$$

$$\Rightarrow \quad Y(z^{-1}) = \frac{R_{xy}(z)}{X(z)}$$

$$\Rightarrow \quad R_{yy}(z) = Y(z^{-1})\, Y(z)$$

$$\Rightarrow \quad r_{yy}(l) = Z^{-1}\left[R_{yy}(z) \right]$$

Student Exercise Follow above steps to arrive at the same answer as the previous page.

3. A causal, LTI system is described by the LCCDE with zero initial conditions:

$$y(n) = -y(n-6) + 2x(n) + x(n-5) \quad \text{—(1)}$$

(i) Determine the following values of the impulse response of the system: h(0), h(1), h(5), h(6), h(11) and h(120). (15 points.) (ii) Prove or disprove that the system is BIBO stable. (5 points.)

(i) Impulse response:

By taking z-transform of (1) with $x(n) = \delta(n)$, we get

$$Y(z) = \frac{(2 + z^{-5})}{(1 + z^{-6})}, \quad |z| > 1 \quad \text{—(2)}$$

Let $H_1(z) = \dfrac{1}{(1+z^{-6})}$.

with $h_1(n) = \begin{cases} h_0(n/6), & n \text{ divisible by } 6 \\ 0, & \text{otherwise} \end{cases}$ —(3)

and $h_0(n) = (-1)^n u(n)$. —(4) [Note! This is addressed in Problem A.1]

But with $x(n) = \delta(n)$ and zero I.C, $H(z) = Y(z)$ (2)

$\Rightarrow$ $h(n) = 2h_1(n) + h_1(n-5)$ —(5)

where $h_1(n) = \begin{cases} (-1)^{n/6}, & n \text{ divisible by } 6, \; n \geq 0 \\ 0, & \text{otherwise.} \end{cases}$ —(6)

From (5) & (6),

$h(0) = 2 \quad h(1) = 0 \quad h(5) = 1, \; h(6) = -2, \; h(11) = -1$

and $h(120) = 2$.

Note that $h(n)$ is the causal part of a "periodic signal" with alternating signature.

$h(n) = [\underset{\uparrow}{2} \; 0 \; 0 \; 0 \; 0 \; 1 \,|\, -2 \; 0 \; 0 \; 0 \; 0 \; 0 \; -1 \,|\, 2 \; 0 \; 0 \; 0 \; 0 \; 0 \; 1 \,|\, -2 \ldots]$

(ii) stability:

Not BIBO stable.

ROC of $H(z)$ does not include the unit circle.

Note: you can calculate the required values from (1) by recursion!

Bonus (10 points)

A discrete time, causal FIR filter is designed to reject (i.e. completely zero out) the dc, 100 Hz and half-sampling frequency components of a CT input signal x (t), sampled at the rate of 600 samples/sec. (a) Sketch the pole-zero diagram of the transfer function H (z) of the FIR filter. (Be sure to indicate any zeros or poles at the origin and/or infinity.), and (b) Determine H (z).

DT frequencies to be zeroed out: $0, \frac{\pi}{3}$ & π rad/sample

$\uparrow \rightarrow \frac{100 \times 2\pi}{600} = \pi/3$.

Thus we need zeros at

$z = 1, \quad z = -1$ and $z = e^{\pm j\pi/3}$.

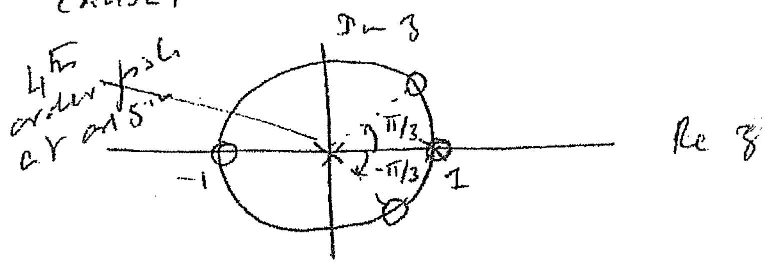
This will give us the non-causal transfer function

$$H_{NC}(z) = (z-1)(z+1)(z-e^{j\pi/3})(z-e^{-j\pi/3})$$

$$= (z^2-1)(z^2-z+1) = \boxed{z^4 - z^3 + z - 1} \quad \text{—(1)}$$

However, we make the system causal, by delaying the impulse response by 4 samples; i.e.

$$H_{causal}(z) = z^{-4} H_{NC}(z) = \boxed{1 - z^{-1} + z^{-3} - z^{-4}} \quad \text{—— (2)}$$



From (1),

$$\boxed{h(n) = \{\underset{\uparrow}{1} \; -1 \; 0 \; 1 \; -1\}}$$

The filter is stable as the impulse response is absolutely summable.