SOLUTION

EE 6360
Exam 2
April 3, 2008

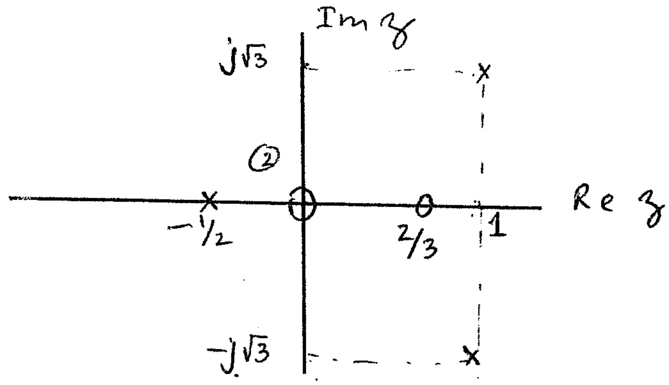
Please Print.

Last Name **First Name**

Instructions

1. Examination Duration: 1 hour 15 minutes. If you arrive 15 minutes early, you can have that extra time.
2. ONE 8.5" x 11"sheet, with both sides of hand-written material, is allowed. No solved problems of any kind are allowed. Violation will result in a grade of zero for this exam, and other disciplinary actions may be taken.
3. One calculator is allowed.
4. Answer in the space/sheets provided.
5. **Answer all 5 questions. There is a bonus question at the end for 10 points.**
6. **DO NOT RELY ON PARTIAL CREDIT, WHICH IS SOLELY AT THE DISCRETION OF THE INSTRUCTOR.**
7. Any copying or cheating will result in appropriate action as per university regulations.

1. (20 pts)
The pole-zero plot of an LTI system is shown below. The gain of the system is 9/8. Determine <u>all possible impulse response sequences</u> associated with the information given. Comment on the stability of each sequence, giving appropriate reason. [Note: The zero at the origin is of multiplicity 2, i.e. it is a double zero at the origin.]



$$H(z) = 9/8 \; \frac{z^2 (z - 2/3)}{(z + 1/2)(z^2 - 2r\cos\omega_0 z + r^2)}$$

where $r = 2$, $\cos\omega_0 = 1/2 \longrightarrow \boxed{\omega_0 = \pi/3 \text{ rad}}$

or
$$H(z) = \frac{9/8\,(1 - 2/3\, z^{-1})}{(1 + 1/2\, z^{-1})(1 - 2z^{-1} + 4z^{-2})}$$

$$= \frac{A}{(1 + 1/2\, z^{-1})} + \frac{Bz^{-1} + C}{(1 - 2z^{-1} + 4z^{-2})} \quad \text{—(1)}$$

$$\boxed{A} = H(z)(1 + 1/2\, z^{-1})\Big|_{z^{-1} = -2} = \frac{9/8\,(1 - 2/3(-2))}{1 - 2(-2) + 4(-2)^2} = \frac{9/8 \times 7/3}{21} = \boxed{\frac{1}{8}}$$

$$\frac{1}{8}(1 - 2z^{-1} + 4z^{-2}) + Bz^{-1} + \frac{1}{2}Bz^{-2} + C + \frac{1}{2}Cz^{-1} = 9/8 - 3/4\, z^{-1}$$

$$\Longrightarrow \boxed{B = -1} \qquad \boxed{C = 1}$$

$$H(z) = \frac{1/8}{(1 + 1/2\, z^{-1})} + \frac{(1 - z^{-1})}{(1 - 2z^{-1} + 4z^{-2})}.$$

ROC 1: $|z| > 2$, causal $h(n)$, unstable (unit circle not in ROC)

$$h(n) = \left[\frac{1}{8}\left(-\frac{1}{2}\right)^n + (2)^n \cos\left(\frac{\pi}{3} n\right)\right] u(n)$$

ROC 2: $|z| < \frac{1}{2}$, $h(n)$ anticausal, unstable
(unit ck not in ROC)

$$\boxed{h(n) = \left[-\frac{1}{8}\left(-\frac{1}{2}\right)^n - (2)^n \cos(\pi/3\, n)\right] u(-n-1)}$$

ROC 3: $\frac{1}{2} < |z| < 2$, $h(n)$ two-sided, stable
(ROC contains unit circle)

$$\boxed{h(n) = \frac{1}{8}\left(-\frac{1}{2}\right)^n u(n) - 2^n \cos(\pi/3\, n)\, u(-n-1)}$$

2. (20 pts)
(a) Design a causal FIR filter of length 3 samples, which notches out the continuous time frequency of 3/8th the sampling frequency, and has a unity gain at 1/4th sampling frequency. Sketch the pole-zero diagram of the filter. (10 points). **(b)** A continuous time signal x (t) = 4 + 2cos (2π 300t) + 2 cos(2π 200t) is C/D converted at the rate of 800 samples per second, and the resulting discrete time signal, x(n), is input to the above FIR filter. The output, y (n), is D/C converted to the continuous time signal y(t) at the same rate as C/D conversion. Determine the signal y (t). (10 points)

a

Notch frequency, $w_0 = 2\pi \cdot \dfrac{3/8\, F_s}{F_s} = \boxed{\dfrac{3\pi}{4} \text{ rad/sec}}$

$$H_c(z) = G\left(1 - 2\cos\left(\tfrac{3\pi}{4}\right) z^{-1} + z^{-2}\right)$$

↑
Causal

$$= \boxed{G\left(1 + \sqrt{2}\, z^{-1} + z^{-2}\right).}$$

$$H_c(w) = G\left(1 + \sqrt{2}\, e^{-jw} + e^{-jw2}\right)$$

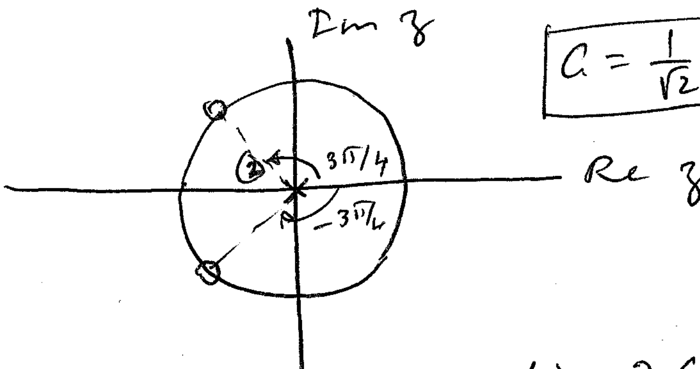
Requirement: $|H_c(\tfrac{\pi}{2})| = 1$

$$H_c(\pi/2) = G\left(1 + \sqrt{2}\, e^{-j\pi/2} + \cancel{e^{-j\pi}}\right)$$

$$= \sqrt{2}\, G\, e^{-j\pi/2}$$

$$\Rightarrow \boxed{G = \frac{1}{\sqrt{2}}.} \Rightarrow \boxed{H_c(\tfrac{\pi}{2}) = 1 \cdot e^{-j\frac{\pi}{2}}}$$

Pole-zero Diagram



$\boxed{G = \frac{1}{\sqrt{2}}}$

b

$$x(t) = 4 + 2\cos(2\pi \cdot 200t) + 2\cos(2\pi\, 300t)$$

↓ $F_s = 800$ Hz

$$x(n) = 4 + 2\cos(2\pi \cdot \tfrac{1}{4} n) + 2\cos(\tfrac{3\pi}{4} n).$$

$$H(0) = \frac{1}{\sqrt{2}}(1+\sqrt{2}+1) = \frac{\sqrt{2}}{\sqrt{2}}(1+\sqrt{2}) = \boxed{1+\sqrt{2}}$$

$$y(n) = \overset{(1+\sqrt{2})}{\cancel{H(0)}}\cdot 4 + \overset{1}{\cancel{\left|H\left(\tfrac{\pi}{2}\right)\right|}}\cdot 2\cos\left(\frac{\pi}{2}n + \overset{-\pi/2}{\cancel{\theta_H\left(\tfrac{\pi}{2}\right)}}\right)$$
$$+ \overset{0}{\cancel{\left|H\left(\tfrac{3\pi}{4}\right)\right|}}\cdot 2\cos\left(\frac{3\pi}{4}n + \theta_H\left(\frac{3\pi}{4}\right)\right)$$

$\Longrightarrow$

$$\boxed{y(n) = 4(1+\sqrt{2}) + 2\cos\left(\frac{\pi}{2}n - \pi/2\right)}$$

$$y(t) = 4(1+\sqrt{2}) + 2\cos\left(2\pi\cdot\frac{1}{4}800t - \pi/2\right)$$

$\Rightarrow$

$$\boxed{y(t) = 4(1+\sqrt{2}) + 2\cos\left(2\pi\, 200t - \pi/2\right)}$$

3. (20 pts)
The signal $y(n) = x(n) + w_p(n)$, where $x(n)$ is the desired signal, and $w_p(n)$ is a periodic interference corrupting the signal. The desired signal is given by

$$x(n) = 2\cos(\pi n/4 + \pi/8)$$

and the interference is given by

$$w_p(n) = \sum_{k=-\infty}^{\infty} w(n - k4), \qquad \text{with } w(n) = \{\underset{\wedge}{3}\ 0\ 1\ 0\}$$

Design a causal FIR filter [i.e. determine H (z)] that will exactly recover x (n) from y(n) while suppressing the interference completely.

The interference is periodic with fundamental period of 4 samples. Thus, there are four harmonics to be suppressed. These are

$$s_0(n) = 1 \quad (D.C)$$

$$s_1(n) = e^{j\frac{2\pi}{4}1.n}$$

$$s_2(n) = e^{j\frac{2\pi}{4}.2n}$$

$$s_3(n) = e^{j\frac{2\pi 3}{4}n}$$

We need to place zeros at $\omega = 0, \pi/2, \pi$ and $\frac{3\pi}{2}$.
To make the system causal, we place 4 poles at the origin. We multiply by a gain factor, G, to make the filter response to be unity at $\omega = \pi/4$, the desired signal frequency.

Thus

$$H(z) = G(1 - z^{-4})$$

$$H\left(\frac{\pi}{4}\right) = H(z)\Big|_{z = e^{j\frac{\pi}{4}}} = G\left(1 - e^{-j\frac{\pi}{4}4}\right) = 2G.$$

If we set $H\left(\frac{\pi}{4}\right) = 1$, then $\boxed{G = \frac{1}{2}}$

Thus the FIR causal filter is

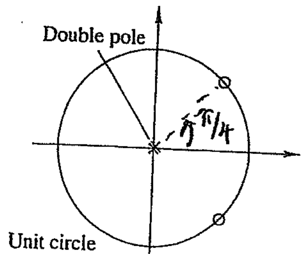
$$\boxed{H(z) = \frac{1}{2}(1 - z^{-4})}$$

This filter will exactly recover $x(n)$.

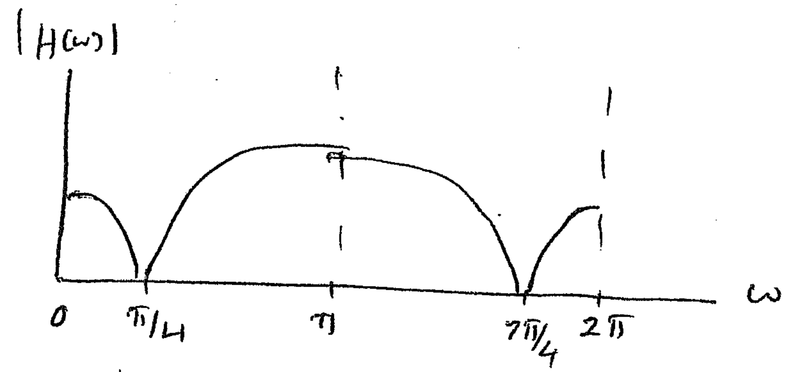
4. (Two Parts)

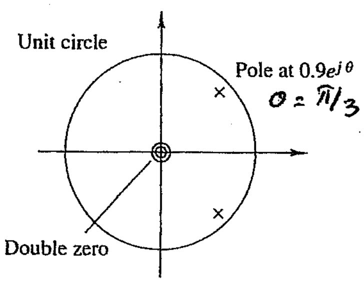
(a) Sketch roughly the magnitude response |H (ω)| corresponding to the following pole-zero patterns. Your sketch should be for $0 \le \omega \le 2\pi$. **(10 points)**

[You must clearly indicate all key frequencies where the peaks and valleys occur].

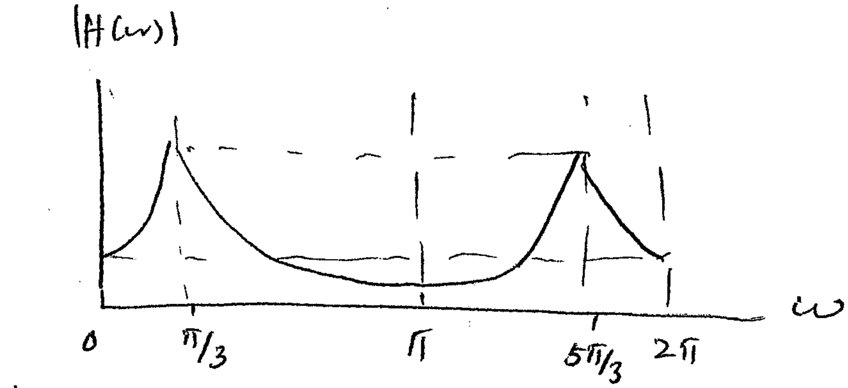


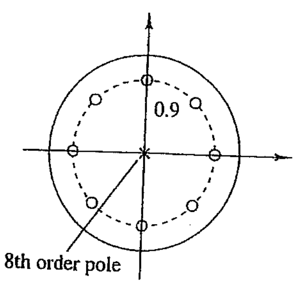
(a)



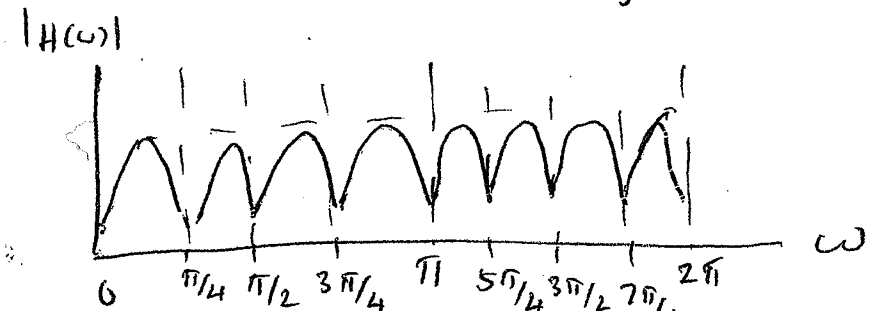


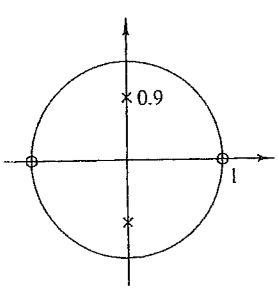
(b)



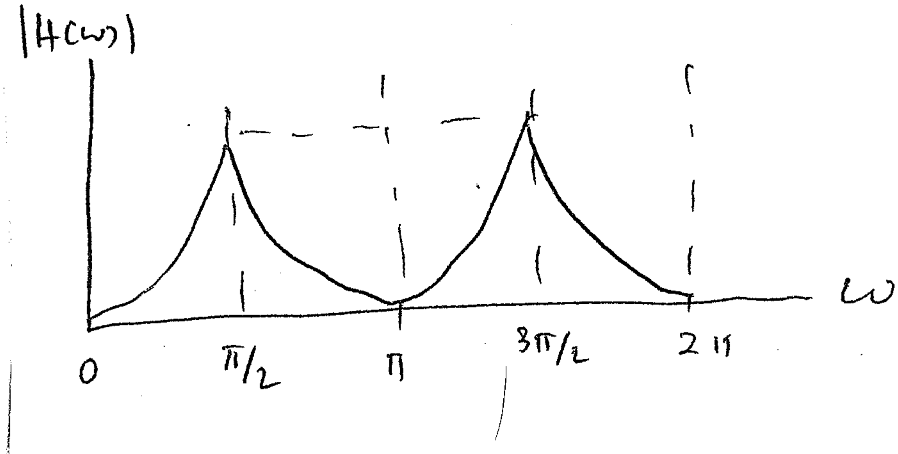


(c)





(d)



(b) [10 points] Determine the DT sequence x(n) that corresponds to the DFS coefficients

$$c_k = 2\cos[(2\pi/3)k] + 2\cos[(2\pi/5)k] \quad \text{—(1)}$$

$$C_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} \quad \text{—(2)}$$

- From (1), because C_k's are periodic in k with a fund. period of N, we note that $\boxed{N=15}$.
- We rewrite (1) using Euler's identities

$$C_k = 1\cdot e^{j\frac{2\pi}{15}k\cdot 5} + 1\cdot e^{-j\frac{2\pi}{15}kn} + 1\cdot e^{j\frac{2\pi}{15}\cdot 3} + 1\cdot e^{-j\frac{2\pi}{15}3} \quad \text{—(3)}$$

- But $e^{j\frac{2\pi}{N}kn} = e^{-j\frac{2\pi}{N}k(N-n)}$ [DFS Trick]

which can be used in (3) to write

$$C_k = 1\cdot e^{-j\frac{2\pi}{15}k\cdot 10} + 1\cdot e^{-j\frac{2\pi}{15}k\cdot 5} + 1\cdot e^{-j\frac{2\pi}{15}k12} + 1\cdot e^{-j\frac{2\pi}{15}k\cdot 3} \quad \text{—(4)}$$

- Comparing (1) & (4), in one period $[n=0,1,2\ldots14]$, $x(n)$ is given by

$x(5) = x(10) = x(3) = x(12) = 15$
All other $x(n) = 0$ in one period.
$x(n)$ repeats every 15 samples

5. **[4 short questions] Write only the required answers. No steps should be written. Credit only for the correct answers.**

5.1 What is the impulse response, h (n), of the ideal band pass filter with a center frequency of $3\pi/8$ rad/sample, and a bandwidth of $\pi/4$ rad/sample?

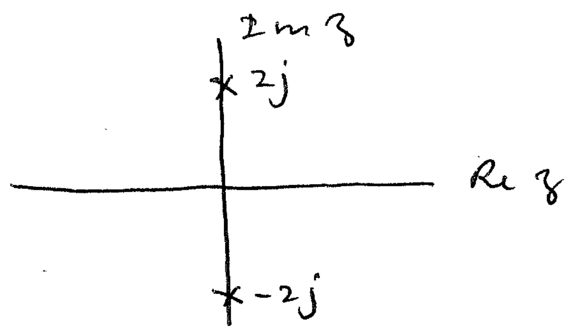
$\boxed{h(n) = 2\,\frac{\sin(\frac{\pi}{8}n)}{\pi n}\cos\left(\frac{3\pi}{8}n\right)}$ OR $\boxed{h(n) = \frac{\sin(\frac{\pi}{2}n) - \sin(\frac{\pi}{4}n)}{\pi n}}$

5.2 What is the DTFT, X (ω), of the DT sequence $x(n) = -(1/2)^n u(-n-1)$?

Does not exist. (unstable sequence).

5.3 The z-transform of a causal sequence x (n) is given by $X(z) = \dfrac{1}{(1 + \frac{1}{4} z^{-2})}$.

Sketch the pole-zero plot of Y (z), the z-transform of y (n) = x (-n).



Gain factor = 4

Two zeros at ∞.

5.4 The DTFT of a DT sequence x (n) is given by X (ω) = 2 – 2sin (3 ω) + 4 cos (4 ω). What is the sequence x (n)?

$\boxed{x(n) = \{2, j, 0, 0, \underset{\uparrow}{2}, 0, 0, -j, 2\}}$

Bonus Question (10 points)

A causal digital filter H (z) is characterized by the following properties:

(1) It is low pass and has one pole and one zero.
(2) The pole is at a distance of 0.5 from the origin of the z-plane.
(3) |H (0)| = 1 (DC gain)
(4) H (π) = 0 (Half sampling frequency response)

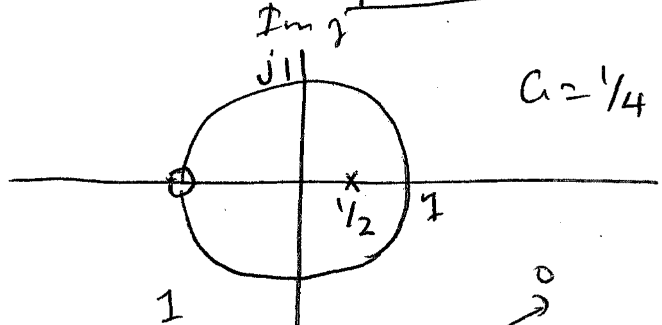
(1) Determine H (z), and sketch its pole-zero plot. (2) Compute the output of the system if the input is

x(n) = 2 + 2 cos (π n + π/4)

Soln. 1

$$H(z) = G\,\frac{(z+1)}{z-1/2} = G\cdot\frac{(1+z^{-1})}{(1-1/2\,z^{-1})} \Rightarrow \boxed{\frac{1}{4}\,\frac{(1+z^{-1})}{(1-1/2\,z^{-1})}}$$

$$|H(0)| = 1 \Longrightarrow \boxed{G = 1/4}$$



$G = 1/4$

$$y(n) = \overset{1}{\cancel{H(0)}}\cdot 2 + \overset{0}{\cancel{|H(\pi)|}}\cdot 2\cdot\cos(\pi n + \pi/4 + \theta_H(\pi))$$

$$\Longrightarrow \boxed{y(n) = 2}$$

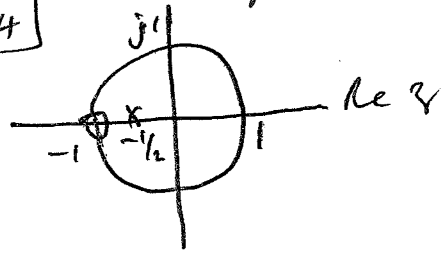
Soln. 2

$$H(z) = G\,\frac{(z+1)}{(z+1/2)} = G\,\frac{(1+z^{-1})}{(1+1/2\,z^{-1})}.$$

$$|H(0)| = 1 \Rightarrow G\,\frac{4}{3} = 1 \Longrightarrow \boxed{G = 3/4}$$

$$\boxed{H(z) = 3/4\cdot\frac{(1+z^{-1})}{1+1/2\,z^{-1}}}$$

$\boxed{G = 3/4}$



$$\boxed{y(n) = \overset{3/4}{\cancel{H(0)}}\cdot 2 = 3/2}$$

because $H(\pi) = 0$.