



Short Notes: Note on a "Square" Functional Equation

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NOTE ON A "SQUARE" FUNCTIONAL EQUATION*

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The ordinary Pascal triangle for the binomial coefficients is well-known; basically, it depends upon the recursion relation

$$f(n, r) + f(n, r - 1) = f(n + 1, r).$$

A natural generalization of this situation occurs if we consider the tableau defined by

$$g(n + 1, r + 1) = g(n, r + 1) + g(n + 1, r) + g(n, r)$$

with the initial conditions

$$g(n, 0) = g(0, r) = 1.$$

We shall develop several expressions for these numbers $g(n, r)$ and show that they have a further combinatorial interpretation, namely, $g(n, r)$ is the volume of an r -sphere in n -space, under the Lee metric.

The tableau for $g(n, r)$ is easily built up, and the first portion of it is reproduced in Table 1. Note that the entry at the lower right corner of a square (of four adjacent entries) is the sum of the entries at the other three corners.

TABLE 1
 $g(n, r)$

1	1	1	1	1	1	1	1
1	3	5	7	9	11	13	15
1	5	13	25	41	61	85	113
1	7	25	63	129	231	377	575
1	9	41	129	321	681	1289	2241
1	11	61	231	681	1683	3653	7183
1	13	85	377	1289	3653	8989	19825
1	15	113	575	2241	7183	19825	48639

Just as in the case of the binomial coefficients, many relations exist among the quantities $g(n, r)$. If we use the usual convention that $\binom{a}{b}$ is defined only for b an integer, and that $\binom{a}{b}$ vanishes for $b < 0$ and for $b \geq a$ (a a positive integer), then we may prove the following.

LEMMA 1.

$$g(n, r) = \sum_{\alpha} \binom{n}{\alpha} \binom{r + \alpha}{n}.$$

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Proof. We readily verify the result for $r = 0$ and $n = 0$. Assume it holds for all i and j with $i \leq n + 1$ or $j \leq r + 1$, with the possible exception $(i, j) = (n + 1, r + 1)$. Then

$$g(n + 1, r + 1) = \sum_{\alpha} \binom{n + 1}{\alpha} \binom{r + \alpha}{n + 1} + \sum_{\alpha} \binom{n}{\alpha} \binom{r + \alpha}{n} + \sum_{\alpha} \binom{n}{\alpha} \binom{r + 1 + \alpha}{n},$$

by virtue of the defining functional equation. Then

$$g(n + 1, r + 1) = \sum_{\alpha} \binom{n + 1}{\alpha} \binom{r + \alpha}{n + 1} + \sum_{\alpha} \binom{n}{\alpha} \binom{r + \alpha}{n} + \sum_{\beta} \binom{n}{\beta - 1} \binom{r + \beta}{n}.$$

The two last terms combine to give

$$\sum_{\alpha} \binom{r + \alpha}{n} \left[\binom{n}{\alpha} + \binom{n}{\alpha - 1} \right] = \sum_{\alpha} \binom{r + \alpha}{n} \binom{n + 1}{\alpha},$$

which combines with the first term to give

$$\sum_{\alpha} \binom{n + 1}{\alpha} \left[\binom{r + \alpha}{n + 1} + \binom{r + \alpha}{n} \right] = \sum_{\alpha} \binom{n + 1}{\alpha} \binom{r + \alpha + 1}{n + 1}.$$

Thus the lemma follows by induction.

We can obtain the $g(n, r)$ as coefficients in a power series in the following.

LEMMA 2. $g(n, r)$ is the coefficient of x^r in the expansion of $(1 + x)^n/(1 - x)^{n+1}$. Or, since $g(n, r) = g(r, n)$, we might use the coefficient of x^n in $(1 + x)^r/(1 - x)^{r+1}$.

Proof.

$$g(n, r) = \sum_{\alpha} \binom{n}{\alpha} \binom{r + \alpha}{n} = \sum_{\alpha} \binom{r + \alpha}{\alpha} \binom{r}{n - \alpha}.$$

Now

$$\binom{-k}{\beta} = (-1)^{\beta} \binom{k + \beta - 1}{\beta};$$

hence

$$\begin{aligned} g(n, r) &= \sum_{\alpha} (-1)^{\alpha} \binom{-(r + 1)}{\alpha} \binom{r}{n - \alpha} \\ &= \text{coefficient of } x^n \text{ in the expansion of } (1 - x)^{-(r+1)}(1 + x)^r. \end{aligned}$$

For actual computation, Lemma 2 is almost as easy as the basic definition; one chooses $\min(n, r)$ for the power a in $(1 + x)^a(1 - x)^{-(a+1)}$.

COROLLARY. If we write $g_n = g(n, r)$, with r fixed, then

$$\sum_{\beta} (-1)^{\beta} \binom{r + 1}{\beta} g_{n-2\beta} = \binom{2r + 1}{n}.$$

Proof.

$$(1 + x)^r = (1 - x)^{r+1} \sum_{\alpha} g_{\alpha} x^{\alpha}.$$

Thus

$$\begin{aligned} (1 + x)^{2r+1} &= (1 - x^2)^{r+1} \sum_{\alpha} g_{\alpha} x^{\alpha} \\ &= \sum_n \left(\sum_{\beta} \binom{r + 1}{\beta} (-1)^{\beta} g_{n-2\beta} \right) x^n. \end{aligned}$$

Comparison of coefficients produces the corollary.

The corollary allows easy computation of the various g_n ; for example,

$$g_0 = 1,$$

$$g_1 = 2r + 1,$$

$$g_2 = 2r^2 + 2r + 1,$$

$$g_3 = (2r + 1)(2r^2 + 2r + 3)/3,$$

$$g_4 = (2r^4 + 4r^3 + 10r^2 + 8r + 3)/3.$$

LEMMA 3.

$$g(n, r) = \sum_{\alpha} 2^{\alpha} \binom{n}{\alpha} \binom{r}{\alpha}.$$

Proof. We have

$$\begin{aligned} g(n, r) &= \sum_{\alpha} \binom{n}{\alpha} \binom{r + \alpha}{n} \\ &= \sum_{\alpha} \binom{n}{\alpha} \sum_{\beta} \binom{r}{\beta} \binom{\alpha}{n - \beta} \\ &= \sum_{\beta} \binom{r}{\beta} \sum_{\alpha} \binom{n}{\alpha} \binom{\alpha}{n - \beta} \\ &= \sum_{\beta} \binom{r}{\beta} \sum_{\alpha} \binom{n}{\beta} \binom{\beta}{n - \alpha} \\ &= \sum_{\beta} \binom{r}{\beta} \binom{n}{\beta} \sum_{\alpha} \binom{\beta}{n - \alpha} \\ &= \sum_{\beta} \binom{r}{\beta} \binom{n}{\beta} 2^{\beta}. \end{aligned}$$

From Lemma 3, we obtain a second combinatorial interpretation for $g(n, r)$; the value

$$\sum_{\beta} \binom{r}{\beta} \binom{n}{\beta} 2^{\beta}$$

is exactly the same value deduced by S. Golomb for the volume of a sphere of radius r in n dimensions (or a sphere of radius n in r dimensions) using the Lee metric¹. This indicates that the numbers $g(n, r)$ possess some applications, in addition to their intrinsic interest.

It is perhaps worth mentioning that diagonal sums are easily computed for the $g(n, r)$ (we recall that such sums produce Fibonacci numbers in the case of the binomial coefficients) by an immediate application of the defining relation to produce the following.

LEMMA 4. If $d_m = \sum_{n+r=m} g(n, r)$, then

$$d_{m+2} = 2d_{m+1} + d_m.$$

¹ Reported at the Conference on Error-Correcting Codes, Madison, Wisconsin, 1968.