SIAM REVIEW
Vol. 12, No. 2, April 1970

NOTE ON A "SQUARE" FUNCTIONAL EQUATION*

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The ordinary Pascal triangle for the binomial coefficients is well-known; basically, it depends upon the recursion relation

$$f(n, r) + f(n, r - 1) = f(n + 1, r).$$

A natural generalization of this situation occurs if we consider the tableau defined by

$$g(n + 1, r + 1) = g(n, r + 1) + g(n + 1, r) + g(n, r)$$

with the initial conditions

$$g(n, 0) = g(0, r) = 1.$$

We shall develop several expressions for these numbers $g(n, r)$ and show that they have a further combinatorial interpretation, namely, $g(n, r)$ is the volume of an r-sphere in n-space, under the Lee metric.

The tableau for $g(n, r)$ is easily built up, and the first portion of it is reproduced in Table 1. Note that the entry at the lower right corner of a square (of four adjacent entries) is the sum of the entries at the other three corners.

TABLE 1
$g(n, r)$

1	1	1	1	1	1	1	1
1	3	5	7	9	11	13	15
1	5	13	25	41	61	85	113
1	7	25	63	129	231	377	575
1	9	41	129	321	681	1289	2241
1	11	61	231	681	1683	3653	7183
1	13	85	377	1289	3653	8989	19825
1	15	113	575	2241	7183	19825	48639

Just as in the case of the binomial coefficients, many relations exist among the quantities $g(n, r)$. If we use the usual convention that $\binom{a}{b}$ is defined only for b an integer, and that $\binom{a}{b}$ vanishes for $b < 0$ and for $b \geqq a$ (a a positive integer), then we may prove the following.

LEMMA 1.

$$g(n, r) = \sum_{\alpha} \binom{n}{\alpha} \binom{r + \alpha}{n}.$$

* Received by the editors October 1, 1968 and in revised form March 7, 1969.
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Proof. We readily verify the result for $r = 0$ and $n = 0$. Assume it holds for all i and j with $i \leqq n + 1$ or $j \leqq r + 1$, with the possible exception $(i,j) = (n + 1, r + 1)$. Then

$$g(n + 1, r + 1) = \sum_\alpha \binom{n+1}{\alpha}\binom{r+\alpha}{n+1} + \sum_\alpha \binom{n}{\alpha}\binom{r+\alpha}{n} + \sum_\alpha \binom{n}{\alpha}\binom{r+1+\alpha}{n},$$

by virtue of the defining functional equation. Then

$$g(n + 1, r + 1) = \sum_\alpha \binom{n+1}{\alpha}\binom{r+\alpha}{n+1} + \sum_\alpha \binom{n}{\alpha}\binom{r+\alpha}{n} + \sum_\beta \binom{n}{\beta-1}\binom{r+\beta}{n}.$$

The two last terms combine to give

$$\sum_\alpha \binom{r+\alpha}{n}\left[\binom{n}{\alpha} + \binom{n}{\alpha-1}\right] = \sum_\alpha \binom{r+\alpha}{n}\binom{n+1}{\alpha},$$

which combines with the first term to give

$$\sum_\alpha \binom{n+1}{\alpha}\left[\binom{r+\alpha}{n+1} + \binom{r+\alpha}{n}\right] = \sum_\alpha \binom{n+1}{\alpha}\binom{r+\alpha+1}{n+1}.$$

Thus the lemma follows by induction.

We can obtain the $g(n, r)$ as coefficients in a power series in the following.

LEMMA 2. *$g(n, r)$ is the coefficient of x^r in the expansion of $(1 + x)^n/(1 - x)^{n+1}$. Or, since $g(n, r) = g(r, n)$, we might use the coefficient of x^n in $(1 + x)^r/(1 - x)^{r+1}$.*

Proof.

$$g(n, r) = \sum_\alpha \binom{n}{\alpha}\binom{r+\alpha}{n} = \sum_\alpha \binom{r+\alpha}{\alpha}\binom{r}{n-\alpha}.$$

Now

$$\binom{-k}{\beta} = (-1)^\beta\binom{k+\beta-1}{\beta};$$

hence

$$\begin{aligned} g(n, r) &= \sum_\alpha (-1)^\alpha \binom{-(r+1)}{\alpha}\binom{r}{n-\alpha} \\ &= \text{coefficient of } x^n \text{ in the expansion of } (1 - x)^{-(r+1)}(1 + x)^r. \end{aligned}$$

For actual computation, Lemma 2 is almost as easy as the basic definition; one chooses $\min(n, r)$ for the power a in $(1 + x)^a(1 - x)^{-(a+1)}$.

COROLLARY. *If we write $g_n = g(n, r)$, with r fixed, then*

$$\sum_\beta (-1)^\beta \binom{r+1}{\beta} g_{n-2\beta} = \binom{2r+1}{n}.$$

Proof.

$$(1 + x)^r = (1 - x)^{r+1} \sum_\alpha g_\alpha x^\alpha.$$

Thus

$$\begin{aligned} (1 + x)^{2r+1} &= (1 - x^2)^{r+1} \sum_\alpha g_\alpha x^\alpha \\ &= \sum_n \left(\sum_\beta \binom{r+1}{\beta}(-1)^\beta g_{n-2\beta}\right) x^n. \end{aligned}$$

Comparison of coefficients produces the corollary.

The corollary allows easy computation of the various g_n; for example,

$$g_0 = 1,$$
$$g_1 = 2r + 1,$$
$$g_2 = 2r^2 + 2r + 1,$$
$$g_3 = (2r + 1)(2r^2 + 2r + 3)/3,$$
$$g_4 = (2r^4 + 4r^3 + 10r^2 + 8r + 3)/3.$$

LEMMA 3.

$$g(n, r) = \sum_{\alpha} 2^{\alpha}\binom{n}{\alpha}\binom{r}{\alpha}.$$

Proof. We have

$$\begin{aligned} g(n, r) &= \sum_{\alpha}\binom{n}{\alpha}\binom{r+\alpha}{n} \\ &= \sum_{\alpha}\binom{n}{\alpha}\sum_{\beta}\binom{r}{\beta}\binom{\alpha}{n-\beta} \\ &= \sum_{\beta}\binom{r}{\beta}\sum_{\alpha}\binom{n}{\alpha}\binom{\alpha}{n-\beta} \\ &= \sum_{\beta}\binom{r}{\beta}\sum_{\alpha}\binom{n}{\beta}\binom{\beta}{n-\alpha} \\ &= \sum_{\beta}\binom{r}{\beta}\binom{n}{\beta}\sum_{\alpha}\binom{\beta}{n-\alpha} \\ &= \sum_{\beta}\binom{r}{\beta}\binom{n}{\beta}2^{\beta}. \end{aligned}$$

From Lemma 3, we obtain a second combinatorial interpretation for $g(n, r)$; the value

$$\sum_{\beta}\binom{r}{\beta}\binom{n}{\beta}2^{\beta}$$

is exactly the same value deduced by S. Golomb for the volume of a sphere of radius r in n dimensions (or a sphere of radius n in r dimensions) using the Lee metric[1]. This indicates that the numbers $g(n, r)$ possess some applications, in addition to their intrinsic interest.

It is perhaps worth mentioning that diagonal sums are easily computed for the $g(n, r)$ (we recall that such sums produce Fibonacci numbers in the case of the binomial coefficients) by an immediate application of the defining relation to produce the following.

LEMMA 4. *If* $d_m = \sum_{n+r=m} g(n, r)$, *then*

$$d_{m+2} = 2d_{m+1} + d_m.$$

[1] Reported at the Conference on Error-Correcting Codes, Madison, Wisconsin, 1968.