APPLICATIONS OF THE ELECTROMAGNETIC HELMHOLTZ RESONATOR

by

Russell Alan Stoneback

APPROVED BY SUPERVISORY COMMITTEE:

______________________________________________
Roderick A. Heelis, Chair

______________________________________________
Gregory D. Earle, Co-Chair

______________________________________________
Cyrus D. Cantrell

______________________________________________
Joseph Izen

______________________________________________
Brian A. Tinsley

1U.S. PATENTS PENDING 20070017344, 20070017345, 20070214940
APPLICATIONS OF THE ELECTROMAGNETIC HELMHOLTZ RESONATOR

by

RUSSELL ALAN STONEBACK, B.S., M.S.

DISSERTATION

Presented to the Faculty of

The University of Texas at Dallas

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY IN PHYSICS

THE UNIVERSITY OF TEXAS AT DALLAS

May, 2009

1U.S. PATENTS PENDING 20070017344, 20070017345, 20070214940
ACKNOWLEDGEMENTS

This dissertation would certainly not exist in this form without the support of my advisers Dr. Roderick A. Heelis and Dr. Gregory D. Earle. When I entered graduate school five years ago, I was determined to investigate a new kind of musical instrument using electromagnetic waves that I had conceived, but not yet built. This electromagnetic musical instrument replaces sound with light, though the electromagnetic instrument still operates with the same kind of behaviors employed in traditional acoustic musical instruments. Initially, my attempts to make these musical instruments the subject of my dissertation failed. After the second rejection, I began looking into possible research in relativity. I mentioned to Dr. Heelis one day that my repeated requests were denied without a chance to explain my reasoning. At the next faculty meeting, a committee was formed to hear my pre-proposal. This presentation went so well that I was allowed to present a formal proposal on a sub-topic of the proposed musical instrument, the electromagnetic Helmholtz resonator. Although I was not present for the faculty discussions regarding these issues, I feel certain that without the advisery support of both Dr. Heelis and Dr. Earle along with Dr. Heelis’ willingness to fund my research, I would not have received approval. Being able to spend 5 years of my life researching electromagnetic musical instruments means more to me than I could possibly say here.
APPLICATIONS OF THE ELECTROMAGNETIC HELMHOLTZ RESONATOR

Publication No. ________________

Russell Alan Stoneback, Ph.D.
The University of Texas at Dallas, 2009

Supervising Professors: Roderick A. Heelis, Chair
Gregory D. Earle, Co-Chair

An electromagnetic Helmholtz resonator comprised of a capacitor with an aperture is investigated theoretically and experimentally. It is proposed that this resonance may be described using effective impedances describing the capacitor and aperture, similar to lumped element descriptions of the acoustic Helmholtz resonator. The dipole impedance of an electromagnetic aperture is derived and verified using the finite element method. Incorporating standard network relations, the aperture impedance can be used to calculate radiated power. Measurements of a capacitor demonstrates that the transmitted voltage through the capacitor is modified by induced charges. An induced voltage is introduced, and predictions agree with observations. Measurements of a capacitor with an aperture in the grounded plate indicate that induced currents cancel the imaginary impedance of the aperture, and double the real impedance. The observed impedance is close to predictions using the derived aperture impedance, confirming the utility of the aperture impedance in describing the system. The numerically obtained aperture electromagnetic fields are similar to the Birkeland current distribution and the cross polar cap potential in the Earth’s polar ionosphere, motivating a
model where the polar ionosphere is treated as an effective aperture. It is proposed that this effective aperture interacts with the capacitor formed between the Earth and ionosphere, creating an electromagnetic Helmholtz resonator. Predictions made with this model agree with measurements of transmitted power and phase velocity by FAST during a geomagnetic substorm, measurements of the Ionospheric Alfvén Resonator, and oscillations recorded by ground based magnetometers. The same effective aperture behavior is expected in sunspots and polar coronal holes. A peak is predicted in Alfvén wave power across the transition region for waves with a 5 min. period that delivers an average power over 100 \( W/m^2 \) to the corona, sufficient to heat the quiet corona and launch the solar wind. Applied to sunspots, a minimum umbral temperature of 3750 K is predicted with a peak in transmitted power at 3 min., consistent with observations. A prototype electromagnetic guitar and associated methods to obtain music are also presented. These instruments replace the acoustic systems normally employed for musical instruments with electromagnetic equivalents and music samples are presented.
# Table of Contents

ACKNOWLEDGEMENTS ........................................ iv  

ABSTRACT ...................................................... v  

LIST OF FIGURES ................................................ ix  

LIST OF TABLES .................................................. xv  

CHAPTER 1. INTRODUCTION ..................................... 1  

CHAPTER 2. LITERATURE REVIEW ................................. 7  

CHAPTER 3. EFFECTIVE APERTURE DIPOLE CIRCUIT ............... 11  
3.1 Method of Moments ........................................ 11  
3.2 Aperture Dipole Impedances .................................. 17  
3.3 Aperture Circuit .............................................. 20  
3.4 Numerical Model .............................................. 22  
3.5 Results ......................................................... 24  
3.6 Discussion ..................................................... 33  
3.7 Conclusion ..................................................... 34  

CHAPTER 4. EXPERIMENTS ....................................... 35  
4.1 Measuring High Frequency Electrical Circuits .................. 36  
4.2 Procedure ..................................................... 39  
4.3 Parallel Plate Capacitor Construction .......................... 41  
4.4 Measurements of a Capacitor .................................. 42  
  4.4.1 Transmitted and Induced Voltage ......................... 47  
  4.4.2 Transmitted and Induced Current ......................... 56  
  4.4.3 Transmitted Power ......................................... 58  
  4.4.4 Derived Impedance Comparison ............................ 61  
  4.4.5 Effective Circuit Descriptions ............................ 69  
  4.4.6 Interactions with a Resonant Coaxial Cable ............... 80
LIST OF FIGURES

3.1 Geometry of original problem and an equivalent situation. .................. 12
3.2 Numerical model geometry. Not to scale. ..................................... 23
3.3 Examples of the element mesh in the numerical model. (a) Overview of model mesh. (b) Finite element mesh of aperture in plane conductor. (c) View of the elements in and around the aperture in the aperture plane. ............... 25
3.4 Magnitude of real (a) and imaginary (b) average impedance of aperture excited by normally incident wave using the presented aperture circuit and FEM. Numerics obtained with aperture radius to thickness ratio \(a/d = 10\). ............... 27
3.5 Comparison of predicted and numerical radiated power into region 2. ......... 28
3.6 Distribution of power within the aperture for a normally incident plane wave with vacuum in both regions. The black line is the poynting flux in the numerically modeled aperture. The blue line is the power transmitted into the aperture using (3.54), the red line is the power radiated into region 2 (3.61), and the green/brown line is the power radiated/absorbed in region 1 (3.62), all calculated using the average ANSYS aperture impedance and \(h = 2.1a\). ................................................................. 28
3.7 Spatial distribution of impedance within aperture for vacuum in both regions. 30
3.8 Percent error between the power radiated into region 2 in the FEM solution and a theoretical description of an aperture in a thick screen [73] excited by a normally incident plane wave (black). Percent error in real/imaginary (blue/red) impedance and power (green) between numerical model and presented aperture circuit for vacuum. ........................................ 31
3.9 Radiated light (blue) for a circular aperture excited by a high energy electron beam measured by Degiron [13] compared to the predicted transmitted aperture power (3.61, red), using \(a = 135\) nm, \(h = 2.1a\). Reprinted and modified from Optics Communications, vol. 239 (1-3), Degiron et al, Optical transmission properties of a single subwavelength aperture in a real metal, pp. 61-66, Copyright (2004), with permission from Elsevier. ............................... 31
3.10 Light radiated by circular aperture excited by high energy electron beam for two different electric field polarizations. Reprinted from Optics Communications, vol. 239 (1-3), Degiron et al, Optical transmission properties of a single subwavelength aperture in a real metal, pp. 61-66, Copyright (2004), with permission from Elsevier. ........................................ 32
3.11 Real part of normal electric fields inside and just outside the aperture boundary for an incident electric field \(E_x = 27\) V/m in region 1, \(ka \approx 1.8\). Peak values reach \(\pm 120\). The fields outside the aperture reverse orientation in region 2. ................................................................. 32
4.1 Electromagnetic guitar being measured. ......................................... 40
4.2 Measured and predicted imaginary impedance of 250 inch coaxial cable terminated with a short circuit. The red line is measured data, and the blue line is predicted using (4.14). ................................................................. 41

4.3 Pictures of the small parallel plate capacitor. ........................................... 43

4.4 Capacitor plate size comparison. ................................................................. 44

4.5 Measuring the reflection coefficient of a capacitor. ...................................... 44

4.6 Measured real (a) and imaginary (b) impedance of parallel plate capacitor with two spacings, $L_z = 2$ inches (Red), $L_z = 8$ inches (Blue). ......................... 45

4.7 Imaginary impedance of small parallel plate capacitor for $L_z = 2$ (red), 4 (green), and 8 (blue). ................................................................. 46

4.8 Comparison of impedance for large (red) and small (blue) capacitor ($L_z = 2''$), where the ratio of plate radius to plate separation is the same. .................. 48

4.9 A closer look at the impedance for large (red) and small (blue) capacitor ($L_z = 2''$), where the ratio of plate radius to plate separation is the same. .. 49

4.10 Measuring the reflected and transmitted voltage........................................... 50

4.11 Comparison of measured voltage reflection (red) and transmission (blue) coefficients. ................................................................. 51

4.12 Comparison of measured (blue) and predicted (4.22, green) transmission voltage coefficients at plate $B$. ................................................................. 54

4.13 Comparison of measured (blue) and predicted (4.21, green) induced capacitor voltage coefficients. ................................................................. 55

4.14 Comparison of predicted transmitted currents obtained with standard network theory (green, 4.8), displacement current (blue, 4.29), and total predicted current at $B$ (red, 4.33). ................................................................. 59

4.15 Comparison of power transmitted through capacitor using standard network theory (green, 4.9) and the power using the combined voltage and current at plate $B$ (blue, 4.34). ................................................................. 60

4.16 Comparison of power transmitted through capacitor using standard network theory (green, 4.9) and the transmitted power calculated by summing the standard network power and the power in the displacement current (blue, 4.36). 62

4.17 Comparison of predicted transmitted power obtained with standard network theory (green, 4.9), induced current (blue, 4.37), and total predicted power at $B$ (red, 4.38). ................................................................. 62

4.18 Comparison of impedance obtained with induced voltage $V_m$ (blue), reflection coefficient $\Gamma_m$ (green), and transmission coefficient (red). Negative real impedances are illustrated with a dashed line. ......................... 64

4.19 A closer look at the imaginary impedance obtained with the induced voltage $V_m$ (blue), the reflection coefficient (green), and the transmission coefficient (red). ................................................................. 65

4.20 Comparison of impedance obtained with induced voltage $V_m$ (4.45, blue), reflection coefficient $\Gamma_m$ (green), and transmission coefficient (4.46, red). Negative real impedances are illustrated with a dashed line. ......................... 68

4.21 Comparison of voltage at plate $B$ using predictions based upon impedance from induced voltage (green), and the measured voltage at $B$ ($T_m$, blue). .... 70
4.22 Circuit used to fit observed low order resonances of parallel plate capacitor.  

4.23 Comparison of the impedance of the circuit in Fig. 4.22 using fitted parameters (dashed, blue) and the measured capacitor impedance (green) using $V_m$.  

4.24 Comparison of the impedance of the circuit in Fig. 4.22 using fitted parameters (dashed, blue) and the measured capacitor impedance (green) using the reflected voltage.  

4.25 Comparison of the impedance of the circuit in Fig. 4.22 using fitted parameters (dashed, blue) and the measured capacitor impedance (green) for plate separation $L_z = 4''$.  

4.26 Alternate experimental configuration. Ground of the coaxial cable is connected to plate $A$, while the center conductor delivers the source wave to plate $B$.  

4.27 Impedance obtained with the measured reflection coefficient of the setup in Fig. 4.26.  

4.28 A closer look at the imaginary impedance obtained with the measured reflection coefficient of the setup in Fig. 4.26.  

4.29 Impedance derived from the measured reflection coefficient of an electromagnetic guitar; configured like Fig. 4.26.  

4.30 A closer look at the imaginary impedance derived from the measured reflection coefficient of an electromagnetic guitar; configured like Fig. 4.26.  

4.31 Impedance derived from the measured reflection coefficient of an electromagnetic guitar with resonant 250 in. coaxial cable; configured like Fig. 4.26.  

4.32 Impedance derived from the measured reflection coefficient of an electromagnetic guitar with resonant 250 in. coaxial cable; configured like Fig. 4.10, with no connection from plate $B$ to the network analyzer.  

4.33 Circuit used to describe electromagnetic guitar, where only one plate is connected to the network analyzer.  

4.34 Postulated coupling of coaxial cable with the electromagnetic guitar.  

4.35 Comparison of fitted circuit to measurements of electromagnetic guitar, with no connection from plate $B$ to the network analyzer.  

4.36 Comparison of predicted impedance to measurements of electromagnetic guitar with a 250 in. cable, with no connection from plate $B$ to the network analyzer.  

4.37 Impedance derived from the measured reflection coefficient of an electromagnetic guitar with (blue) and without (green) an aperture; configured like Fig. 4.26.  

4.38 A closer look at the imaginary impedance derived from the measured reflection coefficient of an electromagnetic guitar with (blue) and without (green) an aperture; configured like Fig. 4.26.  

4.39 Effective circuit description of copper guitar.  

4.40 Comparison of measured impedance (red) of electromagnetic guitar with a copper plate and the effective circuit (blue) in Fig. 4.39. The absolute magnitude of the imaginary impedance is presented.
4.41 Effective circuit description of aluminum guitar with an aperture. .......... 94
4.42 Postulated coupling of electric dipole aperture impedance. ................. 95
4.43 Comparison of measured impedance (red) of aluminum guitar with an aperture and the effective circuit (blue) in Fig. 4.41. The absolute magnitude of the imaginary impedance is presented ........................................... 96

5.1 Resonant modes of air cavity in an acoustic guitar obtained with FEM. Guitar plates are completely rigid and nulls are represented by dark blue. Reprinted from Journal of Sound and Vibration, vol. 252 (3), Elejabarrieta et al., Air Cavity Modes in the Resonance Box of the Guitar: The Effect of the Sound Hole, pp. 584-590, Copyright (2002), with permission from Elsevier. ......... 118

5.2 Admittance of an acoustic guitar with aperture (sound hole/tone hole) open (top) and closed (bottom). Reprinted with permission from Firth. Physics of the guitar at the Helmholtz and first top-plate resonances. The Journal of the Acoustical Society of America, vol. 61 (2) pp. 588-593 Copyright 1977, American Institute of Physics .................................................. 119


5.4 Prototype electromagnetic guitar shaped like an acoustic guitar. .............. 124

5.5 Ground of coaxial cable is connected to the front plate, while the center coaxial conductor is terminated in the back plate. ................................. 125

5.6 Flow chart for the real time processing of live music using an electromagnetic musical instrument. .............................................................. 131

5.7 Demonstration of the influence of separate measurements of the electromagnetic instrument response in the final output audio. .................. 134

5.8 Influence of $m_0$ on output audio. The time scale is the same for the output audio in Fig. 5.7. ................................................................. 134

5.9 Real impedance of copper guitar with no aperture. The actual frequency range of operation is scaled down to the equivalent acoustic frequency by matching the free space wavelengths of acoustic and electromagnetic waves. ............. 137

5.10 Resonant modes of an acoustic guitar obtain with FEM. Guitar plates are assumed to be simply supported to the guitar sidewall. Reprinted with permission from Elejabarrieta et al. Coupled modes of the resonance box of the guitar. The Journal of the Acoustical Society of America, vol. 111 (5) pp. 2283-2292. Copyright 2002, American Institute of Physics. ............... 137

5.11 Power transmitted into electromagnetic guitar at the equivalent acoustic frequencies for source impedance $Z_0 = 50$ Ω (blue), $Z_0 = 1500$ Ω (green). ....... 142

5.12 Power transmitted into electromagnetic guitar at the equivalent frequency $f_a = 0.35c_a/c_e f_e$ for $Z_0 = 50$ Ω (blue), $Z_0 = 1500$ Ω (green). ....... 142

5.13 Power transmitted into electromagnetic guitar at the equivalent frequency $f_a = 3.1c_a/c_e f_e$ for $Z_0 = 50$ Ω (blue), $Z_0 = 625$ Ω (green) configured like Fig. 4.10. ................................................................. 143
6.1 Real part of fields produced in and around aperture excited by a normally incident plane wave, where the aperture is replaced by a conductor containing magnetic currents. The reflection of a normally incident plane wave with magnetic field $\mathbf{H}_0$ from a plane conductor leads to a doubled magnetic field (Gray) and no electric field near the conductor surface. The fields produced by the effective aperture currents (black) are generally constant in the aperture, but the fields created outside the aperture vary strongly as a function of distance.

6.2 Aperture backed by a conductor where the conductor is located within a distance much smaller than the aperture radius. It is proposed that the impedance of the aperture facing the conductor is shorted, while the impedance of the opposite aperture face is unchanged. To satisfy boundary conditions, the remaining aperture impedance is divided into equal impedances for each aperture face.

6.3 Measurements using FAST satellite on Oct. 22, 1999 [14] during the main phase of a geomagnetic storm (*) along with predicted values for transmitted power and Alfvén phase velocity for aperture magnetic (solid) and electric (dashed) dipoles. (a) Power transmitted into Earth (b) Alfvén phase velocity, measured using $\delta E/\delta B$. The FAST measurements are reproduced with permission from Dombeck et al. Alfvén waves and Poynting flux observed simultaneously by Polar and FAST in the plasma sheet boundary layer. Journal of Geophysical Research (2005) vol. 110 (A12) pp. 8

6.4 Fractional power transmitted from Earth to space using the self-resonance of the aperture for electric dipole (solid) and magnetic dipole (dashed) coupling.

6.5 Real tangential vector fields in an aperture backed by a conductor near resonance with conductor: (a) electric field (b) magnetic field (c) associated plasma velocity. The colors of the rainbow are linearly scaled using Mathematica to illustrate the relative magnitudes of each quantity; the smallest values are purple, while the largest values are red.

6.6 Plasma convection pattern near aperture self resonance resonance, just above maximum in radiated power. The colors correspond to a linear mapping of the colors of a rainbow in Mathematica, where purple corresponds to a magnitude of 0 and red is the maximum attained value.

6.7 Real part of electric field normal to the polar cap inside and just outside aperture.

6.8 Numerically determined (a) real impedance (b) imaginary impedance (c) radiated power.

7.1 A view of the Sun in extreme ultraviolet from FE XII on March 19, 2009. Image from SOHO/EIT (ESA and NASA).

7.2 A view of the Sun’s corona in white light using the Mark IV K-coronameter at the Mauna Loa Solar Observatory on January 20, 2009.

7.3 Power transmitted from chromosphere to corona for electric (solid) and magnetic (dashed) dipole coupling.

7.4 Alfvén phase velocity due to the effective polar aperture for electric (solid) and magnetic (dashed) dipole coupling.
7.5 Transmitted power and Alfvén phase velocity from a sunspot assuming aperture impedance using wave speed in chromosphere for electric (solid) and magnetic (dashed) dipole coupling. .................................................. 175

7.6 Transmitted power and Alfvén phase velocity from a sunspot assuming aperture impedance using average wave speed of 50 km/s for electric (solid) and magnetic (dashed) dipole coupling. .................................................. 177
LIST OF TABLES

4.1 Fitted circuit parameters for electromagnetic guitar using impedance from induced voltage. ................................................................. 71
4.2 Fitted circuit parameters for electromagnetic guitar using impedance from reflected voltage. ......................................................... 71
4.3 Fitted circuit parameters for small capacitor. ............................. 73
4.4 Fitted circuit parameters for electromagnetic guitar using impedance from reflected voltage. ......................................................... 87
4.5 Fitted circuit parameters for electromagnetic guitar. .................... 92
CHAPTER 1
INTRODUCTION

The acoustic Helmholtz resonator was introduced in 1875 by Hermann Helmholtz [33], and is exemplified by an empty wine bottle. Blowing over the neck of the bottle produces a low frequency sound, with a wavelength longer than the bottle.

Helmholtz resonators are commonly employed in speaker cabinet designs and stringed musical instruments. Musical instruments employ Helmholtz resonance to create the fundamental resonance of the instrument (like the wine bottle), and allow the instrument cavity to respond at frequencies normally prohibited by the size. Similarly, speaker cabinets employ Helmholtz resonance to improve the bass response of music with a smaller cabinet size.

A common method for describing the response of a Helmholtz resonator employs a circuit analogy. The opening of the wine bottle (aperture) is modeled as an inductor, while the bottle itself (cavity) is modeled as a capacitor; the interaction of these elements naturally leads to resonant behavior. Based upon this circuit analogy, is it possible to build an electromagnetic Helmholtz resonator using a capacitor and an aperture that has a circuit description similar to the acoustic Helmholtz resonator?

To answer this question, a theoretical investigation of the low frequency impedance of an electromagnetic aperture in an infinite conductor is performed in chapter 3. The impedance of the aperture is obtained by building upon previous work that replaces the aperture with currents that recreate the same electromagnetic fields. It is found that the aperture acts as an inductor at low frequencies. As a check on this theoretical work, a numerical analysis of an aperture excited by a plane wave has been performed using the finite element method in §3.4. The derived aperture impedance is consistent with the numerically obtained impedance in §3.5, verifying the theoretical work. In contrast with previously
obtained aperture impedances, it is found that the aperture impedance only depends upon 
the properties of the region not containing the wave source. Additionally, the normalizations 
previously employed to obtain the aperture impedance are not consistent with numerical 
results; corrected factors are obtained.

The derived aperture impedance is used to construct a simple analytical description 
of the power radiated by an aperture with arbitrary shape excited by an arbitrary source 
using standard wave power techniques at the interface of two media in §3.3. This method 
indicates that only a portion of the power transmitted into the aperture radiates through 
the aperture, some of the transmitted power is radiated back into the source region. It 
is postulated that the division of radiated power into each region is determined by image 
currents in the conductor surrounding the aperture.

Experimental and numerical investigations of apertures observe a peak in radiated 
power larger than the power incident upon the aperture. This observed behavior is predicted 
in the presented model because currents around the aperture will absorb power from the 
incident wave surrounding the aperture and transfer it to the other side of the aperture 
(§3.3).

The application of image currents in this manner is consistent with results from recent 
experiments involving plasmon waves on a metal surface. Experiments involving regularly 
spaced arrays of apertures excited at optical frequencies demonstrate that the radiated power 
can be larger than the amount of power incident on the aperture areas. These increases 
in transmission occur at resonances of the plasmons and can be controlled by the periodic 
spacing of the apertures in the array. At resonance, the plasmons increase the power radiated 
by the apertures by transferring some of the power from the areas in between the apertures 
to the other side.

Predictions of the power radiated by an aperture closely follow measurements of a 
circular aperture excited by a high energy electron beam at the peak in dipole transmission 
in §3.5. Since the predicted peak in power strongly depends upon the introduced image
currents, the correspondence with experimental evidence supports the postulated division of power in the aperture using image currents. The spatial distribution of light emitted by the plasmons around the aperture is also consistent with fields observed in the numerical aperture model. These results indicate that the obtained aperture impedance may assist in theoretical investigations of aperture arrays and associated plasmon resonances.

Like the circuit description of the acoustic Helmholtz resonator, a similar circuit is constructed for the expected electromagnetic interaction between a capacitor and an aperture in §5.3. Though the electromagnetic Helmholtz resonator has been studied for both infinite cylinders with slots and for cavities with an aperture, the presented work differs from these investigations by pursuing an effective circuit description of the electromagnetic Helmholtz resonance. To confirm this predicted circuit, a parallel plate capacitor with an aperture has been constructed and characterized with a network analyzer, presented in chapter 4.

Measurements of parallel plate capacitors with the network analyzer clearly indicate that the voltage and current transmitted through the system are not given by standard network relations. Consistent with the known low frequency operation of a capacitor in a standard circuit, charges induced by the capacitor’s voltage must be accounted for. The voltage and current transmitted through the capacitor are derived including the induced charges in §4.4. To verify these relations, the impedance of a capacitor was calculated using measurements of the reflected voltage, the voltage transmitted through the capacitor, and the difference between these measured quantities. The resultant impedances agree, supporting the theoretical work.

Unexpected low frequency resonances starting below 10 MHz are observed for the measured capacitors, with waveguide resonances predicted to begin above 300 MHz. Circuit models fit to these measurements in §4.4.5 indicate an inductance much larger than predicted by waveguide theory. Measurements of the interaction between a resonant coaxial cable and a capacitor in §4.4.6 confirm the fitted circuits, and confirm that the measured impedance corresponds to physical behavior. The source of these low frequency resonances is unknown,
though measurements suggest that the behavior may be two dimensional resonances along the capacitor plate.

Measurements of a capacitor with an aperture do not indicate an additional resonance, though an increase in real impedance is detected in §4.5. Using a fitted circuit from measurements of the system without the aperture a prediction, made using the derived aperture impedances combined with the circuit, is consistent with measurements of the capacitor with the aperture. Since the aperture was located in the same capacitor plate that was grounded, charges induced in this plate cancel the imaginary impedance of the aperture and double the real impedance. These results confirm the analytically derived aperture impedances and the use of the aperture circuit in describing the coupling between a capacitor and an aperture. Though the area of the capacitor plate and the aperture it contains are very different, this difference is not apparent in the measured impedance. Using the understanding gained by these measurements, configurations are presented that should yield the desired Helmholtz resonance in §4.5.1.

Stringed acoustic musical instruments normally employ two kinds of resonance apart from the strings, the resonance of waves within the instrument and the Helmholtz resonance. It is shown that acoustic and electromagnetic cavities resonate at the same wavelengths in chapter 5. In addition, it is shown that an electromagnetic analog of the acoustic Helmholtz resonator could resonate at the same wavelengths if both resonators have the same dimensions in §5.3. These two systems can be combined to create an electromagnetic analog of an acoustic guitar, and a prototype\(^1\) is presented in §5.4. A method\(^2\) is presented in §5.5 that uses measurements of these new musical instruments along with standard music software that enables listeners to hear the sound of electromagnetic musical instruments. Additionally, a method is presented that enables the use of these new instruments for live music in §5.5.1\(^3\). Using a simple guitar recording, the sound of the electromagnetic guitar is introduced in

\(^1\)U.S. Patent Pending 20070017345
\(^2\)U.S. Patent Pending 20070017344
\(^3\)U.S. Patent Pending 20070214940
§5.6.1. The electromagnetic guitar is also used to modify the sound of drums, a keyboard, and electric guitars in a recorded song.

The numerical analysis of an aperture demonstrates that field distributions in the aperture are very similar to the cross polar cap potential and the Birkeland current system observed in the polar region of the Earth’s ionosphere. In chapter 6, it is shown that the polar cap in the Earth’s ionosphere has physical behaviors that mimic the effective aperture currents used to determine the aperture impedance. Thus, a model is constructed where the polar regions are treated as effective apertures. Combined with the capacitor formed between the Earth and ionosphere, the Earth is modeled as a capacitor with two apertures - an electromagnetic Helmholtz resonator. The presented model is consistent with observations of transmitted electromagnetic wave power into the polar region and measured wave phase velocities during the main phase of a substorm by FAST, and with observations by ground based magnetometers.

The effective aperture behavior of the Earth’s ionosphere is found in regions with open magnetic field lines, the same magnetic structure encountered on the Sun in polar coronal holes and sunspots. The use of aperture theory for the solar polar coronal hole in chapter 7 predicts a peak in transmitted power from the chromosphere to the corona at a 5 min. period, consistent with observed Alfvén waves in the corona using Hinode. The phase velocities of the waves created by the effective polar aperture are consistent with the fast and slow solar wind. The polar aperture predicts an averaged power density of $100 - 170 \, \text{W/m}^2$ over the coronal surface for a quiet sun, consistent with the power needed to launch the solar wind and heat the corona.

Treating sunspots as an aperture in §7.2 predicts a peak in transmitted power at 3 min. from the umbra. The resonant period increases moving out into the penumbra, approaching 8 min. at the edge of the sunspot. This change in period is consistent with observations of waves propagating across a sunspot. It is proposed that acoustic waves in the Sun are converted to Alfvén waves via the effective aperture. The radiation of these
Alfvén waves removes energy from the sunspot, lowering the temperature. Assuming a 5 min. source of power from acoustic waves in the Sun, the aperture model predicts an umbral temperature ranging between $4200 - 5000$ K and a penumbral temperature near $5300$ K, consistent with typical temperatures observed in sunspots.
The diffraction of an aperture in a conducting screen is a classic problem in physics. Early work was performed by Huygens, Young, Fresnel, and Kirchoff [38]. A formulation of Kirchoff’s integral by Rayleigh is still found in graduate EM textbooks today [38]. While this theory is sufficient when wavelengths are small compared to the aperture, it predicts too much power for wavelengths larger than the aperture. The first low frequency solution, presented by Bethe [2], found that the basic response of the aperture is like an electric and magnetic dipole. The dipole moments were calculated by replacing the aperture with a conductor containing magnetic currents that reproduced the same field as the aperture. This method simplifies the boundary conditions of the problem, enabling an analytical solution for simple aperture shapes. These moments were corrected and extended by Bouwkamp [37] to include quadrupole and octupole terms for a normally incident plane wave. The aperture dipole moments were generalized by DeMeulenaere [57] to include the effects of different materials on either side of the aperture.

The dipole moments produced by an aperture can be described by an effective polarizability, used to calculate the power radiated by an aperture for an incident plane wave. Exact values can be found for circles and for the dipole along the short side of a rectangle much longer than it is wide. In general, however, these moments must be calculated numerically. Some approximations for some basic shapes such as ellipses, rectangles, diamonds, crosses, etc. are given in [57, 76, 54, 55, 56, 63]. Gluckstern found a general relationship between the electric and magnetic polarizabilities, equivalent to a parallel combination of circuit elements [24], providing insight into the relationship between the electric and magnetic aperture dipoles. Gluckstern also presents numerical results on the dependence of polarizability upon the thickness of the aperture [23].
Harrington developed a unified treatment of matrix methods [26] suitable for solving electromagnetic problems that incorporated the method of moments. Combined with the aperture equivalent currents introduced by Bethe, Harrington developed a mathematical method [29], suitable for computation, that solves for the equivalent current of an aperture and presents results for a narrow slot in a thin [30] and thick [28] conducting plane. These techniques were then applied to investigate the coupling between apertures and cavities [47, 52], and apertures and wires [34]. An improved computational method for arbitrary apertures and conductors is obtained by Wang and Harrington [82].

Apertures are in use today in compact plate and patch antennas. Plates with simple slots are experimentally investigated in [79, 64] and [59] considers multiple rectangular slots. Rahim experimentally investigates different aperture positions in the plate antenna [70]. The use of slots for antennas prompted theoretical investigations of infinite slots [41]. The inclusion of an aperture can increase the bandwidth of an antenna [7], making them suitable for Ultra Wideband operation [49]. The behavior of these systems can be investigated using network terms [43, 35, 42, 39]. The aperture dipole circuit presented here may assist in theoretical descriptions of similar devices.

The general network formulation by Harrington and Mautz [29] can also be used to obtain the dipole impedances of an aperture. For long wavelengths, the magnetic dipoles are inductive, while the electric dipole is capacitive. Several calculations of these impedances are presented over the years, [27, 53, 46, 31]. However, the obtained aperture impedances are not consistent with the aperture impedance obtained from the numerical model presented here. A numerical model performed using the boundary element method [11] is consistent with the results presented here.

The interaction of an inductive aperture and a capacitive body forms the acoustic Helmholtz resonator [33]. Introduced by Helmholtz in 1875, the Helmholtz resonator is currently employed in musical instruments [18], speaker cabinets, and photoacoustic resonators [60]. In an introduction to an advanced treatment of the Helmholtz resonator [72],
Rayleigh notes that the same mathematical problem is found for apertures in both acoustic and electrostatic situations. Harrington uses the low frequency inductance of an aperture to motivate a resonance between a capacitor and an aperture [27], though the resonance frequency is not calculated. An investigation of aperture polarizability considers both acoustic and electromagnetic waves [57]. Gaylschein [20] presents a mathematical treatment of Helmholtz resonance for both acoustic and electromagnetic waves upon infinite cylinders with slits, while [61] only consider EM waves. It is also found that the Helmholtz resonance can split into two separate resonances if the cavity has a lowest resonance with repeated real eigenvalues [19].

The air intake structure on a modern jet can function as an EM Helmholtz resonator when illuminated by RADAR, prompting numerical investigations of apertures backed by cylindrical cavities [8, 9]. In a numerical model performed by Ohnuki [62], a new resonance is observed with the introduction of an aperture to a parallel plate waveguide, and is identified as a Helmholtz resonance. The EM Helmholtz resonance occurs below the resonances for a waveguide alone, consistent, for example, with measurements of an acoustic Helmholtz resonator in an acoustic guitar [18]. A low frequency resonance is also found for a slotted cylinder in a numerical model performed by Rao [71] and Johnson [40].

The recent discovery of a plasmon resonance with an array of apertures [15] has opened new applications [22] for sub wavelength apertures in near field probes. Plasmons are surface waves at the interface between a metal and dielectric. While these waves are normally evanescent, a periodic array of apertures constrains the allowed wave vectors of the plasmon creating a resonance condition. At resonance the apertures couple with the plasmons and transmit more light than predicted by Bethe’s dipole moments.

Similar results are also obtained by using periodic textures on both sides of the aperture and can narrow the beam of light radiated through the aperture [48]. The transmission properties of single apertures at optical wavelengths have been measured along with the light radiated by an aperture excited by an electron beam [13]. The electron beam excites the
dipole moment of the aperture and demonstrates coupling of the aperture with plasmons on the surface. The spatial distribution of the light radiated by these plasmons is very similar to field distributions obtained in the numerical model presented here. The dependance of the periodic spacing of the apertures in the array upon transmitted light is thoroughly investigated in [12].
CHAPTER 3
EFFECTIVE APERTURE DIPOLE CIRCUIT

Previous work applies the method of moments [29] to the electromagnetic boundary conditions in an aperture to construct a general network formalism describing the aperture. In the long wavelength limit, this framework may be used along with known electric and magnetic aperture dipole moments [2][37] to calculate the dipole impedance of the aperture. However, the impedances [27][53][31] are not consistent with results obtained with a numerical model performed using the Finite Element Method (FEM) - suitable modifications are presented here. For clarity, the derivation of the general formalism is included. The updated aperture impedances are combined in parallel to create an aperture dipole circuit, and the power radiated by the aperture using the derived circuit can be calculated incorporating standard wave techniques at the interface of media with different impedances.

3.1 Method of Moments

Consider a plane, perfect conductor with an aperture in Fig. 3.1. Region 1 to the left of the conductor is characterized by \( \mu_1, \epsilon_1 \) and region 2 to the right is characterized by \( \mu_2, \epsilon_2 \), with incident waves in both regions. Bethe [2] found that if the aperture is small compared to the incident wavelength, this problem may be solved by considering an equivalent situation: the same plane conductor with the aperture replaced by a perfect conductor containing magnetic currents. An advantage of this method is that the incident plane wave reflects from a uniform perfect conductor.

The equivalence of a magnetic current (current comprised of magnetic monopoles) to the fields produced by the aperture may be motivated by the behavior of time harmonic magnetic fields and electric currents at the surface of a perfect conductor. Since a magnetic
field does not penetrate a perfect conductor, the boundary conditions on electromagnetic fields require a surface current at the field discontinuity. In mathematically describing this situation, using either the surface current or the magnetic field as a source yields the same physical result as long as

\[ \mathbf{J} = \hat{n} \times \mathbf{H} \] (3.1)

where \( \mathbf{J} \) is a linear electric current density (amps per meter), and \( \mathbf{H} \) is the magnetic field. Similarly, to ensure correspondence with the original aperture problem using magnetic currents,

\[ \mathbf{M} = \mathbf{E} \times \hat{n}_{12} \] (3.2)

where \( \mathbf{M} \) is the equivalent linear magnetic current density (volts per meter), \( \mathbf{E} \) is the electric field in the aperture in the original problem, and \( \hat{n}_{12} \) is a unit vector normal to the conductor pointing from region 1 to region 2.

To construct the basic framework necessary to obtain the aperture impedance, it is sufficient to consider behavior in the static limit. Using the framework detailed by Harrington and Mautz [53], suppose that the electric and magnetic fields in region 1 may be written

\[ \mathbf{E}_1 = \mathbf{E}_1(-\mathbf{M}) + \mathbf{E}_{1s} \] (3.3)

\[ \mathbf{H}_1 = \mathbf{H}_1(-\mathbf{M}) + \mathbf{H}_{1s} \] (3.4)
where $E_1$ is the total field in region 1 in the original problem, $E_{1s}$ is the field produced by the incident plane wave and its reflection from the plane conductor with shorted aperture (aperture replaced with conductor), and $E_1(-M)$ is the field produced by the magnetic current in place of the aperture in region 1. Similarly, in region 2 let

$$E_2 = E_2(M) + E_{2s} \quad (3.5)$$
$$H_2 = H_2(M) + H_{2s} \quad (3.6)$$

where $E_2$ is the total field in region 2, $E_{2s}$ is the field produced by the incident plane wave and its reflection from the plane conductor with shorted aperture, and $E_2(M)$ is the field produced by the magnetic current in place of the aperture in region 2. The continuity of the total tangential electric field is ensured with $-M$ in region 1 and $M$ in region 2.

The continuity of the tangential component of the magnetic field requires that

$$-H_1^{tan}(M) - H_2^{tan}(M) = -(H_1^{tan} - H_2^{tan}) \quad (3.7)$$

and continuity of the normal component of the displacement field requires that

$$\epsilon_1 E_1^{nrm}(M) + \epsilon_2 E_2^{nrm}(M) = \epsilon_1 E_{1s}^{nrm} + \epsilon_2 E_{2s}^{nrm} \quad (3.8)$$

where the superscripts tan/nrm denote the tangential and normal components of the field respectively.

Given that the aperture may be considered as an electric and magnetic dipole in the low frequency limit \((ka << 1)\) [2], where \(a\) is the aperture radius, one may separate the current $M$ into three components [53],

$$M = V_e \hat{M}_e + V_{h\beta} \hat{M}_{h\beta} + V_{h\gamma} \hat{M}_{h\gamma}. \quad (3.9)$$

$V_e$ is the magnitude of the magnetic surface current density (volts per meter) that produces the electric dipole and $V_{h\beta}, V_{h\gamma}$ are magnetic currents that produce the magnetic dipoles along \(\hat{\beta}\) and \(\hat{\gamma}\) in the aperture plane. $\hat{M}_e, \hat{M}_{h\beta},$ and $\hat{M}_{h\gamma}$ are orthogonal basis functions describing
the direction and distribution of these currents over the aperture area. The directions \( \hat{\beta}, \hat{\gamma} \)
are assumed to diagonalize the aperture polarizability tensor.

Let the symmetric product be defined

\[
< A, B > = \frac{\int A \cdot B \, dS}{S} \tag{3.10}
\]

where \( S \) denotes the surface area of the aperture. Substitution of (3.9) into (3.7) and taking
the symmetric product with \( \hat{M}_{h\beta} \) yields

\[
V_{h\beta} < H_{1s}^{tan}(\hat{M}_{h\beta}) + H_{2s}^{tan}(\hat{M}_{h\beta}), \hat{M}_{h\beta} > = < H_{1s}^{tan} - H_{2s}^{tan}, \hat{M}_{h\beta} > \tag{3.11}
\]

where the linearity of the magnetic field allows one to write

\[
H_{1}^{tan}(V_{h\beta}\hat{M}_{h\beta}) = V_{h\beta}H_{1}^{tan}(\hat{M}_{h\beta}). \tag{3.12}
\]

Since \( V_{h\beta} \) carries units volts per meter, \( H_{1}^{tan}(\hat{M}_{h\beta}) \) is lacking these quantities, identifying

\[
[Y_1 + Y_2]_{h\beta} = - < H_{1s}^{tan}(\hat{M}_{h\beta}) + H_{2s}^{tan}(\hat{M}_{h\beta}), \hat{M}_{h\beta} > \tag{3.13}
\]

as an averaged admittance (siemens) of the aperture to magnetic fields. Furthermore,

\[
I_{h\beta} = - < H_{1s}^{tan} - H_{2s}^{tan}, \hat{M}_{h\beta} > \tag{3.14}
\]

where \( I_{h\beta} \) is the averaged tangential magnetic field along \( \hat{M}_{h\beta} \) exciting the aperture and
carries units amps per meter. \( I_{h\beta} \) is interpreted to be an effective current exciting the
magnetic dipole along \( \hat{\beta} \) in the aperture. Substitution into (3.11) yields

\[
V_{h\beta} [Y_1 + Y_2]_{h\beta} = I_{h\beta} \tag{3.15}
\]

relating the magnetic current representing the aperture magnetic dipole (\( V_{h\beta} \)) to the effective
current representing the input magnetic field (\( I_{h\beta} \)). Similarly, replacing \( \beta \to \gamma \) yields

\[
V_{h\gamma} [Y_1 + Y_2]_{h\gamma} = I_{h\gamma}. \tag{3.16}
\]
Following the same procedure for the electric field, the magnetic current expansion (3.9) is substituted into (3.8). Taking the symmetric product with $\hat{M}_e$ yields

\[ V_e < \epsilon_1 E_{1s}^{nrm} (\hat{M}_e) + \epsilon_2 E_{2s}^{nrm} (\hat{M}_e), \hat{M}_e > = \epsilon_1 E_{1s}^{nrm} - \epsilon_2 E_{2s}^{nrm}, \hat{M}_e > . \] (3.17)

As in the magnetic case just discussed, it is desired that the right hand side of this equation represent an input current (amps per meter) to the aperture. However, as written, the units on the right hand side correspond to the electric displacement field. A suitable transformation may be obtained by appealing to the equivalence of certain magnetic and electric currents. Consider that the electric dipole field created by a magnetic current loop $K$ (volts) surrounding area $S$ will also be produced by an electric current $I$ (amps) over some length $l$ normal to $S$ [32, pg. 135] when

\[ II = -j\omega \epsilon_i KS. \] (3.18)

Strictly, the equivalence of the fields produced by these currents is only maintained when $l$ is infinitesimally short, and $K, I$ are infinitesimally thin. However, the restrictions may be relaxed if one only requires equivalence for fields far from the currents. Thus, supposing the aperture dipoles have a finite length allows one to divide both sides by $l$, yielding an admittance

\[ \frac{I}{K} = -j\omega \epsilon_i \frac{S}{l_e} \] (3.19)

that relates a magnetic current to an equivalent electrical current in the aperture. Further, for long wavelengths ($ka \ll 1$), the infinitesimal electric and magnetic current elements may be replaced by electric ($J$) and magnetic ($\bar{M}$) current densities over the aperture,

\[ \frac{J}{M} = -j\omega \epsilon_i b \] (3.20)

where $b$ is a associated length of the aperture,

\[ b = \frac{S}{l_e}. \] (3.21)

The admittance (3.20) converts incident electric fields to equivalent aperture input currents.
The total length of the currents for the magnetic dipoles $l_n$ can be obtained by integrating the path length of the magnetic currents across both sides of the aperture,

$$l_n = 2 \int \int \hat{M}_n \cdot d\hat{l}_n \, d\theta$$  \hspace{1cm} (3.22)

where $\theta = 0$ is perpendicular to $\hat{M}_n$, for $n = \hat{\beta}, \hat{\gamma}$.

However, since the electric dipole is normal to the aperture, the effective dipole current does not direct a path suitable for (3.22). Consider electric dipoles in the plane of the aperture by substituting $\hat{J}_n$ for $\hat{M}_n$ in (3.22). Rotating these electric dipole currents out of the aperture plane leads to a normal electric dipole with two possible current lengths, $l_\beta$ and $l_\gamma$. Since the electric dipole could have components along both $\hat{\beta}$, $\hat{\gamma}$ without altering the radiated power, both lengths may be present, and the electric dipole length is presumed to be

$$\frac{1}{l_e} = \frac{1}{l_\beta} + \frac{1}{l_\gamma}.$$ \hspace{1cm} (3.23)

This parallel distribution of lengths is consistent with the the parallel distribution of aperture polarizability [23]. For a circular aperture, $l_e = 4a$, and $b = \pi a / 4$, the same effective thickness as an infinitely thin acoustic circular aperture radiating into a half space [50].

Applying the method of moments to the electric field including (3.20) yields

$$V_e[Y_1 + Y_2]_e = I_e$$ \hspace{1cm} (3.24)

where

$$[Y_1 + Y_2]_e = -j\omega b < \epsilon_1 \mathbf{E}_1^{nrm}(\hat{M}_e) + \epsilon_2 \mathbf{E}_2^{nrm}(\hat{M}_e), \hat{M}_e >$$ \hspace{1cm} (3.25)

is the averaged admittance of the aperture for electric fields,

$$I_e = - < j\omega \epsilon_1 b \mathbf{E}_1^{nrm} - j\omega \epsilon_2 b \mathbf{E}_2^{nrm}, \hat{M}_e >$$ \hspace{1cm} (3.26)

is the equivalent input current density for the source electric field, and $V_e$ is the aperture equivalent magnetic current density.
Adopting the normalization $\hat{M}_{h\beta} = \hat{\beta}, \hat{M}_{h\gamma} = \hat{\gamma},$ and $\hat{M}_e = \hat{z},$ the input currents reduce to

$$I_{h\beta} = -|H_{1s}^{tan} - H_{2s}^{tan}| \cos \theta$$

(3.27)

$$I_{h\gamma} = -|H_{1s}^{tan} - H_{2s}^{tan}| \sin \theta$$

(3.28)

$$I_e = -j\omega b(\epsilon_1 E_{1s}^{nrm} - \epsilon_2 E_{2s}^{nrm})$$

(3.29)

where $\theta$ is the angle between $\hat{\beta}$ and $H_{1s}^{tan} - H_{2s}^{tan}$. The total current exciting the magnetic dipole is

$$I_h = \sqrt{I_{h\beta}^2 + I_{h\gamma}^2} = |H_{1s}^{tan} - H_{2s}^{tan}|.$$

(3.30)

### 3.2 Aperture Dipole Impedances

Given the continuity of fields within the aperture [53, eq. 3], the aperture impedance must be the same in both regions 1 and 2. However, the impedances listed in [53, eqns. 33-34] vary across the aperture. Considering only the source in region 1, it is proposed that the dipole produced in the source region (region 1) is an image of the dipole created in region 2, and that the impedance of the aperture is determined solely by the material properties of region 2.

To achieve this, the difference between the half space dipole moments [57] produced on either side of the aperture is used to determine the aperture impedance. For a source in region 1, radiating into region 2, these moments are

$$\mathbf{p}_e = \mathbf{p}_{e2} - \mathbf{p}_{e1} = \alpha_e (\epsilon_1 E_{1s}^{nrm} - \epsilon_2 E_{2s}^{nrm}) \hat{n}_{12}$$

(3.31)

$$\mathbf{p}_m = \mathbf{p}_{m2} - \mathbf{p}_{m1} = - (\alpha_{m\beta} \cos \theta \hat{\beta} + \alpha_{m\gamma} \sin \theta \hat{\gamma}) I_h$$

(3.32)

with equivalent currents

$$\mathbf{I}_l = j\omega \mathbf{p}_e$$

(3.33)

$$\mathbf{K}_l = j\omega \mu_2 \mathbf{p}_m$$

(3.34)
where $\alpha_m, \alpha_e$ are the magnetic and electric aperture polarizabilities, and satisfy [24]

$$\frac{1}{\alpha_e} = \frac{1}{\alpha_{m\beta}} + \frac{1}{\alpha_{m\gamma}}.$$  \hfill (3.35)

The dipole moments for a source in region 2 only differ by sign.

The fictional magnetic currents ($\mathbf{M}$) in the aperture will produce the same fields far from the aperture as the dipole currents (3.33, 3.34) if $\int \mathbf{K} d^3r = \int \mathbf{M} d^3r$ within the aperture, or

$$\mathbf{M} = \frac{\mathbf{K} l}{S}.$$

The continuity of the tangential electric field within the aperture requires that $\mathbf{M}$ be used in region 2. Therefore, the equivalent magnetic current density for the magnetic dipole in region 2 is given by

$$\mathbf{M}_{h\beta} + \mathbf{M}_{h\gamma} = \mathbf{V}_{h\beta} \dot{\mathbf{M}}_{h\beta} + \mathbf{V}_{h\gamma} \dot{\mathbf{M}}_{h\gamma} = \frac{\mathbf{K} l}{S}.$$  \hfill (3.37)

Since $\dot{\mathbf{M}}_{h\beta}, \dot{\mathbf{M}}_{h\gamma}$ are assumed to be $\dot{\beta}, \dot{\gamma}$ respectively,

$$\mathbf{V}_{h\beta} = j\omega \mu_2 \alpha_{m\beta} I_{h\beta}$$  \hfill (3.38)

$$\mathbf{V}_{h\gamma} = j\omega \mu_2 \alpha_{m\gamma} I_{h\gamma}.$$  \hfill (3.39)

Using (3.15), the total imaginary admittance along $\dot{\beta}$ is

$$Im[\mathbf{Y}_1 + \mathbf{Y}_2]_h^\beta = \frac{S}{j\omega \mu_2 \alpha_{m\beta}}$$

with a similar relation along $\dot{\gamma}$.

For the real admittance, consider that the time average power radiated by a magnetic current [32, pg. 79] into a half space $j$ is,

$$P_j = \frac{k_j^2}{12\pi Z_j} \frac{|2\mathbf{K} l|^2}{2}$$

where the magnetic current element has been doubled due to image currents in the plane conductor. Normalizing this power by the aperture area,

$$\frac{P_{2m}}{S} = \frac{1}{2} \frac{S k_j^2}{3\pi Z_j} \frac{|\mathbf{K} l|^2}{S}.$$  \hfill (3.42)
substituting (3.37) and comparing the resultant equation to the power dissipated in a circuit,

\[ P = \frac{1}{2} V^2 Y \]  

(3.43)

one obtains a real admittance

\[ \text{Re}[Y_1 + Y_2]_h = \frac{S k_2^2}{3 \pi Z_2}. \]  

(3.44)

For the electric dipole admittance, divide (3.18) by \( b \), substitute in (3.33) and (3.36) with \( M = V_e \hat{M}_e \), and solve for \( V_e \). Using (3.24) and the input current (3.29), the imaginary admittance is

\[ \text{Im}[Y_1 + Y_2]_e = \frac{j \omega \varepsilon_2 S b^2}{\alpha_e}. \]  

(3.45)

Similarly, using the radiated power density of an electric dipole current over the aperture, (3.18), and (3.36), the real admittance is

\[ \text{Re}[Y_1 + Y_2]_e = \frac{S b^2 k_2^4}{3 \pi Z_2}. \]  

(3.46)

The aperture impedance for a source in either region is distributed between both sides of the aperture. A parallel distribution is adopted in [53], though no justification is offered. Consider the dipole equivalent currents [53, eqns. 27-28] for the individual aperture dipoles in each region [53, eqns. 19-20],

\[ K l_1 = -K l_2 \]  

(3.47)

\[ \varepsilon_1 I_2 = -\varepsilon_2 I_1. \]  

(3.48)

The magnetic dipole current (3.47) is the same on both sides of the aperture, implying a series distribution of impedance. The distribution of electric dipole current (3.48) is expected for a parallel combination of capacitors with different dielectrics, \( \varepsilon_1 \) and \( \varepsilon_2 \) respectively, indicating a parallel distribution of aperture electric dipole impedance.

The magnetic dipole impedance is obtained by combining (3.40) and (3.44), inverting, and separating into two equal parts assuming a series distribution,

\[ Z_{1h}^\beta = Z_{2h}^\beta = \frac{1}{2} \left( \frac{S k_2^2}{3 \pi Z_2} + \frac{S}{j \omega \mu_2 \alpha_m \beta} \right)^{-1}. \]  

(3.49)
Similarly, the electric dipole impedance assuming a parallel distribution is
\[ Z_{1e} = Z_{2e} = 2 \left( \frac{Sb^2k^4}{3\pi Z_2} + \frac{j\epsilon_2\omega Sb^2}{\alpha_e} \right)^{-1} \] (3.50)
where only the properties of the region not containing the source is used.

### 3.3 Aperture Circuit

The impedances derived for the aperture suggest that an effective dipole description of the aperture is possible using circuit elements. Since both the electric and magnetic dipoles are excited simultaneously, the effective impedances representing those aperture dipoles are combined in parallel. For a source in region 1, the aperture impedance for the dipole along \( n = \hat{\beta}, \hat{\gamma} \) is given by the dipole impedances for region 2 (3.49, 3.50)
\[ Z^n_a = Z^{n}_{2h} \| Z_{2e} = \frac{Z^n_{2h}Z_{2e}}{Z^{n}_{2h} + Z_{2e}}. \] (3.51)
Since the aperture impedance depends upon the region containing the source, a source in region 2 should be considered independently.

The fractional power transmitted for an incident wave at the interface of two different impedances, \( Z_1 \) for region 1, and \( Z^n_a \) for the aperture impedance is
\[ P^n_{impd} = Re \left( \frac{4Z_1Z^*_a \cos^2 i}{|Z_1 \cos i + Z^*_a \cos r|^2} \right) \] (3.52)
for parallel polarization, and
\[ P^n_{impd} = Re \left( \frac{4Z_1Z^*_a \cos^2 i}{|Z_1 \cos r + Z^*_a \cos i|^2} \right) \] (3.53)
for perpendicular polarization, where * denotes the complex conjugate, \( i \) is the angle of incidence, and \( r \) is the angle of refraction. Through comparison to the radiated dipole power into region 2 using moments [57, eqns. 8, 31], (3.52-3.53) are modified by the materials in region 1 and 2. Including the power in the incident wave with magnetic field \( H_{10} \), doubled at the conductor surface, the total radiated power is
\[ P^n_{mag} = \frac{\mu^2}{(\mu_1 + \mu_2)^2} \frac{Z_1SP^n_{impd}}{2} |< 2H_{10}, \hat{M}_{hn} >|^2. \] (3.54)
The equivalence of the parallel distribution of magnetic polarizability (3.35) to the electric polarizability reduces the power radiated by the aperture electric dipole. However, given the parallel combination of electric and magnetic dipole impedances, only the magnetic dipole real impedance is observed below aperture resonance. Thus, to obtain an electric dipole power indicated by the square of (3.35)

\[ P_e \propto \frac{\alpha_m^2 \alpha_{m\gamma}^2}{(\alpha_{m\gamma} + \alpha_{m\beta})^2} \]  

some combination of magnetic dipole powers must be used. Since the fractional power (3.52) is proportional to \( \alpha_{mn}^2 \), to obtain (3.55), the electric dipole power is postulated to be

\[ P_e = \frac{\epsilon_2}{(\epsilon_1 + \epsilon_2)^2} \frac{S}{2Z_1} \left| < 2E_{10}, \hat{M}_e > \right|^2 \frac{P_{impd}^\beta \alpha_{m\gamma}^2 \cos^2 \theta + P_{impd}^\gamma \alpha_{m\beta}^2 \sin^2 \theta}{(\alpha_{m\gamma} + \alpha_{m\beta})^2} \]  

The product \( P_{impd}^\beta \alpha_{m\gamma}^2 \) replicates the numerator in (3.55), and similarly when \( \beta \rightarrow \gamma \). The electric dipole power is divided based upon the components of the incident magnetic field along \( \hat{\beta} \) and \( \hat{\gamma} \). This ensures that the electric dipole displays the same resonance characteristics of the magnetic dipoles that are also excited by the incident wave.

Not all of the power transmitted into the aperture radiates into region 2, some is radiated back into region 1. Supposing that the power in the region 1 dipole is the same as the power in the image field in region 1, the total power radiated by the aperture is

\[ P_e + P_{mag} = P_{2ap} + P_{2img} + 2P_{1img} \]  

where \( P_{2ap} \) is the power directly radiated by the aperture in region 2, \( P_{iimg} \) is the power in the image field in region \( i \),

\[ P_{2img} = (g(k_2) - 1) P_{2ap} \]  
\[ 2P_{1img}^i = 2 (g(k_1) - 1) P_{2ap} \]  

and \( g(k)_n \) is the frequency dependence of image charge for a dipole field [32, eq. 3-11],

\[ g(k) = \left( 1 - \frac{3 \cos 2k_i h}{(2k_i h)^2} + \frac{3 \sin 2k_i h}{(2k_i h)^3} \right) \]
where \( h \) is an effective distance of the aperture dipole from the plane conductor. Since the magnetic currents recreating the aperture have an associated length, and the electric dipole is normal to the aperture, it will appear to the far field observer to be located away from the aperture. The minimum distance for a dipole of length \( l \) to adopt any orientation without intersecting the plane conductor is \( h = l_e/2 \). For a circular aperture, \( h = 2a \).

Substituting (3.58-3.59) into (3.57) to solve for \( P_{2ap} \), the total power radiated into region 2 is

\[
P_2 = \frac{g(k_2)}{2g(k_1) + g(k_2)} \left( P^\gamma_{mag} + P^\beta_{mag} + P_e \right)
\]  

(3.61)

while the power radiated into region 1 is

\[
P_1 = \frac{2g(k_1) - 2}{2g(k_1) + g(k_2)} \left( P^\gamma_{mag} + P^\beta_{mag} + P_e \right).
\]  

(3.62)

At low frequencies, equal power is radiated into each region. The power reflected back into region 1 as a plane wave is

\[
P_{refl} = \frac{S|E_0|^2}{2Z_1} - P_1 - P_2.
\]  

(3.63)

3.4  Numerical Model

The behavior of an aperture in a plane conductor was investigated using ANSYS® Academic Research, v. 11.0, a commercial finite element method (FEM) software package. A square transverse electromagnetic (TEM) waveguide was used to direct an input wave towards a circular aperture in a perfect conductor, which radiates into an approximate half space. The half space is constructed using perfectly matched layers (PML), a computational method that absorbs incident radiation but produces little reflection [1], approximating an infinite space.

The geometry is illustrated in Fig. 3.2. The aperture of radius \( a \) and thickness \( d \) utilizes a curved transition to the plane conductor in an effort to minimize gradients in the calculated field and maintain numerical accuracy. Due to symmetry of the fields and geometry, only one quarter of the system is modeled (not shown).
The model is solved over a range of frequencies by applying a series of harmonic single frequency TEM waves. The TEM waveguide is constructed using perfect electric (PEC) and magnetic conductors (PMC) along the transverse boundaries as appropriate. The input wave is created using an impedance matched port which also absorbs incident radiation without producing a reflection.

The overall mesh of the model is illustrated in Fig. 3.3(a). The top region is the incident waveguide, the aperture is in the left middle area, and the lower region approximates an infinite half space. The bulk of the input waveguide and radiation region was meshed using brick elements while the aperture used a tetrahedral element. A transition region utilized pyramid shaped elements to change smoothly from the tetrahedrals in the aperture to the bricks outside. The use of pyramids limits all elements to first order - one component of the electric field per element edge, and one normal component per element face. There is a minimum of 4 elements along \( \hat{z} \) through the aperture. Due to nodal count restrictions, this could not be increased.

The mesh in the aperture is illustrated in Fig. 3.3(b), and was kept constant over the frequency range investigated. The thickness of the plane conductor surrounding the aperture is not meshed since a PEC boundary condition is applied to elements on the surface.
of the conductor. The U shaped notch in the upper left defines the aperture boundary and corresponds to the plane conductor (PEC label) in Fig. 3.2. A view of the elements along the aperture sidewall is in Fig. 3.3(c). Note the large number of elements and the uniformity of the mesh.

The power radiated by the aperture was calculated by adding the time average poynting flux through the elements in a planar slice through the middle of the aperture. This method was verified by comparison to an alternative method supplied by ANSYS. The electromagnetic fields on a surface located half a wavelength away from the aperture are converted to equivalent electric and magnetic currents. ANSYS uses these equivalent currents to calculate the amount of power in the far field. The surface is located half a wavelength away to limit near field effects.

The impedance of the aperture was calculated using the field definitions of voltage and current

\[ - \int_{l_1}^{l_2} \mathbf{E} \cdot d\mathbf{l} = V \]

\[ \oint \mathbf{H} \cdot d\mathbf{l} = I \]

in the plane of the aperture and using \( V/I = Z \). The paths for the electric and magnetic fields are along the incident fields through the aperture center. The closed loop for the magnetic field traverses both sides of the aperture if only the field produced by the aperture is modeled. If the total field is present, both the incident wave plus the aperture fields, then the magnetic field in the source region is dominated by the incident field. Thus, the magnetic field is integrated once across the aperture and doubled to include the oppositely directed aperture magnetic field from the other side.

3.5 Results

Consider the circuit described in §3.3, Fig. 3.4 compares the aperture circuit and ANSYS average real and imaginary impedances for an aperture excited by a normally incident plane
Figure 3.3. Examples of the element mesh in the numerical model. (a) Overview of model mesh. (b) Finite element mesh of aperture in plane conductor. (c) View of the elements in and around the aperture in the aperture plane.
wave in region 1. For low frequencies, the impedance of the aperture circuit corresponds extremely well with the numerical model. The aperture circuit resonances in Fig. 3.4(b) for various materials in region 2 occur near the numerically obtained resonances and similar results are observed for additional material configurations not shown. The increasing difference between the dipole model and the numerical aperture impedance below resonance suggests that the impedance of higher order moments (quadrupole, etc.) combine in series with the dipole impedance.

The power radiated into region 2 from the aperture circuit model and ANSYS is compared in Fig. 3.5. The long dashed black line is the transmission coefficient as determined by Bouwkamp [37], which includes dipole, quadrupole, and octupole radiation. The circuit power matches the power radiated by the aperture dipoles in the low frequency limit. As expected, at higher frequencies, the dipole moment is not sufficient. The maximum in radiated power varies due to the material constants in (3.54, 3.56), suggesting that the impedance of higher order moments should be considered individually when calculating radiated power.

Figure 3.6 compares the poynting flux in the aperture obtained with ANSYS to the predicted transmitted (3.52) and radiated (3.61, 3.62) power using the average ANSYS aperture impedance for \( Z_a \). The power transmitted into the aperture (3.52) depends only upon the aperture and incident wave impedance. The transmitted power is double the poynting flux at low frequencies and similar at high frequencies, consistent with the assumed power distribution using image theory.

The association of the region 1 dipole with an image field (3.62) leads to oscillations between radiating and absorbing power as the aperture transitions from low frequency dipole behavior to higher frequencies. The choice \( h = 2.1a \) in (6.5) leads to a null in power for the region 1 dipole at the intersection between the predicted transmitted power and measured poynting flux for vacuum near \( k_1a \approx 1 \), satisfying conservation of power with the numerical model. Since the material constants in (3.54, 3.56) reduce the maximum power radiated by the derived dipoles, a meaningful comparison can not be made for other configurations. The
Figure 3.4. Magnitude of real (a) and imaginary (b) average impedance of aperture excited by normally incident wave using the presented aperture circuit and FEM. Numerics obtained with aperture radius to thickness ratio $a/d = 10$. 

Figure 3.5. Comparison of predicted and numerical radiated power into region 2.

Figure 3.6. Distribution of power within the aperture for a normally incident plane wave with vacuum in both regions. The black line is the poynting flux in the numerically modeled aperture. The blue line is the power transmitted into the aperture using (3.54), the red line is the power radiated into region 2 (3.61), and the green/brown line is the power radiated/absorbed in region 1 (3.62), all calculated using the average ANSYS aperture impedance and $h = 2.1a$. 
power radiated into region 2 using (3.61) and the ANSYS impedance matches the poynting
flux at low frequencies, and is generally similar at higher frequencies. This correspondence
supports the derived power distribution using image currents.

Figure 3.7 relates the spatial distribution of impedance for an aperture where both
regions are characterized by vacuum. The displayed impedances are normalized by the max-
imum impedance at each particular frequency, and the radial location (\(\rho\)) is normalized by
the aperture radius \(a\). Note the opposite orientation of \(k_1a\) between Figs. 3.7(a) and 3.7(b).
The resonance peaks near the middle of Fig. 3.7(a) are due to waveguide modes excited
by the wave source in region 1. Notice that at low frequencies the impedance distribution
is nearly constant, with the maximum impedance always at the aperture edge. The real
and imaginary aperture impedance vary by less than 15% and 10% respectively over the
whole aperture area, with less than 5% variation between \(0 < \rho < .8a\). In this regime, an
average impedance description is justified. As the aperture transitions through resonance
\((\log(k_1a) \approx .4)\), the outer edge resonates first, and moves inward towards the center with
increasing frequency. Above aperture resonance the impedance of the aperture varies signif-
ically with position, and at \(\log(k_1a) \approx 1\) an average impedance description may no longer
be appropriate.

The accuracy of the numerical model has been estimated by comparison to a theoret-
ical description of a circular aperture in a thick screen given by Roberts [73]. The method
utilizes a sum of circular waveguide modes to describe the aperture; the first fifteen modes
were used to generate the solution for comparison with ANSYS (Fig. 3.8, black). In the low
frequency regime the error is less than 6%, reaching a maximum less than 20% as the aper-
ture passes through resonance. The generally low percent error validates the finite element
model constructed in ANSYS.

The large percent error in radiated power between ANSYS and the aperture circuit is
due to the reduction in power from the thickness of the aperture in ANSYS [11][73]. However,
the percent error for both power and aperture real impedance are nearly the same, confirming
Figure 3.7. Spatial distribution of impedance within aperture for vacuum in both regions.
Figure 3.8. Percent error between the power radiated into region 2 in the FEM solution and a theoretical description of an aperture in a thick screen [73] excited by a normally incident plane wave (black). Percent error in real/imaginary (blue/red) impedance and power (green) between numerical model and presented aperture circuit for vacuum.

Figure 3.9. Radiated light (blue) for a circular aperture excited by a high energy electron beam measured by Degiron [13] compared to the predicted transmitted aperture power (3.61, red), using \( a = 135 \text{ nm}, \ h = 2.1a \). Reprinted and modified from Optics Communications, vol. 239 (1-3), Degiron et al, Optical transmission properties of a single subwavelength aperture in a real metal, pp. 61-66, Copyright (2004), with permission from Elsevier.
Figure 3.10. Light radiated by circular aperture excited by high energy electron beam for two different electric field polarizations. Reprinted from Optics Communications, vol. 239 (1-3), Degiron et al, Optical transmission properties of a single subwavelength aperture in a real metal, pp. 61-66, Copyright (2004), with permission from Elsevier.

Figure 3.11. Real part of normal electric fields inside and just outside the aperture boundary for an incident electric field $E_x = 27 \text{ V/m}$ in region 1, $ka \approx 1.8$. Peak values reach $\pm 120$. The fields outside the aperture reverse orientation in region 2.
the presented aperture real impedance. Note that peaks in the real impedance (blue), but not observed in the power (green), are due to waveguide modes excited by the aperture in the incident TEM waveguide. The percent error between ANSYS and the aperture circuit for the imaginary impedance is under 10% below $ka \approx 0.7$, at which point higher order moments become important. These results validate the aperture dipole circuit.

The derived aperture circuit is compared to experimental results [13] of a circular aperture ($a = 135 \text{ nm}$, $d = 200 \text{ nm}$) in a suspended silver metal film excited by a high energy electron beam. The electron beam is an input current that should reveal the circuit properties of the aperture. The light radiated by the aperture due to this input current is reproduced in Fig. 3.9, along with the predicted transmitted power of the aperture. The dipole aperture circuit yields a radiated power that closely matches the experimental results, up to a constant multiplicative factor because of the presentation of arbitrary units for the experimental results.

The peak in transmission is associated with a localised surface plasmon resonance, and images of the resonance confirm a dipole distribution, Fig. 3.10 [13]. The observed distribution is consistent with normal electric fields around an aperture in a plane conductor, due to the change in the tangential magnetic field from the values inside the aperture to those outside, Fig. 3.11.

3.6 Discussion

The presented aperture circuit builds upon previous work [53] to derive dipole impedances with the correct magnitude. To achieve this, the symmetric product (3.10), the equivalence between aperture and dipole currents (3.36), and the total power radiated by the effective aperture dipoles are all normalized by the aperture area. The impulsive magnetic currents [53, eq. 3] are replaced by currents over the aperture area (3.9) and the admittance [53, pg. 43] used to convert incident electric fields to equivalent input currents is modified to include the effective thickness of the aperture (3.20).
The aperture impedances [53, eqs. 31, 64] previously depended upon both regions surrounding the aperture, and were distributed in parallel. The presented aperture circuit (3.49, 3.50) only depends upon the region not containing the source, introduces additional factors of $S$ and $b$, and the electric/magnetic dipoles are distributed in parallel/series. The electric and magnetic impedances are combined in parallel, leading to a radiated dipole power that goes to 0 as $\omega \to \infty$. The derived impedances have been confirmed by a numerical investigation using ANSYS.

The power radiated by the aperture is calculated using the impedance of the aperture and source region, and standard techniques at an impedance mismatch between media. It is postulated that this power is divided between both sides of the aperture using image charge effects. Since the aperture impedance is nearly constant over the aperture, and the dependence of the source field is accounted for, the presented aperture circuit should closely approximate the response of an aperture for arbitrary sources.

These developments may assist in the study of systems interacting with an aperture. In particular, the correspondence of the dipole aperture circuit and a plasmon resonance of an optical aperture suggests that the dipole circuit can be used in studying plasmon resonances of planar aperture arrays.

3.7 Conclusion

The presented aperture circuit offers a simple and effective method for describing the complex impedance and the power radiated by an aperture in a plane conductor for long wavelengths.
CHAPTER 4
EXPERIMENTS

To investigate the interaction of an aperture backed by a capacitor, the impedance of a capacitor with and without an aperture has been measured using a network analyzer. Both circular parallel plate capacitors as well as an electromagnetic guitar configured like a capacitor §5.4 are measured. Measurements of the reflected and transmitted voltage for the capacitor indicate that the standard impedance relations for these coefficients at the interface of two or three media are insufficient to explain observed behavior. Consistent with low frequency electrostatic descriptions of a capacitor, an induced voltage due to image currents is introduced to the system and predictions using this method agree with measurements.

Measurements also indicate a series of unexpected resonances for the capacitor plates well below the predicted resonances using waveguide theory. An accurate determination of effective circuit parameters for these low frequency resonances is obtained. The structure of this circuit has been found to be physically significant through investigations of the coupling between the capacitor and a resonant coaxial cable. Based upon the fitted circuit models, the low frequency resonances have an inductance that is several orders of magnitude larger than expected from waveguide theory. The source of these resonances is unknown.

There are no large changes in imaginary impedance observed with the introduction of an aperture to the grounded plate of a capacitor, although an increase in real impedance is detected. Though not known at the time, the derived transmitted voltage relations for a capacitor predict that the image currents will cancel the imaginary impedance of the aperture and double the real impedance. Combining the aperture impedances in chapter 3 with a fitted circuit model of a capacitor with no aperture, the predicted impedance of a capacitor with an aperture matches observations. These results confirm the derived impedances of the
aperture and confirm that the aperture circuit can be combined with a circuit description of a system to predict behavior. Even though the area of the capacitor plate and aperture are different, this does not alter the coupling of impedance, though the power radiated by the aperture is limited. Based upon these results, configurations where a resonance between an aperture and capacitor should be observed are discussed.

4.1 Measuring High Frequency Electrical Circuits

The impedance of a low frequency circuit, where the wavelengths are much larger than the components, can be easily determined by using a known current source and measuring the voltage across the individual circuit components. All of the current and voltage delivered by an ideal source enter the circuit, regardless of the impedance of the circuit components. As frequencies increase and wavelengths decrease, the wave nature of voltage and current becomes apparent. In this regime, the impedance between the source and the circuit must match for all the source current and voltage to enter the system. If the circuit impedance is different than the source, a portion of the source power is reflected from the circuit’s input. The larger the difference between the source and circuit, the larger the fraction of power that is reflected back towards the source.

Since the voltage and current waves in a coaxial cable are Transverse Electromagnetic (TEM) waves, the reflection of voltage and current at the interface of the cable and circuit is determined by the reflection of normally incident electromagnetic waves at the interface of two media. The reflection coefficient for the electric field can be shown [38, pg. 306] to be

\[
\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{E^{-}}{E_{0}}. \tag{4.1}
\]

for a wave in impedance \(Z_0\) normally incident upon impedance \(Z_L\), where \(E_0, E^{-}\) are the complex amplitudes of the incident and reflected electric field, and the impedances for each region are defined

\[
Z_i = \sqrt{\frac{\mu_i}{\epsilon_i}}. \tag{4.2}
\]
The total electric field in the incident region can be written as the sum of the incident and reflected waves traveling normal ($\hat{z}$) to the interface,

$$E = E_0 \left( e^{-jkz} + \Gamma e^{jkz} \right).$$  \hspace{1cm} (4.3)

Note that the complex number $j$ is used instead of $i$, where $j = -i$, consistent with notation commonly employed in electrical circuits. For TEM waves, the magnetic field can be written

$$H_0 = \frac{1}{Z} \hat{k} \times E_0$$  \hspace{1cm} (4.4)

so the total magnetic field in the incident region is

$$H = \frac{1}{Z_0} \hat{z} \times E_0 \left( e^{-jkz} - \Gamma e^{jkz} \right).$$  \hspace{1cm} (4.5)

Through comparison to (4.3), the reflection coefficient for the magnetic field

$$\Gamma_c = -\Gamma = \frac{Z_0 - Z_L}{Z_0 + Z_L}$$  \hspace{1cm} (4.6)

is the negative of the voltage reflection coefficient.

Since the tangential components of the electric and magnetic fields are continuous across the boundary between $Z_0$ and $Z_L$, the transmission coefficients are

$$T_v = 1 + \Gamma = \frac{2Z_L}{Z_L + Z_0}$$  \hspace{1cm} (4.7)

$$T_c = 1 + \Gamma_c = \frac{2Z_0}{Z_L + Z_0}.$$  \hspace{1cm} (4.8)

Since voltage is proportional to the electric field and current is proportional to the magnetic field, the transmission coefficient for the electric and magnetic fields also applies to the transmitted voltage and current into high frequency electrical devices. The transmitted voltage and current coefficients describe the fraction of the incident source voltage $V_0$ and current $I_0$ that enters the load. The fractional power transmitted from source impedance $Z_0$ into a general circuit load $Z_L$ is

$$P_{\text{trans}} = T_v T_c^* = \text{Re} \left( \frac{4Z_L Z_0^*}{|Z_L + Z_0|^2} \right)$$  \hspace{1cm} (4.9)
where only the real part is taken. The power reflected back towards the source is

\[ P_{\text{refl}} = -\Gamma \Gamma^* = \text{Re} \left( \frac{-|Z_0 - Z_L|^2}{|Z_0 + Z_L|^2} \right) \]  \hspace{1cm} (4.10)

where the negative sign indicates the change in direction in power. These equations for transmitted voltage, current, and power are referred to here as standard network relations.

Consider a network analyzer delivering an excitation signal via a coaxial cable with impedance \( Z_0 \) to a circuit with impedance \( Z_L \). Since the voltage reflected back towards the source (4.1) is determined by \( Z_0 \), which is known, and the circuit impedance, \( Z_L \), which is generally unknown, measurements of the total voltage by the network analyzer can be used to determine the unknown circuit’s impedance. However, the total voltage in the coaxial cable depends upon location. Choosing the interface between the cable and load as the reference plane, \( d = \ell - z \), where \( \ell \) is the length of the cable, and substituting the reflection coefficient (4.1) into the total electric field (4.3), the observed voltage at the network analyzer \( (z = 0) \) is

\[ V = V_0 \left( \frac{2Z_L \cos k\ell + 2jZ_0 \sin k\ell}{Z_L + Z_0} \right) \]  \hspace{1cm} (4.11)

and the observed current using (4.5) is

\[ I = I_0 \left( \frac{2Z_0 \cos k\ell + 2jZ_L \sin k\ell}{Z_L + Z_0} \right) \]  \hspace{1cm} (4.12)

for a source voltage and current \( V_0 \), \( I_0 \).

To remove the source cable length dependence for accurate measurements, network analyzers are calibrated using a set of known impedances. The cable is measured with a short \( (Z_L = 0) \), open \( (Z_L = \infty) \), and a matched load \( (Z_L = Z_0) \). These measurements are able to characterize the cable joining the network analyzer and the unknown circuit, and the reflection coefficient output by the network analyzer is the coefficient at the interface with the load, \( z = \ell \). Using this complex reflection coefficient and rewriting (4.1), the load impedance is

\[ Z_L = Z_0 \frac{1 + \Gamma}{1 - \Gamma}. \]  \hspace{1cm} (4.13)
In general, however, the device undergoing testing will also have an additional internal propagation distance, $\ell'$. Using the reflection coefficient measured by the network analyzer, the observed impedance at the interface between cable and circuit will be

$$Z_{\text{obs}} = \frac{Z_0}{1 - \Gamma} = \frac{Z_0}{1 - \Gamma} \frac{Z_L \cos k\ell' + j Z_0 \sin k\ell'}{Z_0 \cos k\ell' + j Z_L \sin k\ell'}$$  \hspace{1cm} (4.14)$$

which is certainly not equal to $Z_L$. If this internal propagation distance is small compared to the wavelength, then $k\ell' \ll 1$, and $Z = Z_L$. If this internal distance is known, then the load impedance is

$$Z_L = \frac{Z_0 Z_{\text{obs}} - j Z_0 \tan k\ell'}{Z_0 - j Z_{\text{obs}} \tan k\ell'}.$$  \hspace{1cm} (4.15)$$

4.2 Procedure

A rented\(^1\) Agilent 8753ES network analyzer was used to measure the reflection coefficient of various parallel plate capacitors to determine if there is an interaction between a capacitor and an aperture. Many electrical devices that are measured with network analyzers have a narrow bandwidth, so the maximum number of data points that can be measured over a frequency range is 1601. Since the network analyzer will also be used to measure an electromagnetic guitar, the frequency resolution required is dictated by audio applications.

The impulse response of the electromagnetic guitar that will be calculated from the network analyzer will only have as many data points measured by the analyzer. For standard CD audio, 1601 data points will only yield .05 s of audio. Since acoustic instruments have a longer reverb time §5.5, more points must be measured. Therefore, the total range of the analyzer $100\text{KHz} - 6\text{GHz}$ was broken into 100 $\text{MHz}$ sections. The lowest frequency range $100\text{KHz} - 100\text{MHz}$ was calibrated twice using both a logarithmic and linear distribution of frequencies, while the remaining ranges only used a linear distribution. The total range of the instrument required 60 calibrations, providing over 96,000 measurements, slightly more than 2 seconds of audio. This is more samples than strictly required, however, measurements could

\(^1\)Funds provided by Dr. Roderick A Heelis.
Measurements of the reflection and transmission coefficients were performed in an electromagnetic anechoic chamber at the University of Texas at Dallas between June and August of 2008. The chamber is only designed to be anechoic at frequencies 800 MHz and higher, though measurements were performed between 100 KHz – 6 GHz. The device under test was suspended above the floor by a polypropylene twine connected to a PVC frame, Fig. (4.1). Extra electromagnetic absorbing panels were placed under the devices. Though located on campus, access to the measurement room was restricted, and less than 10 rounds of measurements were performed.

The calibrations were performed with the cable tied to the PVC frame, in the same configuration used when making measurements. To perform the calibration, multiple loads must be attached to the coaxial cable and measured. In this process, the cable inevitably
moves, slightly changing the coaxial cable’s properties. Therefore, a single calibration was performed from $100kHz - 1GHz$ with a logarithmic distribution in frequency to provide a check on the results using multiple calibrations. This calibration was also used in situations not requiring the accuracy of audio applications.

To ensure that the measurement equipment was operating (and being operated) correctly, the impedance of a long coaxial line was measured, terminated with a short, Fig. 4.2. The predicted impedance is generated using (4.14), where $Z_L = 0$ and $\ell = 6.35$ m (250 in.) is the length of the coaxial cable. The predicted impedance (blue) closely matches the impedance obtained using the network analyzer (red).

4.3 Parallel Plate Capacitor Construction

The parallel plate capacitor in Fig. 4.3 was constructed using the same techniques employed in the electromagnetic guitar in §5.4. Acrylic plates support pure aluminum circular plates, separated using nylon threaded rods, nuts, and bolts, with additional acrylic rods for lateral support. Acrylic and nylon are low loss dielectrics, with a stable response over the measured
frequency range. The use of the threaded rods allows the plate separation to be varied, and two plate sizes were investigated, with radii $R_c = 8.5, 17$ in. These capacitors were constructed with the assistance of Keith Swaim\textsuperscript{2}.

To couple the network analyzer to the capacitor, an SMA connector was attached to both capacitor plates. The SMA standard is a compact coaxial cable connector suitable for frequencies in the $GHz$ range. Since aluminum is not easily soldered to, a conductive epoxy\textsuperscript{3} was used to attach the center conductor of the SMA connector to the capacitor plate, leaving the ground free. The ground of the connector is separated from the instrument by a short section of coaxial cable with no outer conductor. This ground is located close to the plate, leading to a possible interaction between the two. A variation was also investigated where the ground of the coaxial cable is coupled to one plate and the center conductor is coupled to the other plate, requiring a coaxial cable in the capacitor. The length of this internal cable must be accounted for when interpreting measurements.

4.4 Measurements of a Capacitor

Consider a small parallel plate capacitor configured like Fig. 4.5 and measurements as the plate separation was varied in Figs. 4.6-4.7. SMA connectors were attached to each plate, with the center conductor electrically connected to an aluminum plate, and the ground was left free. Each plate of the capacitor was connected to the analyzer and the reflected voltage coefficient from interface $A$ was measured. Voltage transmitted into the capacitor from $A$ is either radiated, or travels through the capacitor and towards a matched load in port 2 of the network analyzer. The connection at interface $B$ is used to ground the capacitor plate.

Assuming an ideal circular capacitor made of a perfect conductor, the resonance frequencies of the system are determined by the roots of the Bessel function derivative [10].

\textsuperscript{2}Machinist, W.B. Hanson Center for Space Sciences
\textsuperscript{3}MGChemicals Silver Conductive Epoxy 8331
Figure 4.3. Pictures of the small parallel plate capacitor.
Figure 4.4. Capacitor plate size comparison.

Figure 4.5. Measuring the reflection coefficient of a capacitor.

The lowest frequency series resonance \((\text{Im}(Z) = 0)\) occurs at

\[
f_0 = \frac{1.841}{2\pi \sqrt{\mu \epsilon R_c}}
\]  \hspace{1cm} (4.16)

where \(R_c\) is the radius of the capacitor in meters. For \(R_c = 8.5\) in. \(\approx .216\) m, the predicted resonance frequency \((f_0 \approx 400\) MHz\) is considerably higher than the observed resonance near 10 MHz. Previous experiments \([75]\) of a capacitor with varying plate separations \((L_z/R_c \approx .02 - .15)\) indicate that the observed resonance frequency will always be lower than ideal predictions, approximately written

\[
f = f_0 \sqrt{\frac{2L_z \epsilon_0}{\pi R_c \epsilon} \left( \ln \frac{\pi R_c}{2L_z} + 1.7726 \right)}^{-1}
\]  \hspace{1cm} (4.17)

Since the capacitor presented here has a minimum \(L_z/R_c = .25\), (4.17) may not apply, though a decrease in resonance frequency of 20% is predicted. Though (4.17) tends to underestimate
Figure 4.6. Measured real (a) and imaginary (b) impedance of parallel plate capacitor with two spacings, $L_z = 2$ inches (Red), $L_z = 8$ inches (Blue).
the decrease in resonance by a few percent, it is still insufficient to explain the 98% reduction in frequency observed. Additionally, the observed resonances decrease with a decreasing plate separation, in contrast with [75] which finds decreases in resonance frequency with increasing plate separation. Given the large discrepancy in resonance frequency, and the differing behavior as plate separation varies, the observed resonances are not believed to be standard waveguide resonances.

The impedance of the small and large capacitor plates are compared in Fig. 4.8, with a closer look in Fig. 4.9. The low frequency resonances are all lower for the larger capacitor. Though the larger capacitor has a radius twice that of the small cap, the resonance frequencies do not decrease by a factor of 2 predicted by 4.17. Using the ideal capacitance of parallel plates,

$$C = \frac{A\epsilon}{L_z}$$

(4.18)

the change in capacitance for an increase in area by a factor of 4 is the same as when the plate separation decreases from $L_z = 8 \rightarrow 2$ in. In Fig. 4.6, the change in plate separation yields a change in frequency similar to observations when increasing the plate size. The capacitance of the small plate with $L_z = 2$ in. is comparable to the large plate with $L_z = 8$
in., and the two systems have a similar resonance frequency. Since the resonance frequency of the small capacitor with \( L_z = 2 \) in. is nearly the same as the large capacitor with \( L_z \approx 8 \) in., these results suggest the two systems have a similar inductance.

4.4.1 Transmitted and Induced Voltage

As a check on the reflection measurements for the capacitors, the voltage transmitted through a capacitor (EM guitar body §5.4) was also measured. The incident, reflected and transmitted waves in the EM instrument are depicted in Fig. 4.10, where the propagation factor \( e^{-j(kz-\omega t)} \) is suppressed. A portion of the incident wave \( V_0 \) reflects at the capacitor plate and is measured by the network analyzer, output as calibrated reflection coefficient \( \Gamma_m \). Using continuity, the fraction of the incident wave that is transmitted into the capacitor is \( 1 + \Gamma_m \) (4.7). The wave that emerges from the capacitor \( (T_m) \) travels along a coaxial cable to port 2 on the analyzer and is also measured, although some of the power transmitted into the system could be radiated away. The wave \( V_{\text{ind}} \) is postulated to be the voltage induced on plate \( B \) by the charges present on \( A \). Assuming the phase velocity in the capacitor is given by vacuum, for measurements below 300 MHz, wavelengths in the capacitor are greater than \( 10L_z \) and \( kL_z << 1 \), so it may be possible to ignore the spatial dependence of these waves along \( \hat{z} \).

Measurements of the reflected and transmitted waves presented in Fig. 4.11 clearly demonstrates that the wave that travels through the capacitor is not given by (4.7), so

\[
T_m \neq 1 + \Gamma_m. \tag{4.19}
\]

At low frequencies, the capacitor impedance is large and leads to an expected real reflection coefficient of 1 in Fig. 4.11(a), the limit of an open load. This reflection coefficient should lead to a transmitted voltage \( 1 + \Gamma_m = 2 \), however, the observed transmitted voltage \( (T_m) \) is zero.

At 6 MHz, the transmitted voltage peaks near one and the reflected voltage has a minimum near zero. This behavior is consistent with a matched load, indicating the TEM wave from port 1 travels through the capacitor with no modification towards port 2. Starting near 20
Figure 4.8. Comparison of impedance for large (red) and small (blue) capacitor ($L_z = 2''$), where the ratio of plate radius to plate separation is the same.
Figure 4.9. A closer look at the impedance for large (red) and small (blue) capacitor ($L_z = 2\pi$), where the ratio of plate radius to plate separation is the same.
MHz, there are some scattered peaks and the transmitted voltage dips below zero. Another peak in transmission occurs at 30 Mhz, and continues beyond the measurement range. While the measured and reflected coefficients generally act out of phase, the variations near 20 Mhz are in phase, and the variations in reflection above 50 Mhz are not observed in the transmitted voltage. These differences could arise due to losses in the system due to radiation and Ohmic loss, or are the result of external sources.

In Fig. 4.11(b), the imaginary components of $Im(1 + \Gamma_m) = Im(\Gamma_m)$ are out of phase with the measured transmitted field, but it is expected that these fields should be equal. The peak in real transmission has a corresponding zero for imaginary voltage at 6 MHz, indicating the system is at resonance. Like the deviations observed for the real components, the variations near 20 MHz are in phase, while the remaining behavior is out of phase. The real peak in transmission at 30 MHz has a corresponding zero in the imaginary component, indicating another resonance.

One may suppose that the system at hand is truly a three medium problem, the first medium is the coaxial cable, the second the capacitor, with the third being another coaxial cable. However, the transmission coefficient in this case [3, pg. 164, eq. E-9] (the same for acoustics)

$$T' = \frac{2Z_0}{2Z_0 \cos kL_z + j(Z_L + Z_0^2/Z_L) \sin kL_z} \quad (4.20)$$

yields $T_m = 1$ for $kL_z << 1$, which is also not consistent with Fig. 4.11, or the known behavior of a capacitor.
Figure 4.11. Comparison of measured voltage reflection (red) and transmission (blue) coefficients.
The voltage that is experimentally known to be applied to plate $A$, $T_v$ (4.7), induces a voltage in plate $B$, driving currents in the plate. The resultant traveling wave reflects at the interface of plate $B$ with the cable connected to port 2. Applying (4.1) with $Z_L$, $Z_0$ reversed, the reflection coefficient at this interface is $-\Gamma$, leading to a reflected voltage $-\Gamma T_v$. If the voltage reflected at the interface between $B$ and the cable is then assumed to travel back towards plate $A$ and interact, the three medium transmission coefficient will eventually be obtained and has already been rejected. Therefore, it is postulated that the voltage created by reflection is replaced by an equivalent voltage created by image currents, induced in plate $B$ due to the charges in plate $A$. When solving for the field produced by a charge above a grounded conductor, the boundary conditions enforced by the conductor are reproduced by introducing a charge with equal and opposite magnitude. Equivalently, the boundary conditions at plate $B$ are satisfied if there is an image source from port 2, with a voltage coefficient

\[ V_{ind} = -\Gamma T_v = -\Gamma(1 + \Gamma). \]  

(4.21)

Since $V_{ind}$ is associated with image charge, it only exists when induced by the voltage in plate $A$ to satisfy boundary conditions and is therefore is not treated as a normal wave. It does not reflect or transmit at the interface between $B$ and the coaxial cable since it is not actually a wave driven by a source in port 2.

The effect of this induced voltage maintains the ground on plate $B$ for capacitive fields. By continuity, the sum of the incident plus induced voltage is equal to the transmitted voltage (at plate $B$), predicted to be

\[ T_{vB} = T_v + V_{ind} = 1 - \Gamma^2 = \frac{4Z_LZ_0}{(Z_L + Z_0)^2}. \]  

(4.22)

This is nearly the same as the power transmitted at the interface of two media (4.9), but there is no complex conjugate and both the real and imaginary parts are present. For an infinite load, the reflection coefficient is one and the voltage at plate $B$ is zero. At a matched load, the voltage at $B$ is the same at plate $A$, leading to no voltage difference between the plates.
A comparison of predicted and measured voltage at plate $B$ is in Fig. 4.12. The predicted transmitted voltage uses the impedance derived from reflection in (4.22). Predictions match measurement at low frequencies, with increasing differences with frequency. No voltage is transmitted at low frequencies which is consistent with plate $B$ being grounded. At 6 MHz, the real part of the measured transmitted voltage peaks below one, while the predicted voltage is greater than one above and below 6 MHz and one at resonance. The measured real transmitted voltage becomes negative near 15 MHz and approaches zero near 100 MHz, not reflected in predictions. The general behavior is reproduced, though there are differences between the two curves. Predictions are generally greater than the measurements which may be due to additional power measured at plate $A$ (§4.4.3) due to the displacement current (§4.4.3). Since the transmitted voltage at $B$ is $1 - \Gamma^2$, reductions in the observed reflection will lead to an increase in the predicted transmitted voltage.

Rewriting (4.22), the difference between the total voltage on plate $B$ and the voltage on plate $A$ is the induced voltage on plate $B$. Since both the voltages at plate $A$ ($1 + \Gamma_m$) and $B$ ($T_m$) have been measured, the difference between these measured quantities gives an experimental determination of the induced voltage,

$$V_m = T_m - (1 + \Gamma_m).$$

A comparison between the measured and predicted induced voltage (4.21) using only the measured reflection coefficient is in Fig. 4.13. The measured induced voltage is smoother than the measured transmitted voltage. At low frequencies, both the measured and predicted real component of the induced voltage are equal and opposite the total voltage on plate $A$, and the overall behavior is similar at higher frequencies. The measured induced voltage goes to zero near 6 MHz, where the measured reflected voltage also goes to zero, consistent with predictions. The predicted imaginary component generally follows measurements, although predictions are a bit larger than observations. The correspondence between measurements and predictions confirms the derived induced voltage coefficient.
Figure 4.12. Comparison of measured (blue) and predicted (4.22, green) transmission voltage coefficients at plate $B$. 
Figure 4.13. Comparison of measured (blue) and predicted (4.21, green) induced capacitor voltage coefficients.
4.4.2 Transmitted and Induced Current

In a capacitor, the insulative layer separating the plates prevents current from flowing directly between them. An effective current is formed by oscillations of the electric field in time, known as a displacement current [38]. Ignoring edge effects, the electric field in an ideal parallel plate is constant, leading to a constant displacement current density,

\[ \mathbf{J}_D = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = j\omega \epsilon_0 \mathbf{E}. \]  

(4.24)

The electric field in the capacitor is given by the voltage difference between the plates divided by the distance between them. The total voltage applied to plate A is \(1 + \Gamma\), and the total voltage on the grounded plate is \(1 - \Gamma^2\), leading to

\[ \mathbf{E} = \frac{((1 + \Gamma) - (1 - \Gamma^2)) V_0}{L_z} \hat{z} = \Gamma(1 + \Gamma) \frac{V_0}{L_z} \hat{z}. \]  

(4.25)

Substituting into the displacement current (4.24) and integrating over the capacitor area \(S_c\), the total displacement current within the capacitor is

\[ I_c = S_c \mathbf{J}_D = \frac{j\epsilon_0 \omega S_c}{L_z} (\Gamma + \Gamma^2) V_0. \]  

(4.26)

Since the impedance of an ideal capacitor is

\[ Z_c = \frac{L_z}{j\epsilon_0 \omega S} \]  

(4.27)

the current may be rewritten

\[ I_c = \frac{V_0}{Z_c} (\Gamma + \Gamma^2) = I_0 \frac{Z_0}{Z_c} (\Gamma + \Gamma^2). \]  

(4.28)

If losses are included by modifying the free space permittivity to include a complex term [38], then the purely imaginary capacitor impedance \(Z_c\) will also have a real component. Since power losses in the system are equivalently described by the real impedance of the capacitor, the imaginary impedance \(Z_c\) is replaced with the load impedance \(Z_L\) and the current in the system can be rewritten

\[ I_c = \Gamma(1 - \Gamma) = \frac{2Z_0}{Z_0 + Z_L} \Gamma. \]  

(4.29)
Note that the displacement current $I_c$ can also be obtained by calculating the induced currents, obtained similarly to the induced voltage. The current transmitted into the capacitor $(1 - \Gamma)$ reflects at plate $B$ with the negative of the regular current reflection ($\Gamma_c \rightarrow -\Gamma$). The currents induced in the capacitor are supposed to recreate this boundary condition and the induced current is

$$I_{\text{ind}} = \Gamma (1 - \Gamma) = I_c. \quad (4.30)$$

Using the induced current (4.30) and voltage (4.21), including the source voltage and current terms, the impedance of the induced current

$$\frac{V_{\text{ind}}}{I_{\text{ind}}} = \frac{-\Gamma (1 + \Gamma) V_0}{\Gamma (1 - \Gamma) I_0} = -\frac{Z_L Z_0}{Z_0} = -Z_L \quad (4.31)$$

is the negative of the load impedance. The negative impedance reflects that the induced waves travel along $-\hat{z}$, while the network analyzer delivers a signal along $\hat{z}$.

By continuity, the total current into the system must be the same as the total current out of the system. Using standard network relations, a portion $1 - \Gamma$ of the incident current transmits into the system. Though this current cannot flow between the plates, boundary conditions are maintained by the radiative electromagnetic fields produced. If these waves travel towards the opposite plate, the fields will drive currents in the plate and the wave can continue towards the matched load in port 2. This is observed in Fig. 4.12 at 6 MHz, where the voltage at plate $B$ is nearly the same as the voltage on plate $A$. Waves that don’t encounter the other plate radiate away.

In addition to this wave solution, voltage differences between the plates lead to an electric field normal to the capacitor plates. This changing electric field is like an effective current, with magnitude $\Gamma (1 - \Gamma) I_0$. By continuity, the displacement current must be present at both plates, as well as the space in between. The continuity of displacement current within the capacitor and the induced currents at $B$ has been shown explicitly. The continuity of the displacement current at plate $A$ follows from the normal electric field in the capacitor.
At the surface of a conductor, a normal electric field leads to a charge density

$$\mathbf{E} \cdot \hat{n} = \frac{\sigma}{\varepsilon_0}. \quad (4.32)$$

The electric field is ideally constant throughout the capacitor, leading to equal and opposite charge densities in each plate. Since these densities change equally with time, the currents are also the same on each plate. The total current is

$$T_{cB} = T_{cA} + I_{ind} = 1 - \Gamma^2. \quad (4.33)$$

The standard network relations predict a current (4.8) at plate A of $1 - \Gamma$, and the displacement current is reduced in comparison by a factor $\Gamma$. These currents along with the predicted total current at $B$ are compared in Fig. 4.14. The combination of the displacement current as well as the current from network theory drives the predicted total current above unity, similar to the predicted voltage at $B$. Using standard network theory, the transmitted current can vary between $[-2, 2]$, so predictions above one are allowed.

### 4.4.3 Transmitted Power

Since the displacement current and the transmitted current using network theory correspond to different modes of operation for the capacitor, the power in each mode should be calculated separately. Consider a matched load, $Z_0 = Z_L$, there is no reflection ($\Gamma = 0$) and the displacement current in the capacitor is $I_c = 0$. The current transmitted into plate $A$ using network relations $(1 - \Gamma)$ is the full source current. If the total voltage at plate $B$, $T_vB$, is used along with the total current at $B$, then the transmitted power

$$Re \left[ T_{vb}T_{cB}^* \right] = \frac{16|Z_0Z_L|^2}{|Z_0 + Z_L|^4} \quad (4.34)$$

goes above unity in Fig. 4.15. More power exits the system than is input, violating the conservation of power. Since (4.34) is calculated using the total voltage and current at $B$, and predictions for both voltage $T_vB$ and current $T_{cB}$ are greater than one, so is the calculated power. This violation may be a result of uncertainty in measurements, but demonstrates
Figure 4.14. Comparison of predicted transmitted currents obtained with standard network theory (green, 4.8), displacement current (blue, 4.29), and total predicted current at B (red, 4.33).
Figure 4.15. Comparison of power transmitted through capacitor using standard network theory (green, 4.9) and the power using the combined voltage and current at plate $B$ (blue, 4.34).

that (4.34) is not bounded by one. Since power must always be conserved, (4.34) is incorrect, and confirms that each mode should be considered separately.

Consider a calculation of the power in the displacement field alone at plate $A$. Using the product of the voltage difference between the plates and the displacement current, the power is

$$\text{Re} [(V_A - V_B)I_c^*] = |\Gamma|^2(1 + \Gamma)(1 - \Gamma)^* = \text{Re} \left[ \frac{4Z_LZ_0^*}{|Z_0 + Z_L|^2} \frac{Z_L - Z_0}{Z_L + Z_0} \right].$$

(4.35)

Compared to the power transmitted via network theory (4.9), the power via the displacement current is reduced by a factor $|\Gamma|^2$. Including the standard transmitted power (4.9), the total power entering the system at $A$ is

$$\text{Re} [T_{vA}T_{cA}^* + (V_A - V_B)I_c^*] = \text{Re} \left[ \frac{4Z_LZ_0}{|Z_0 + Z_L|^2} (1 + |\Gamma|^2) \right].$$

(4.36)

illustrated in Fig. 4.16. The radiated power is now bound by one and is consistently larger than the power transmitted into the system indicated by network theory.

Consider a calculation of the power in the induced currents in plate $B$ using the
induced voltage and current,

\[ Re[V_{ind}I^*_ind] = -Re[\Gamma(1 + \Gamma)\Gamma^*(1 - \Gamma)^*] = -Re\left[ \frac{4Z_LZ_0^*}{|Z_0 + Z_L|^2} |\Gamma|^2 \right] \]  \hspace{1cm} (4.37)

which differs from (4.35) by a negative sign. Combined with the power transmitted from \( A \),

\[ Re[V_{ind}I^*_ind + T_{vA}T^*_cA] = Re\left[ \frac{4Z_LZ_0^*}{|Z_0 + Z_L|^2} (1 - |\Gamma|^2) \right] \]  \hspace{1cm} (4.38)

and a comparison of these powers is in Fig. 4.17, calculated using the measured reflection coefficient. The power from the induced currents is out of phase with the power transmitted from \( A \), reducing the apparent power transmitted through the system. At plate \( A \), the relative phase of the modes increases the apparent power into the system. This asymmetric distribution does not violate the conservation of power.

Consider a right circular cylindrical surface enclosing the capacitor, such that the long axis is aligned with \( \hat{z} \). The power in the displacement field has an electric field along \( \hat{z} \), and a magnetic field along \( \hat{\theta} \). Computing the Poynting flux,

\[ S = \frac{1}{2} E \times H^* = \frac{EH^*}{2} \hat{r} \]  \hspace{1cm} (4.39)

the power is directed along the radial direction. The displacement field contributes (4.35) along \( \hat{r} \) at plate \( A \), and the image currents contribute an equal an opposite power at plate \( B \) (4.37), leaving zero net flux in the capacitor along \( \hat{r} \), satisfying the conservation of power. The power transmitted along \( \hat{z} \) into the capacitor via the standard relations,

\[ Re[T_{vA}T^*_cA] = Re\left[ \frac{4Z_LZ_0^*}{|Z_0 + Z_L|^2} \right] \]  \hspace{1cm} (4.40)

is already known to satisfy the conservation of power. So although there appears to be an excess of power at plate \( A \) and a lack of power at plate \( B \) compared to (4.40), the total power in the capacitor is given by (4.40).

4.4.4 Derived Impedance Comparison

The derived current and voltage on plate \( B \) has been supported by comparison to measured quantities in Figs. 4.12, 4.13. However, additional checks can be made by calculating the
Figure 4.16. Comparison of power transmitted through capacitor using standard network theory (green, 4.9) and the transmitted power calculated by summing the standard network power and the power in the displacement current (blue, 4.36).

Figure 4.17. Comparison of predicted transmitted power obtained with standard network theory (green, 4.9), induced current (blue, 4.37), and total predicted power at B (red, 4.38).
impedance of the system using the reflected, transmitted, and induced voltage. If the derived
transmitted and induced voltages are correct, the impedance calculated with these quantities
should be consistent with the impedance from reflection (4.13).

Since the measured induced capacitor voltage in Fig. 4.13 may remove some effects
of losses and noise sources, it may be useful in obtaining a more accurate impedance of the
system than using the reflected voltage. Substituting (4.1) into (4.21) and solving for the
load impedance $Z_L$ using the quadratic equation,

$$ Z_L = \frac{Z_0}{2 + V_{\text{ind}}} \left( 1 - V_{\text{ind}} \pm \sqrt{1 - 4V_{\text{ind}}} \right). $$

(4.41)

In the low frequency limit, the induced voltage approaches $V_{\text{ind}} \to -2$ and the two possible
values for $Z_L$ approach $\infty$ (+ solution) and zero (− solution). Since a capacitor also has an
infinite impedance for DC signals, only the additive solution is used.

The transmitted voltage from plate $B$ can also be used to determine the unknown
impedance of the capacitor. Solving (4.22) for the load impedance yields

$$ Z_L = \frac{Z_0}{2 - T_{\text{vB}}} \left( 2 - T_{\text{vB}} \right) \pm \frac{2\sqrt{1 - T_{\text{vB}}}}{T_{\text{vB}}}. $$

(4.42)

For the open load DC impedance of the capacitor, the positive solution is taken.

The impedance obtained with (4.41), where $V_{\text{ind}}$ is replaced with measured $V_m$, is
compared to the impedance obtained with the reflection coefficient (4.13) at $A$ and the
impedance obtained solely using the transmission coefficient (4.42) at $B$ in Fig. 4.18. At
low frequencies the magnitude of imaginary impedance from reflection is less than the value
obtained from the induced voltage, which is less than the value obtained from transmission.
Since the imaginary impedance of the load is much larger than the cable impedance, a
reduction in the observed reflection due to the additional power at plate $A$ is interpreted
as a reduced imaginary impedance. Similarly, the reduction in power observed at plate $B$
is interpreted as an increased load impedance. Thus, the true load impedance will be in
between these two limits, consistent with the impedance from the induced voltage. These
results support the asymmetric power distribution in §4.4.3.
Figure 4.18. Comparison of impedance obtained with induced voltage $V_m$ (blue), reflection coefficient $\Gamma_m$ (green), and transmission coefficient (red). Negative real impedances are illustrated with a dashed line.
Above 10 MHz, the real impedances obtained from these three different values diverge. The real impedance from transmission goes negative, but the impedances from the induced voltage and reflection are similar. The change in sign implies that power is traveling from plate $B$ towards plate $A$, instead of towards the matched load in port 2. The peaks in impedance from each measured quantity are at different frequencies, but share a generally similar form. The imaginary impedances from the three methods also differ, and a closer look at these impedances is in Fig. 4.19. The resonances from the reflection coefficient are made very clear using the induced voltage. The impedance from transmission is positive above 40 MHz, while the other methods yield a negative impedance. These disparities may indicate that the derived relations for transmitted voltage at plate $B$ are incorrect. However, since the impedances obtained at low frequencies are generally consistent for each method, it appears the derived quantities are correct in this regime, although there remains an unaccounted for frequency dependence.

Including the phase lag due to propagation between the plates (along $\hat{z}$), the voltage measured at plate $B$ can be written

$$T_{vB} = T_{vA}e^{-jkL_z} + V_{ind}e^{jkL_z} = (1 - \Gamma^2) \cos k_z L_z - j(1 + \Gamma)^2 \sin k_z L_z$$  \hspace{1cm} (4.43)
where the wave from A travels distance $L_z$ and the induced wave travels distance $z = -L_z$. In terms of impedance

$$T_{vB} = \frac{4Z_0Z_L}{(Z_L + Z_0)^2} \cos k_zL_z - j \frac{4Z_L^2}{(Z_L + Z_0)^2} \sin k_zL_z.$$  (4.44)

As required, for $k_zL_z << 1$, (4.44) reduces to (4.22). Since the induced voltage is due to an image of the wave from A, the wave number $k$ for both waves is taken to be the same. At $k_zL_z = \pi/2$, $3\pi/2$, the real part of the voltage goes to zero, and similarly for the imaginary part at $k_zL_z = \pi$. For the first zero, the real part goes from positive to negative, and the measured crossing at 12.4 MHz may correspond to the zero at $k_zL_z = \pi/2$. This predicts a positive to negative transition for the imaginary component at 24.8 MHz, near an observed positive to negative transition at 24.2 MHz. However, to achieve a resonance at these frequencies due to propagation along $\hat{z}$, the phase velocity of electromagnetic wave must be 2% of the value for vacuum.

If there is indeed a reduced phase velocity and associated waveguide modes, then the impedance in Fig. 4.18 is no longer valid near 12 MHz and above, and deviations due the propagation along $\hat{z}$ will be incorrectly attributed to the load impedance. Using 4.43, the load impedance is

$$Z_L = Z_0 \frac{2 \cos k_zL_z - (1 + V_m) \pm \sqrt{(1 + V_m - 2 \cos k_zL_z)^2 - V_m(V_m + 2 + 4j \sin k_zL_z)}}{V_m + 2 + 4j \sin k_zL_z}.$$  (4.45)

which reduces to (4.13) when $k_zL_z << 1$ as required. The corresponding equation for the impedance from the measured transmitted voltage is

$$Z_L = Z_0 \frac{2 \cos k_zL_z - T_m \pm \sqrt{(T_m - 2 \cos k_zL_z)^2 - T_m(T_m + 4j \sin k_zL_z)}}{T_m + 4j \sin k_zL_z}.$$  (4.46)

In Fig. 4.20, the load impedances including the spatial dependence along $\hat{z}$ obtained from the transmitted and induced voltage are much closer to the impedances obtained from reflection. The real impedance from transmission still goes negative, indicating this behavior is not due to a phase lag from plate separation. The first transition from a positive to negative real
impedance now occurs with the parallel resonance, where phase changes would be expected. The magnitude of the real impedance from transmission is still similar to the other derived real impedances. Though the other real impedance zeros are near parallel resonances, this association is not as strong as for the first resonance.

It was calculated earlier (4.31) that the impedance of the induced wave is the negative of the load impedance. If the observed resonances radiate a significant amount of power out of the system, the wave from $A$ using standard network relations will not drive currents in $B$. However, any voltage difference between the plates would still drive induced currents. Thus, measurements at plate $B$ would be dominated by the induced currents, with associated negative impedance. The measured negative impedance supports the derived relations for the induced currents, and confirms the system is radiating energy. Above 40 MHz, the real impedance from transmission heads towards zero, in contrast with the other impedances.

For the imaginary impedance, the first series and parallel resonance for each impedance occurs near the same frequency, and much greater similarity is found for the resonance near 40 MHz. Though the imaginary impedance from transmission was very different from the other methods in Fig. 4.18 above 10 MHz, all methods now yield similar results.

To obtain this correspondence, the phase velocity of electromagnetic waves in vacuum is used, a suitable approximation for air. These results confirm that the phase velocity of the system along $\hat{z}$ is not reduced. Further, the observed variations in reflection and transmission are due to actual changes in the systems impedance and not merely apparent changes due to propagation along $\hat{z}$. The general correspondence of the three methods in determining the unknown load impedance supports the presented relations for voltage and current observed at plate $B$.

Supported by the correspondence in derived impedance, the predicted transmitted voltage using the impedance obtained from the induced voltage is compared to measurements in Fig. 4.21. Compared to the prediction using the reflection coefficient in Fig. 4.12, using the impedance from the induced voltage improves the correspondence between measurements and
Figure 4.20. Comparison of impedance obtained with induced voltage $V_m$ (4.45, blue), reflection coefficient $\Gamma_m$ (green), and transmission coefficient (4.46, red). Negative real impedances are illustrated with a dashed line.
predictions. The real part of transmission is reduced from previous predictions near 100 MHz, and the imaginary component now has a much better correspondence with measurements. The close correspondence of both the real and imaginary parts confirms the derived relations for transmitted voltage (4.43).

Rewriting the currents in the system (4.33) to include the phase lag due to propagation along \( \hat{z} \),

\[
T_{cB} = (1 - \Gamma)e^{-jk_z L_z} + \Gamma(1 - \Gamma)e^{jk_z L_z}
\]

which can be written

\[
T_{cB} = (1 - \Gamma^2) \cos k_z L_z - j(1 - \Gamma^2) \sin k_z L_z.
\]

Rewritten in terms of impedance,

\[
T_{cB} = \frac{4}{(Z_L + Z_0)^2} \left( Z_0 Z_L \cos k_z L_z - j Z_0^2 \sin k_z L_z \right).
\]

Since the power is calculated for each mode in the capacitor individually using \( V e^{-jkz}(I e^{-jkz})^* \), the phase changes cancel, leaving the transmitted power unchanged from (4.40).

**4.4.5 Effective Circuit Descriptions**

The derived imaginary impedances obtained from measurements clearly suggests the circuit description in Fig. 4.22. These circuit parameters were determined using the FindFit routine in Mathematica 6.0 and the imaginary impedance of the system derived from measurements. A comparison of the impedance for a circuit with the values in table 4.1 and the impedance from \( V_m \) (4.45) is in Fig. 4.23. The fitted circuit closely matches the imaginary impedance and closely follows the real impedance below 50 Mhz. The real impedance was not used in the circuit fit and provides a check on the obtained values. The peak in real impedance near 70 MHz was not fitted because it was not present in the imaginary impedance.

The capacitance of the guitar can be estimated assuming an ideal parallel plate capacitor, \( C = A \varepsilon / L_z \), and is on the order of 10 pF. Because the guitar also includes a conductive
Figure 4.21. Comparison of voltage at plate $B$ using predictions based upon impedance from induced voltage (green), and the measured voltage at $B$ ($T_m$, blue).
sidewall, the expected capacitance is larger, consistent with the fitted capacitance $C_1$ near 75 pF. The inductance expected is $L = \mu L_z/8\pi$ [38], on the order of 10 nF, much smaller than the observed low frequency circuit inductance $L_2 + L_3 \approx 7.5 \mu$H. The inductance of the 10 foot LMR 240 coaxial cable between the analyzer and circuit has an inductance $L_{cable} \approx .6 \mu$H, insufficient to explain the observed inductance. Thus, these low frequency resonances are characterized by an increased inductance of unknown origin.

\[ \text{Figure 4.22. Circuit used to fit observed low order resonances of parallel plate capacitor.} \]

Table 4.1. Fitted circuit parameters for electromagnetic guitar using impedance from induced voltage.

<table>
<thead>
<tr>
<th>$C_1$ (pF)</th>
<th>$L_2$ ($\mu$H)</th>
<th>$C_2$ (pF)</th>
<th>$R_2$ (Ω)</th>
<th>$L_3$ ($\mu$H)</th>
<th>$C_3$ (pF)</th>
<th>$R_3$ (Ω)</th>
</tr>
</thead>
<tbody>
<tr>
<td>74.8</td>
<td>6.82</td>
<td>16.3</td>
<td>42.28</td>
<td>0.713</td>
<td>25.7</td>
<td>21.9</td>
</tr>
</tbody>
</table>

Table 4.2. Fitted circuit parameters for electromagnetic guitar using impedance from reflected voltage.

<table>
<thead>
<tr>
<th>$C_1$ (pF)</th>
<th>$L_2$ ($\mu$H)</th>
<th>$C_2$ (pF)</th>
<th>$R_2$ (Ω)</th>
<th>$L_3$ ($\mu$H)</th>
<th>$C_3$ (pF)</th>
<th>$R_3$ (Ω)</th>
</tr>
</thead>
<tbody>
<tr>
<td>112.1</td>
<td>3.55</td>
<td>32.1</td>
<td>30.59</td>
<td>0.774</td>
<td>16.55</td>
<td>52.5</td>
</tr>
</tbody>
</table>

A similar procedure was applied using the impedance from the reflected voltage, a comparison is in Fig. 4.24 and coefficients are listed in table 4.2. There are some differences between the values in table 4.1, but this is expected since the derived impedance from reflection will be altered by the extra power stored in the displacement field. The peaks in imaginary impedance are lower than the induced voltage impedance which leads to an
Figure 4.23. Comparison of the impedance of the circuit in Fig. 4.22 using fitted parameters (dashed, blue) and the measured capacitor impedance (green) using $V_m$. 

(a) Real Impedance

(b) Imaginary Impedance
increase in resistance in the fitted parameters. This resistance is larger than the observed real impedance at low frequency, but matches the peaks. This may reflect an increase in power loss at resonance.

Similarly, the fitted parameters for the small circular capacitor circuit obtained using the measured reflection coefficient are listed in table 4.3. Like for the electromagnetic guitar, the obtained capacitance is generally consistent with expectations and the inductance is several magnitudes larger than predicted by waveguide theory. The total inductance $L_2 + L_3$ is inversely proportional to the plate separation $L_z$, suggesting the capacitor is not coupling with a constant parasitic inductance. The inductance associated with waveguide modes increases with plate separation, although the obtained inductances decrease with increasing $L_z$. The large resistance obtained for $L_z = 2$ in. is not consistent with the measured real impedance at low frequencies.

Table 4.3. Fitted circuit parameters for small capacitor.

<table>
<thead>
<tr>
<th>$L_z$ (in)</th>
<th>$C_1$ (pF)</th>
<th>$L_2$ (µH)</th>
<th>$C_2$ (pF)</th>
<th>$R_2$ (Ω)</th>
<th>$L_3$ (µH)</th>
<th>$C_3$ (pF)</th>
<th>$R_3$ (Ω)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>43.5</td>
<td>3.04</td>
<td>32.6</td>
<td>27.8</td>
<td>1.3</td>
<td>86.7</td>
<td>134.5</td>
</tr>
<tr>
<td>4</td>
<td>31.3</td>
<td>2.12</td>
<td>45.7</td>
<td>18.8</td>
<td>0.55</td>
<td>22.5</td>
<td>31.9</td>
</tr>
<tr>
<td>8</td>
<td>25.</td>
<td>1.24</td>
<td>75.6</td>
<td>11.2</td>
<td>0.45</td>
<td>27.1</td>
<td>22.4</td>
</tr>
</tbody>
</table>

The resonances observed in Fig. 4.18 are near resonances of the cable lengths employed to measure the constructed capacitors, with an example of a resonant cable in Fig. 4.2. If the observed low frequency resonances were from a cable with a fundamental of 20 MHz, then harmonics are expected at 60 and 100 MHz. However, the resonances in Fig. 4.18 occur near 20 and 45 MHz, not consistent with the harmonic distribution expected from a resonant cable. Further, the coaxial cables employed in the experiment were calibrated to remove cable resonances and the proper operation of the equipment was verified. To investigate the possibility that the observed resonances could be the interaction of the capacitor plates with the ground of the coaxial cables left free at the interface between cable and capacitor,
Figure 4.24. Comparison of the impedance of the circuit in Fig. 4.22 using fitted parameters (dashed, blue) and the measured capacitor impedance (green) using the reflected voltage.
Figure 4.25. Comparison of the impedance of the circuit in Fig. 4.22 using fitted parameters (dashed, blue) and the measured capacitor impedance (green) for plate separation $L_z = 4''$. 
a more traditional configuration was employed. The new configuration electrically connects the ground of the source cable to plate $A$ of the capacitor, while the center conductor is electrically connected to plate $B$, Fig. 4.26. Since plate $A$ and the shielding of the coaxial cable are at the same potential, a possible parasitic capacitance between the systems should be reduced.

The connection from plate $A$ to plate $B$ used an SMA flange mount connector with an extended post. The calibration performed on the coaxial cable sets the reference plane measured by the network analyzer at plate $A$, though the reflection of the incident wave from the system occurs at plate $B$. To remove this internal propagation distance, zeros in measurements of the reflected voltage along with (4.15) can be used to determine this distance. Though the physical length of the connector is known, due to the design of the SMA connector, the electrical length is slightly different.

![Figure 4.26. Alternate experimental configuration. Ground of the coaxial cable is connected to plate $A$, while the center conductor delivers the source wave to plate $B$.](image)

Measurements of this system for the small capacitor are in Fig. 4.27. The low frequency peaks in real impedance are still present, though reduced in magnitude. Corresponding variations are also observed in the imaginary impedance, though the system does not have zeros in imaginary impedance. The expected waveguide modes ($\S 5.2$) are now present, observed around 500 MHz. A closer look at the waveguide impedance is in Fig. 4.28. The lowest series resonance of the system (150 MHZ) is still lower than expected (400 MHz), but the parallel resonance below 700 MHz corresponds well with the expected peak at 675
MHz using the next largest root for circular resonances \((1.814 \rightarrow 3.05)\) in (4.16). While the imaginary impedance in Fig. 4.6 appears dominated by the low frequency modes, the imaginary impedance in Fig. 4.27 only has small deviations from the expected capacitive impedance.

Similar results are observed for the electromagnetic guitar in Fig. 4.29, excited in the same manner depicted in Fig. 4.26. The peaks in real impedance are still observed and variations in imaginary impedance are reduced. Since the guitar plates are several inches apart, an SMA cable was used inside the instrument to convey the signal to plate \(B\). Since this cable is relatively long for the frequency range, it has a strong effect on the measured impedance that must be removed using (4.15).

The observed change in impedance for the low frequency modes does not necessarily indicate an interaction with the cable. In the previous setup in Fig. 4.5, the network analyzer was connected to plate \(A\) on the side facing away from plate \(B\). Since current is confined near the surface [38], to deliver charge to the other side of plate \(A\) and excite the capacitive mode of the system, current must propagate around the plate to reach the other side. If energy is radiated before the signal can drive currents on the inner conductive surfaces of the capacitor, no capacitive field will be observed. Measurements of the transmitted voltage at plate \(B\) yield a negative load impedance in Fig. 4.20 above 20 MHz (ignoring peak near 40 MHz), consistent with radiated power from plate \(A\) out of the system. Above this frequency, the system no longer responds with the expected capacitor impedance. When the excitation setup is changed to Fig. 4.26 and charge is delivered directly to the inner capacitor surfaces, the influence of the low frequency resonances is reduced and waveguide modes are observed. Even if the low frequency resonances remove energy from the system, the applied charge will still induce charge on the opposite plate, and behavior may be dominated by standard capacitive operation. Thus, the observed changes in low frequency mode impedance could be due to changes in the applied signal.

Waveguide modes are expected for the configuration in Fig. 4.5, though none are
Figure 4.27. Impedance obtained with the measured reflection coefficient of the setup in Fig. 4.26.
observed. Even if plate \(A\) transmits no excitation to plate \(B\) preventing three dimensional resonances, two dimensional resonance modes for the plate alone should be observed. Thus, either the observed low frequency resonances prohibit the expected waveguide modes at frequencies well above the low frequency resonances, or the low frequency resonances are the two dimensional waveguide resonances for the plate. Consider a cubical cavity with a side length of 1 m and a square piece of the same material comprising the cavity with the same side length. The two dimensional resonances of the cavity (no variation along one direction) would be determined by the properties within the cavity, while resonances within the square are determined by the properties of the material. If the phase velocity in the material is lower than the cavity, the resonance frequencies are correspondingly lower.

In high conductivity metals, like the pure aluminum used here, currents are limited to a thin layer termed the skin depth \(\delta\) \([38]\)

\[
\delta = \sqrt{\frac{2}{\mu \sigma \omega}}
\]

(4.50)

where \(\sigma\) is the conductivity of the material. Using a conductivity of \(3 \times 10^7\) S/m for aluminum and the vacuum permeability \(\mu_0\), the skin depth at \(\omega = 2\pi\) 10 MHz is \(3 \times 10^{-5}\) m, and at \(\omega = 2\pi\) 500 MHz, the skin depth is \(4 \times 10^{-6}\) m. Given the thinness of this layer, currents may

![Figure 4.28. A closer look at the imaginary impedance obtained with the measured reflection coefficient of the setup in Fig. 4.26](image-url)
be considered a surface current at high frequencies. Though these currents travel slowly in the material itself, given the proximity to the much faster electromagnetic waves just outside the conductor, fields propagate at near the speed of light in vacuum. Thus, for the observed waveguide modes near 500 MHz, it is sufficient to use the speed of light for vacuum when calculating resonance frequencies. Perhaps the increase in thickness of the skin depth by a factor of 10 for the observed low frequency resonances is sufficient to reduce the frequencies of two dimensional waveguide modes, though this is a conjecture. Measurements indicate that the dependence of the low frequency modes with variations in plate separation is inverse of the dependence for waveguide modes. However, the expected waveguide resonances involve fields between the plates, while the observed low frequency resonances may be confined to the plate. If confined to the plate, the same dependence with plate spacing is not expected, since the resonance no longer explicitly involves fields between the plates.

In support of the low frequency resonances being a reduced frequency waveguide mode, the observed impedance for the waveguide modes is very similar to the lower frequency resonances in Fig. 4.19. Though the third parallel resonance is better represented in Fig. 4.28 at 1.5 GHz, indications of a similar resonance can be found in both Fig. 4.19 and Fig. 4.25. It must be noted that the similarity of these impedances (apart from frequency) is shared by any system described by the circuit in Fig. 4.22.

4.4.6 Interactions with a Resonant Coaxial Cable

To further investigate the observed low frequency resonances, a resonant coaxial cable was connected to the same guitar face excited by the network analyzer. The coaxial cable was attached to the instrument using an SMA connector, with the outer conductor electrically connected to the instrument face. The center conductor extends into the instrument face and is left free. The hole in the instrument face matches the size of the outer shield of the SMA conductor. The voltage applied to the plate leads to a voltage difference between the ground and center conductor, driving waves into the cable. Similar behavior is also found if
Figure 4.29. Impedance derived from the measured reflection coefficient of an electromagnetic guitar; configured like Fig. 4.26.
the center conductor is connected to the instrument face and the ground is left free. In both cases, the connection between instrument and cable was located near the connection to the network analyzer.

The cable was terminated with a short, leading to a resonant cable with the impedance depicted in Fig. 4.2. The impedance of the system plus cable is in Fig. 4.31. The coupling of the cable impedance with the low frequency resonances leads to an additional resonance below 10 MHz. Further, the peak in real impedance at 4 MHz now has a corresponding resonance (zero imaginary impedance) at the same frequency. The cable’s influence decreases with frequency, near 100 Mhz most indications of its presence is gone. Similarly, for the imaginary impedance, strong interactions are observed at low frequency, but above 20 MHz the deviations in imaginary impedance disappear.

These results suggest that the cable is coupled in series with the inductor $L_2$ in the circuit depicted in Fig. 4.22. When the parallel $R_2L_2C_2$ circuit including cable passes resonance, the impedance of the coaxial cable becomes shielded from the rest of the circuit. When the coaxial cable resonates, its imaginary impedance could cancel the capacitance $C_2$ enough for the coaxial real impedance to continue to produce small peaks at 30, 50, 70 and 90
MHz. Since the coaxial cable is driven by the voltage in the instrument plate, the reduced interaction of the cable also confirms that the voltage distribution along the instrument plate is changing as a function of frequency, consistent with the distribution required by the derived impedance.

The coupling of a coaxial cable and the low frequency resonances was also investigated in a setup similar to Fig. 4.10, though plate B was not connected to the network analyzer. Results are in Fig. 4.32 and the additional resonance frequency just below 10 MHz is due to the additional resonance from the coaxial cable. The cables resonances at 70 and 90 MHz split the previous resonance at 80 MHz. The increasing number of resonances in the cable are visible above 100 MHz in both the real and imaginary impedance, consistent with the previous case.

An effective circuit in Fig. 4.33 was fitted to the impedance in Fig. 4.32 without a cable and the obtained values are in table 4.4. Only the lowest three resonances are represented, but this is sufficient to determine the coupling with the resonant coaxial cable. A comparison of the fitted circuit to measurements is in Fig. 4.35. Using measurements of the cable impedance, the cable is inserted into the circuit in Fig. 4.33, given in Fig. 4.34. The capacitive coupling between the plate and cable increases the low frequency capacitance and it is observed that $C_1 \rightarrow 203 \, \text{pF}$. The specifications of the LMR 240 cable used has a capacitance of approximately 465 pF for a 20 foot cable. The observed capacitance is less than half this value, and may be due to specifics in the coupling between cable and guitar. The value of $C_1$ has little effect on the location of the resonance frequencies.

The predicted impedance of the guitar with the cable is in Fig. 4.36. The coupling of the fitted guitar circuit (with modified $C_1$), plus the measured impedance of resonant coaxial cable, reproduces the general features of the observations. The peaks at 10 MHz in both the real and imaginary impedance are due to the cable. Measurements indicate an increase in the peak magnitude of the resonance near 15 MHz that is not observed in the predicted circuit. This behavior could be explained by a reduction in the real impedance of
Figure 4.31. Impedance derived from the measured reflection coefficient of an electromagnetic guitar with resonant 250 in. coaxial cable; configured like Fig. 4.26.
Figure 4.32. Impedance derived from the measured reflection coefficient of an electromagnetic guitar with resonant 250 in. coaxial cable; configured like Fig. 4.10, with no connection from plate \( B \) to the network analyzer.
the instrument due to the coupling with the cable, not reflected in the predictions obtained by simply inserting the cable impedance. The measured parallel resonance below 20 MHz also reduces in frequency with the inclusion of the coaxial cable and may be due to an increase in $C_3$ similar to the observed increase in $C_1$ with cable. Though the predicted behavior does not exactly match observations due to the limited applicability of the originally fitted circuit, the general features are reproduced. These experimental results confirm that the observed low frequency resonances are the result of physical behavior, though the mechanism is unknown.

A more complicated circuit fit was attempted that included the next two resonances of the system without aperture. However, these attempts shifted the inductance away from $L_3$ to the introduced additional inductance $L_4$, significantly changing the predicted interaction between the system and cable. This failing may be a limitation of the procedures used to fit circuit values to the measured impedance, rather than an indication of actual physical behavior.
Table 4.4. Fitted circuit parameters for electromagnetic guitar using impedance from reflected voltage.

<table>
<thead>
<tr>
<th></th>
<th>$C_1$ (pF)</th>
<th>$L_2$ ($\mu$H)</th>
<th>$C_2$ (pF)</th>
<th>$R_2$ ((\Omega))</th>
<th>$L_3$ ($\mu$H)</th>
<th>$C_3$ (pF)</th>
<th>$R_3$ ((\Omega))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>29.1</td>
<td>1.95</td>
<td>804.3</td>
<td>8.17</td>
<td>1.63</td>
<td>60.7</td>
<td>24.22</td>
</tr>
</tbody>
</table>

Figure 4.35. Comparison of fitted circuit to measurements of electromagnetic guitar, with no connection from plate $B$ to the network analyzer.
Figure 4.36. Comparison of predicted impedance to measurements of electromagnetic guitar with a 250 in. cable, with no connection from plate $B$ to the network analyzer.
4.5 Measurements of a Capacitor with an Aperture

Consider measurements of an electromagnetic guitar with and without an aperture in Fig. 4.37. The electrical configuration of the guitar is in Fig. 4.26, where plate $A$ is grounded and contains the aperture. A coaxial cable delivers the signal to plate $B$ through the instrument. No additional resonances are detected with the inclusion of an aperture. The first series resonance of the two systems in Fig. 4.38 occur at the same frequency, demonstrating no increase in inductance due to aperture. However, the peak at 150 MHz is broader and lower in magnitude, consistent with increased radiation by the instrument. This could simply be the leakage of fields from the aperture, but the real impedance for both systems becomes the same near 200 MHz and consistent thereafter. Since the increased radiation is confined to frequencies near the expected EM Helmholtz resonance, these results suggest that the predicted aperture and capacitor interaction is still occurring.

Consider the effective circuit in Fig. 4.39 obtained from measurements of the copper guitar without an aperture, with the fitted component values in table 4.5. A comparison of the fitted circuit to measurements is in Fig. 4.40 and closely matches observations. Only the measured imaginary impedance is used in the circuit fit, so the correspondence in real impedance further verifies the fitted circuit parameters. The resistor $R_1$ was added manually since it’s impedance does not effect the imaginary components and was not determined in the circuit fit. Note that the resistances in Fig. 4.39 are associated with the capacitors, but the resistances associated with the low frequency resonances are associated with the inductors. The resistor was moved since a satisfactory numerical fit could not be obtained with a resistor associated with the inductor. This may reflect different physics between the systems, or it may be a limitation of the numerical fit.

To include the effect of an aperture in the system, the aperture response to both the voltage transmitted from plate $B$ to $A$, plus the voltage induced on plate $A$ must be considered. Since the voltage induced on plate $A$ is due to image charges, this wave is out of phase with the voltage from $B$ and appears to be incident from the opposite side of the
Figure 4.37. Impedance derived from the measured reflection coefficient of an electromagnetic guitar with (blue) and without (green) an aperture; configured like Fig. 4.26.
Figure 4.38. A closer look at the imaginary impedance derived from the measured reflection coefficient of an electromagnetic guitar with (blue) and without (green) an aperture; configured like Fig. 4.26.

If the voltage at B is high, the induced voltage is low. Since these waves appear to originate from sources on opposite sides of the plate and the voltages are out of phase, the resultant electric fields created by these waves on plate B will have the same orientation. Since these two waves come from different sides of the aperture, with the same electric field orientation, the interaction leads to an altered aperture impedance.

Using the standard aperture theory [2], the aperture dipoles produced by these waves cancel and there is no effect from the aperture at all. However, in the capacitor with an aperture, this does not occur. Normally, the waves incident upon both sides of the aperture are independent. However, in the capacitor, the induced voltage is driven by voltage on plate B and must be considered using the same phase reference plane.

Consider the waves from plate B to A, which will experience aperture impedance (3.49) for tangential magnetic fields and (3.50) for normal electric fields. For the induced voltage waves that appear to originate from the other side of the plate, the aperture impedance must be calculated using the same reference phase for the aperture current, though the dipoles produced have opposite orientation. Since the imaginary impedance is proportional
Figure 4.39. Effective circuit description of copper guitar.

Table 4.5. Fitted circuit parameters for electromagnetic guitar.

<p>| | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$ (pF)</td>
<td>$R_1$ (Ω)</td>
<td>$L_2$ (nH)</td>
<td>$C_2$ (nF)</td>
<td>$R_2$ ($mΩ$)</td>
<td>$L_3$ (nH)</td>
<td>$C_3$ (pF)</td>
<td>$R_3$ (Ω)</td>
<td></td>
</tr>
<tr>
<td>75.9</td>
<td>1.2</td>
<td>0.858</td>
<td>1.2</td>
<td>18.8</td>
<td>98.1</td>
<td>3.</td>
<td>5.25</td>
<td></td>
</tr>
</tbody>
</table>

to the dipole moment, the impedance is equal and opposite to the impedance for waves from plate $B$. The aperture real impedance is proportional to the dipole moment squared and is equal to the impedance for waves from $B$.

The aperture impedance for the electric and magnetic aperture dipoles are combined in parallel in chapter 3 because the plane wave exciting the aperture has electric and magnetic components in phase. The electric and magnetic fields in the capacitor exciting the aperture are out of phase, with energy oscillating between electric and magnetic fields. These fields will excite a similar oscillation in the aperture, so the electric and magnetic dipole impedances are combined in series to reflect the energy oscillation between these moments. The series combination of impedances for waves on both sides of the aperture results in a doubled real impedance and no imaginary impedance.

The real parts of the aperture impedance are included in the circuit for the guitar with no aperture in Fig. 4.41. The real impedance for the magnetic dipole (3.49) and electric dipole (3.50) are labeled as $R_{ap,h}$, $R_{ap,e}$, respectively. Both the magnetic and electric dipoles are excited in the capacitor due to the normal electric fields and tangential magnetic fields in the capacitor. The magnetic dipole impedance is in series with the inductance of the
Figure 4.40. Comparison of measured impedance (red) of electromagnetic guitar with a copper plate and the effective circuit (blue) in Fig. 4.39. The absolute magnitude of the imaginary impedance is presented.
lowest parallel resonance near 150 MHz. It is thought that the electric dipole impedance has a similar coupling with the resonance near 190 MHz, with values determined by $RLC_{2.5}$ in Fig. 4.42. Since this resonance does not appear in the measured imaginary impedance, it isn’t present in the fitted circuit. However, the electric dipole impedance should not be included in the parallel $R_3L_3C_3$ resonance, so it is placed in series outside these resonances.

A comparison of the circuit in Fig. 4.41 to the measurements of the aluminum electromagnetic guitar with an aperture is in Fig. 4.43 and predictions are very similar to measurements. The increase in real impedance between 50 − 125 MHz compared to no aperture is due to the electric dipole impedance and the reduction in the peak magnitude at 150 MHz is due to the magnetic dipole impedance. This frequency dependence can not be obtained with a constant real impedance. The excess real impedance above this resonance is due to the real impedance of the electric dipole placed in series with $C_1$. If the dipole impedance was associated with the resonance near 200 MHz, its influence would be limited above resonance. Further, since the electric dipole real impedance is larger than the other real impedances, the peak in real impedance near 200 MHz would be significantly reduced. Otherwise, the predicted circuit closely matches observations. The slope of the real impedance is partly determined by the effective aperture dipole length (3.21), calculated to be $b = \pi a / 4$. Decreases in $b$ increase the real impedance and measurements confirm that $b$ is less than the aperture radius. These results not only support the derived aperture impedance, but confirm the viability in using the aperture impedance to describe the interaction of a system with an aperture.
Though the area of the aperture and the capacitor are different, this fact does not appear to impact the observed coupling of the capacitor and aperture impedance. However, it is known that not all of the power in the capacitor can excite the aperture. Thus, it is presumed that the aperture and capacitor impedance are actually combined via a transformer. This transformer accounts for the difference in capacitor and aperture area limiting the power radiated by the aperture. Since the power radiated by the aperture is determined by the magnitude of the normal electric and tangential magnetic fields exciting the aperture, the transformer coupling the aperture and capacitor may be determined by calculating the power radiated by the aperture. The impedances derived earlier treated incident fields as a current source and the same technique is used here. In contrast with standard transformers employed in circuits that preserve power but alter the observed impedance, it is shown that the transformer coupling the aperture and capacitor will preserve impedance, but not power.

The power radiated by the aperture electric dipole is given by the average normal electric field over the aperture area, converted to an equivalent input current density (amps per meter) using the relation (3.26)

\[ I_{ap} = bJ_D \]  \hspace{1cm} (4.51)

where \( J_D \) is the displacement current in the capacitor. Including the boundary conditions for electric fields within the aperture, the radiated power is predicted to be

\[ P' = \frac{\epsilon_2^2}{(\epsilon_1 + \epsilon_2)^2} \frac{S}{2} |b^2J_D|^2 Re[Z_{2e}] \]  \hspace{1cm} (4.52)
Figure 4.43. Comparison of measured impedance (red) of aluminum guitar with an aperture and the effective circuit (blue) in Fig. 4.41. The absolute magnitude of the imaginary impedance is presented.
Using the total displacement current over the capacitor area, \( I_c = S_c J_D \), the radiated power can be rewritten

\[
P' = \frac{\varepsilon_2^2}{(\varepsilon_1 + \varepsilon_2)^2} \frac{Sb^2}{S_c^2} \frac{1}{2} |I_c|^2 Re[Z_{2e}] \tag{4.53}
\]

where \( S \) is the aperture area, \( Z_{2e} \) is the impedance of the aperture electric dipole, and \( S_c \) is the capacitor area. Note that in contrast to the power obtained in §3.3, the reflection of waves upon the aperture is not used. Rather, the current density \( I_{ap} \) is used directly.

The power actually radiated by the aperture (4.53) is only a fraction of the power \( P \) dissipated by the aperture impedance in the capacitor with aperture circuit

\[
P = \frac{1}{2} |I_c|^2 Re[Z_{2e}] \tag{4.54}
\]

The excess power dissipated by the aperture impedance in the capacitor circuit is presumed to reflect the additional power transmitted into the system due to the aperture impedance, but is not radiated by the aperture. Rewriting (4.53),

\[
|I'|^2 Z_{2e} = \frac{\varepsilon_2^2}{(\varepsilon_1 + \varepsilon_2)^2} \frac{Sb^2}{S_c^2} |I_c|^2 Z_{2e} \tag{4.55}
\]

must be equivalent to

\[
V' I'^* = \frac{\varepsilon_2^2}{(\varepsilon_1 + \varepsilon_2)^2} \frac{Sb^2}{S_c^2} V I_c^* \tag{4.56}
\]

where \( V, I_c \) are the voltage and current associated with the aperture in the combined capacitor with aperture circuit, and \( V', I' \) are the respective values for determining aperture power. Equating the right hand sides of these equivalent descriptions of power and simplifying,

\[
V = I_c Z_{2e} \tag{4.57}
\]

indicating that the observed impedance in the capacitor with aperture circuit is the aperture impedance.

Since the aperture impedance is conserved, the transformer scales voltages and currents with factor

\[
V' = \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \frac{\sqrt{Sb}}{S_c} V \tag{4.58}
\]

\[
I' = \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \frac{\sqrt{Sb}}{S_c} I_c. \tag{4.59}
\]
Since the electric field in the capacitor excites the aperture, and not the voltage between the capacitor plates, the voltage associated with the aperture is not fixed by the capacitor voltage.

The magnetic field at \( r \) within a circular parallel plate capacitor with radius \( R_c \) where \( r < R_c \) is

\[
H = \frac{I_c r}{2S_c} \hat{\theta}
\]  

and can drive the aperture magnetic dipole. The total power radiated by the aperture is determined by the average magnetic field over the aperture area. Assuming the aperture radius is much smaller than both \( r \) and \( R_c \), then the average is simply the value at \( r \). Assuming that the magnetic field is purely along \( \hat{\beta} \) in the aperture, then the radiated power is predicted to be

\[
P = \frac{\mu_1^2}{(\mu_1 + \mu_2)^2} \frac{S r^2}{4S_c^2} \frac{1}{2} |I_c|^2 \text{Re}[Z_{2h}] \]  

and the scale factors for the voltage and current from the transformer are supposed to be

\[
V' = \frac{\mu_1}{(\mu_1 + \mu_2)} \frac{\sqrt{S r}}{2S_c} V
\]

\[
I' = \frac{\mu_1}{(\mu_1 + \mu_2)} \frac{\sqrt{S r}}{2S_c} I_c.
\]

Consider this postulated transformer and the observed cancellation of the imaginary impedance of an aperture in a grounded plane. Even if the induced voltage is not equal and opposite to the applied voltage, the impedance contribution from each wave will have the same magnitude. Thus, for an aperture in a grounded plane conductor, the imaginary aperture impedance will be zero at all frequencies.

4.5.1 Proposed Configuration for Observing Resonance

Measurements of a capacitor with an aperture confirm that the derived aperture circuit can be used to describe the interaction between the two systems. The electromagnetic instrument measured had an aperture in the ground plane, which, due to the presence of induced charges,
leads to a doubled real impedance and no imaginary impedance. Since induced charges are not present on the plate directly excited by a source, an aperture in this plate may have a non-zero imaginary impedance. Thus, obtaining a resonance between the capacitor and aperture may be as simple as not placing the aperture in the ground plane.

Alternatively, rather than utilize a parallel plate capacitor configuration, all parts of the instrument can be electrically connected. In this configuration, low frequency capacitive behavior arises from the fields produced on the outside surface of the instrument with respect to the ground at infinity. Fundamentally, this is the same capacitance experienced by a single conductive plate. However, changes in the net potential of the instrument will also lead to changes in the aperture potential, and the fields produced by the aperture will apply a voltage to the inside of the cavity. Since all of the conductors inside the instrument are connected and at the same potential, these fields would normally be excluded from inside the instrument. However, since the aperture is not at the same potential as the inner surface of the cavity, a voltage difference does exist and fields are possible within the instrument if those fields oscillate between the aperture inductance and the capacitance of the cavity. Since the voltage across the aperture will be the voltage applied inside the instrument, these two elements are placed in parallel. The excitation applied to the aperture must go through the impedance of the capacitive fields on the outer surface of the instrument, so the parallel combination of aperture and inner cavity impedance is placed in series with the impedance on the outer surface of the instrument. This description is consistent with the acoustic guitar measurements both with and without an aperture in [18], discussed in §5.3.

Though the capacitance of the guitar will be reduced if all parts are electrically connected, this change in capacitance can be tolerated. Measurements of the electromagnetic guitar configured like Fig. 4.10 with plate $B$ either connected to the network analyzer or left free have been performed. Without a connection to charge, the interaction between the plates is reduced, and the capacitance is slightly less than half the capacitance of the system configured normally.
Unfortunately, measurements of the capacitor with an aperture were not understood during the time the rental equipment was available. Thus, these presented configurations had yet to be identified as preferred and were not measured due to lack of time with the measurement equipment.

4.6 Conclusion

Measurements of the reflected and transmitted voltage of a parallel plate capacitor demonstrate that the standard relations describing these quantities are insufficient for the grounded plate. It is postulated that the voltage induced on the ground plane by image charges must also be accounted for and new relations are derived. Using these derived quantities, the impedance from the measured reflection, transmission, and induced voltage is calculated. The magnitude of these impedances are similar for all methods, though there are differences in the low frequency impedance. These differences are consistent with the predicted power distribution in the capacitor. Since antennas are commonly employed above ground planes, measurements of the reflected power may be reduced based upon coupling with the ground plane presented here. Stored energy in the field between the antenna element and the ground plane can alter the ground plane’s potential, leading to a modified reflected voltage. The presence of induced currents for a standard configuration employed for antennas has been verified by the lack of imaginary aperture impedance for an aperture in a grounded plane.

The derived real impedance from measurements of the transmitted voltage through a capacitor is negative for some frequencies, with the first transition to a negative impedance at an unexplained low frequency parallel resonance near 20 MHz. For comparison, the first measured parallel waveguide resonance is near 400 MHz. Based upon circuit models fit to the derived impedance of the system, the apparent low frequency resonant modes have an associated inductance that is several magnitudes larger than expected from waveguide theory. The inductance is inversely proportional to distance, compared to a proportional relationship expected from waveguide modes.
The low frequency resonances are verified through measurements of the interaction between the capacitor and a resonant coaxial cable. The measured impedance of the combined system can be predicted by using a combination of the measured impedance of the coaxial cable and a circuit model of the measured capacitor without cable. The effect of the coaxial cable on the measured impedance is significantly reduced after the first parallel resonance of the aperture, consistent with a changing distribution of voltage and current. This is reflected in the postulated circuit description by placing the cable impedance in the first parallel $RLC$ resonance.

The mechanisms behind the observed resonances are unknown. It is believed that the resonances correspond to two dimensional resonances of current in the conductor plates, although this requires a reduced surface wave speed. Plasmons (surface waves) do have a slightly reduced phase velocity in the capacitor plates, but is on the order of a few percent \cite{22}, not nearly the 98% reduction required. The inductance associated with these reduced resonances is larger than the inductance of the cables coupling the system to the network analyzer and varies with the capacitor plate separation, suggesting the inductance is not due to an external parasitic inductance.

Measurements of a capacitor with an aperture in the ground plane indicate an increase in real impedance given by twice the magnitude of the impedances calculated in chapter 3 and no additional resonances. This behavior is explained by appealing to the voltage wave induced in the ground plane, which appears as a wave incident from outside the capacitor. The aperture dipole excited by the field driving the capacitor is out of phase with the aperture dipole created by the induced voltage. Because the imaginary aperture impedance is proportional to the dipole moment, while the real impedance is proportional to the dipole moment squared, the series combination of aperture impedances for both waves cancels the imaginary impedance and doubles the real impedance. A circuit model fit to the system without an aperture, plus the derived real aperture impedances predicts an impedance consistent with measurements of the system with an aperture. Though the aperture and capacitor have dif-
ferent areas, this does not alter the coupling of impedance, only the radiated power. Though the electromagnetic Helmholtz resonance was not observed, because the interaction of a system with an aperture can be described simply by combining the derived aperture impedance with the system’s impedance without the aperture, it is expected that a configuration where the resonance is observed could also be described using the aperture impedance.
CHAPTER 5
ELECTROMAGNETIC MUSICAL INSTRUMENTS

At a first glance, electromagnetic and acoustic fields may appear very different. Acoustic fields are comprised of longitudinal pressure and velocity waves, while electromagnetic fields are comprised of transverse electric and magnetic fields. Despite these differences, acoustic and electromagnetic waves are described by the same wave equation in sourceless situations and share the same resonant wavelengths in enclosed cavities. In 1954, M. R. Schroeder [69] used the resonance of Electromagnetic fields to determine acoustic properties of a cavity. He states, “There is no difficulty in transferring the results from [Electromagnetic] cavity resonators to acoustic spaces; all we have to remember is that the number of normal frequencies (NF) in a given interval is halved.” This correspondence has also been used by S. Sridhar and Stockmann [77, 78] to solve both acoustic and quantum mechanical problems.

This general correspondence suggests that the systems employed to create acoustic instruments have electromagnetic equivalents that can be used to create an electromagnetic musical instrument. The vibrations of sound and wood in a traditional stringed acoustic instrument are replaced with oscillations of current and electromagnetic fields in a metal instrument body. The acoustic Helmholtz resonance is replaced with the equivalent electromagnetic Helmholtz resonance and the sounds produced by acoustic instruments are replaced with electromagnetic waves. In order to hear the music produced by the electromagnetic instrument, the radiated waves can be measured and converted into sounds. Since electromagnetic and acoustic fields are generally similar, the sound of an electromagnetic musical instrument will be similar to an acoustic instrument. However, the use of resonant electromagnetic fields opens new materials and possibilities for musical instruments.
In the following sections, both acoustic and electromagnetic waveguides are discussed and shown to share the same resonance behaviors. Using the circuit description of the acoustic Helmholtz resonator as a general guide, a simple circuit description of the electromagnetic Helmholtz resonator is proposed using the aperture impedance from chapter 3 and the known capacitance of a parallel plate capacitor. The predicted resonance for the electromagnetic resonator produces a radiated wavelength similar to the acoustic Helmholtz resonator for cavity dimensions employed by acoustic musical instruments. The construction of an electromagnetic musical instrument is discussed and an introduction to the computer audio techniques used to hear the sound of an electromagnetic musical instrument is given.

Measurements of the prototype electromagnetic guitar have also been performed and these measurements are used to generate the impulse response of the guitar. The impulse response of the instrument can be used with standard audio software to modify music, making it sound as if the music originated from the electromagnetic musical instrument. Many impulse responses can be generated from one set of measurements and these variations are discussed. Examples of the music produced by the instrument are introduced.

5.1 Acoustic Waveguides

This abbreviated derivation of the acoustic wave equation is adapted from Fundamentals of Physical Acoustics by Blackstock, an introductory text on acoustics [3]. Acoustic waves are comprised of oscillations in bulk particle velocity and pressure. The velocity field of an acoustic wave may be described via a velocity potential, defined in the same manner as the electric scalar potential,

\[ \mathbf{V} = -\nabla \phi \]  \hspace{1cm} (5.1)

where \( \mathbf{V} \) is velocity, and \( \phi \) is the velocity potential. This velocity potential may also be related to pressure by

\[ p = \rho_0 \frac{\partial \phi}{\partial t} \]  \hspace{1cm} (5.2)
where \( p \) is the deviation from rest pressure and \( \rho_0 \) is the deviation from the rest mass density of the gas. Note that this is only strictly true when one is dealing with an irrotational field. This is generally true in acoustics, although rotational flows are encountered near boundaries [3]. Because both velocity and pressure may be related to the velocity potential, acoustic fields are completely characterized using the velocity potential alone. For a sourceless, linearized, small signal, isentropic acoustic field (generally true in human experience), acoustic behavior is described by

\[
\nabla^2 \phi - \frac{1}{c_a^2} \frac{\partial^2 \phi}{\partial t^2} = 0,
\]

known as the wave equation, where

\[
\nabla^2 = \nabla \cdot \nabla.
\]

The wave equation is a second order differential equation and applies to many different physical situations; some examples include oscillating strings, gases, and electromagnetic fields. The wave equation leads to a class of solutions which may be broken into sub classes based upon the behavior of the solution as one approaches a boundary. To obtain a solution for a particular problem, choose from the available solutions those which match the boundary conditions of the physical problem.

For example, at a rigid surface, the normal component of velocity must go to zero. Mathematically,

\[
\mathbf{V} \cdot \hat{n} = -\hat{n} \cdot \nabla \phi = -\frac{\partial \phi}{\partial n} = 0
\]

(5.5)

where \( \hat{n} \) is normal to the surface in question. Boundary conditions requiring a spatial derivative to be zero are sometimes referred to as a free boundary condition, based upon the similarity to the boundary condition on a string free to slide on a frictionless rod. Alternatively, at a soft surface (no rigidity), the deviation from rest pressure is zero,

\[
p = \rho_0 \frac{\partial \phi}{\partial t} = 0
\]

(5.6)
This condition is normally enforced by requiring $\phi = 0$, known as a fixed boundary condition. If $\phi$ remains zero, then $\partial \phi / \partial t$ is also zero.

The general solution for the wave equation in one dimension with rigid walls at $x = 0, L_x$ may be written

$$\phi = (A_1 \cos kx + A_2 \sin kx) e^{i\omega t} \quad (5.7)$$

where $A_1, A_2$ are coefficients to be determined, $k$ is the wavenumber, and $t$ is time. The oscillatory time behavior of this solution is accounted for in the exponential term, where $i = \sqrt{-1}$ and $\omega$ is the angular frequency, related to frequency by $\omega = 2\pi f$. The frequency $f$ is the number of oscillations of the wave per second. The spatial dependence of the wave is accounted for in the cosine and sine terms, where the wavenumber $k$ is related to wavelength $\lambda = k/2\pi$.

The boundary condition upon rigid walls requires $\partial \phi / \partial x = 0$ at $x = 0, L_x$

$$\frac{\partial \phi}{\partial x} = -A_1 k \sin kx + A_2 k \cos kx. \quad (5.8)$$

At $x = 0$, the cosine term is one, requiring $A_2 = 0$ to satisfy the boundary condition $\partial \phi / \partial x = 0$. At $x = L_x$, the remaining sine term will only be zero if $kL_x = m\pi$, where $m = 0, 1, 2 \cdots$, and the velocity potential is

$$\dot{\phi} = A_1 \cos \left( \frac{m\pi x}{L_x} \right). \quad (5.9)$$

Substituting this solution into the wave equation (5.3) and calculating the time and spatial derivatives,

$$\left( \frac{m^2 \pi^2}{L_x^2} - \frac{\omega^2}{c_a^2} \right) \phi = 0. \quad (5.10)$$

Since this equation must be satisfied for all times and all locations along $\hat{x}$, the coefficients in the parentheses are required to be zero, and

$$\frac{m^2 \pi^2}{L_x^2} = \frac{\omega^2}{c_a^2}. \quad (5.11)$$
Since \( k = \frac{m\pi}{L_x} \), only a particular set of waves are allowed, with wavelengths \( \lambda = \frac{2L_x}{m} \) and associated frequencies

\[
\omega = kc_a = \frac{m\pi c_a}{L_x}.
\]

(5.12)

Solving for \( \omega \), there are a discrete set of resonance frequencies at which the system has a solution. For frequencies away from these resonances, the waves do not satisfy the boundary conditions, and no coherent response of the system will be observed.

Let’s consider the behavior of a source less acoustic field within a closed, rigid rectangular box with dimensions \( L_x, L_y \), and \( L_z \). Since the wave equation can be separated into independent functions along each direction, the resonant behavior of the box along \( \hat{y}, \hat{z} \) is obtained using the same method as the one dimensional example along \( \hat{x} \). Enforcing the boundary condition \( \partial \phi / \partial n = 0 \) at each surface, the velocity potential is

\[
\phi_{lmn}(x, y, z) = A_{lmn} \cos \left( \frac{l\pi x}{L_x} \right) \cos \left( \frac{m\pi y}{L_y} \right) \cos \left( \frac{p\pi z}{L_z} \right) e^{-i\omega_{lmn}t} \tag{5.13}
\]

where \( l, m, p \) label the solution along each direction. Substituting this solution into the wave equation (5.3), the angular frequencies of the resonant modes are

\[
\omega_{lmn} = c_a \sqrt{\left( \frac{l\pi}{L_x} \right)^2 + \left( \frac{m\pi}{L_y} \right)^2 + \left( \frac{p\pi}{L_z} \right)^2}. \tag{5.14}
\]

The spatial dependence of velocity along each direction is obtained using (5.1),

\[
V_{lmn}^x = A_{lmn} \frac{l\pi}{L_x} \sin \left( \frac{l\pi x}{L_x} \right) \cos \left( \frac{m\pi y}{L_y} \right) \cos \left( \frac{p\pi z}{L_z} \right) e^{-i\omega_{lmn}t} \tag{5.15}
\]

\[
V_{lmn}^y = A_{lmn} \frac{m\pi}{L_y} \cos \left( \frac{l\pi x}{L_x} \right) \sin \left( \frac{m\pi y}{L_y} \right) \cos \left( \frac{p\pi z}{L_z} \right) e^{-i\omega_{lmn}t} \tag{5.16}
\]

\[
V_{lmn}^z = A_{lmn} \frac{n\pi}{L_z} \cos \left( \frac{l\pi x}{L_x} \right) \cos \left( \frac{m\pi y}{L_y} \right) \sin \left( \frac{p\pi z}{L_z} \right) e^{-i\omega_{lmn}t} \tag{5.17}
\]

where \( l, m, p = 0, 1, 2 \cdots \). The mode \( l = m = p = 0 \) is excluded since it is static (\( \omega = 0 \)) and the velocity field is zero everywhere.

For a soft rectangular box, the fixed boundary condition that \( \phi = 0 \) at all surfaces is satisfied by the velocity potential

\[
\phi_{lmn}(x, y, z) = A_{lmn} \sin \left( \frac{l\pi x}{L_x} \right) \sin \left( \frac{m\pi y}{L_y} \right) \sin \left( \frac{p\pi z}{L_z} \right) e^{-i\omega_{lmn}t} \tag{5.18}
\]
with the corresponding velocity field,

\[ V_{xlm} = -A_{lmn} \frac{l\pi}{L_x} \cos \left( \frac{l\pi x}{L_x} \right) \sin \left( \frac{m\pi y}{L_y} \right) \sin \left( \frac{p\pi z}{L_z} \right) e^{-i\omega_{lm}t} \]  

(5.19)

\[ V_{ylm} = -A_{lmn} \frac{m\pi}{L_y} \sin \left( \frac{l\pi x}{L_x} \right) \cos \left( \frac{m\pi y}{L_y} \right) \sin \left( \frac{p\pi z}{L_z} \right) e^{-i\omega_{lm}t} \]  

(5.20)

\[ V_{zlm} = -A_{lmn} \frac{n\pi}{L_z} \sin \left( \frac{l\pi x}{L_x} \right) \sin \left( \frac{m\pi y}{L_y} \right) \cos \left( \frac{p\pi z}{L_z} \right) e^{-i\omega_{lm}t} \]  

(5.21)

for \( l, m, p = 1, 2, 3 \cdots \). The resonance frequencies are given by the same function as the case for rigid walls (5.14), but the limits on the indexes for the soft shell have a minmum resonance \( \omega_{111} \), where the rigid cavity can also have \( \omega_{100}, \omega_{010}, \text{and } \omega_{001} \).

Since the wave equation is second order, the boundary condition \( \phi = 0 \) at a soft surface can be replaced with

\[ \frac{\partial^2 \phi}{\partial n^2} = \frac{\partial (\mathbf{V} \cdot \hat{n})}{\partial n} = 0. \]  

(5.22)

Solutions for two dimensional problems, such as the head of a drum (idealized), can be obtained simply by removing the \( \hat{z} \) dependence from the solutions. It will be shown that velocity field in both rigid and soft cavities have the same dependence as the magnetic field in the transverse electric mode and the electric field in the transverse magnetic mode respectively.

For plates, the thickness of the plate becomes important, and the corresponding resonances are determined using [25, eq. 4.2.24]

\[ \left( D \nabla^4 + \rho h \frac{\partial^2}{\partial t^2} \right) Y = 0 \]  

(5.23)

where \( Y \) is the displacement of the plate, \( \nabla^4 = \nabla^2 \nabla^2 \), \( h \) is the thickness of the plate, and

\[ D = \frac{E h^3}{12(1 - \nu^2)} \]  

(5.24)

characterizes the plate material; \( E \) is the modulus of elasticity, a measure of stiffness, and \( \nu \) is Poisson’s ratio, relating loads on the material transverse to the applied force.

If the plate is simply supported, the position of the plate is fixed, but it is allowed to bend. Mathematically, a simply supported plate has boundary conditions \( Y = 0 \) at the
support. For this condition, the resonant modes have the same form as the resonant modes for a thin two dimensional membrane [25, pg. 248]. However, the resonant frequencies are

$$\omega_{lm} = \pi^2 \left( \frac{l^2}{L_x^2} + \frac{m^2}{L_y^2} \right) \sqrt{\frac{D}{h}}. \quad (5.25)$$

Note that these frequencies do not have the same square root dependence found in the resonant waveguide modes (5.14). Thus, though the waveforms are shared with a membrane, the resonance frequencies are not.

5.2 Electromagnetic Waveguides

This introduction to electromagnetic fields in waveguides has been adapted from *Classical Electrodynamics* by Jackson [38]. The equations that describe the behavior of electromagnetic fields are collectively known as Maxwell’s equations,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (5.26)$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J} \quad (5.27)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon} \quad (5.28)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (5.29)$$

where $\mathbf{E}, \mathbf{B}$ represent the electric and the magnetic induction fields, $\mathbf{J}$ is the free current density, $\rho$ is the free charge density, $\varepsilon$ and $\mu$ represent the permittivity and permeability of the space, and $t$ is time. Analytically, these equations can be combined to produce wave equations,

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \mathbf{E} = \frac{\mu_0 \mathbf{J}}{\partial t} - \nabla \frac{\rho}{\varepsilon} \quad (5.30)$$

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \mathbf{B} = -\nabla \times \mu_0 \mathbf{J} \quad (5.31)$$

and the techniques presented for acoustic fields can be used to solve electromagnetic systems. Appropriate boundary conditions for the physical situation are assumed and one looks for a set of equations that satisfy both the wave equation and the boundary conditions. To
determine the resonant modes of an ideal cavity, the free charge density $\rho$ and current $\mathbf{J}$ can be set to zero. This can be done since applying the wave equation to the resonant fields yields zero and therefore can always be included with the fields directly created by a source in the cavity.

At the surface of a perfect conductor, the tangential components of the electric field and the normal component of the magnetic field go to zero,

\[
\mathbf{E} \times \hat{n} = 0 \tag{5.32}
\]
\[
\mathbf{B} \cdot \hat{n} = 0 \tag{5.33}
\]

where $\hat{n}$ is normal to the conductor. The fixed condition on the magnetic field is the same boundary condition for velocity at a rigid surface (5.5). Substituting (5.32) into (5.28) with no free charge density requires that

\[
\frac{\partial \mathbf{E} \cdot \hat{n}}{\partial n} = 0 \tag{5.34}
\]

at the conductor surface, the same free boundary condition for velocity at a soft surface (5.22).

While an acoustic field in a rigid or soft cavity has only one boundary condition at the walls, electromagnetic fields in a conductive cavity have two independent conditions. The two boundary conditions (5.33, 5.34) lead to two independent resonant modes, designated Transverse Magnetic (TM) and Transverse Electric (TE) modes. These two operating modes are developed for an electromagnetic (EM) waveguide.

Consider an rectangular waveguide, infinitely long in the $\hat{z}$ direction. Generally, there could be both an electric and magnetic field propagating along $\hat{z}$,

\[
\mathbf{E} = \mathbf{E}(x, y)e^{i(k_z z - \omega t)} \tag{5.35}
\]
\[
\mathbf{B} = \mathbf{B}(x, y)e^{i(k_z z - \omega t)} \tag{5.36}
\]

Substituting these equations into (5.30, 5.31), it can be shown [38, pg. 358] that all of the
components transverse to \( \hat{z} \) can be calculated with knowledge of the fields along \( \hat{z} \),

\[
E_t = \frac{i}{\mu \varepsilon \omega^2 - k_z^2} (k_z \nabla_t E_z - \omega \hat{z} \times \nabla_t B_z) \tag{5.37}
\]

\[
B_t = \frac{i}{\mu \varepsilon \omega^2 - k_z^2} (k_z \nabla_t B_z + \mu \varepsilon \omega \hat{z} \times \nabla_t E_z) \tag{5.38}
\]

where

\[
\nabla_t = \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y}
\]

is the gradient in components transverse to \( \hat{z} \), \( k_z \) is the wavenumber along \( \hat{z} \), and \( E_t, B_t \) are the field components along \( \hat{x} \) and \( \hat{y} \).

The six separate unknowns in (5.30) and (5.31), one for each field along each direction, have been reduced to two independent unknowns, the electric and magnetic fields along \( \hat{z} \). This can be achieved since all of the electric and magnetic field components are not independent. If there are no components of \( E \) along \( \hat{z} \), the mode is called transverse electric because there are only tangential components of electric field compared to the direction of propagation. Similarly, the transverse magnetic mode has no magnetic fields along \( \hat{z} \). These last two unknown fields, \( E_z \) and \( B_z \), must be determined using the wave equation.

For the TE mode, \( E_z = 0 \), so \( \nabla_t E_z = 0 \) everywhere and the transverse fields are determined by \( \nabla_t B_z \),

\[
E_t = \frac{i \omega (\hat{z} \times \nabla_t B_z)}{\mu \varepsilon \omega^2 - k_z^2} \tag{5.40}
\]

\[
B_t = \frac{i k_z \nabla_t B_z}{\mu \varepsilon \omega^2 - k_z^2} \tag{5.41}
\]

To solve the wave equation for \( B_z \), the boundary conditions on \( B_z \) at each boundary are required. Along the transverse boundaries of the waveguide, the normal component of the magnetic field goes to zero (5.33), along with the transverse component of the electric field (5.32). However, neither of these conditions directly apply to the magnetic field along \( \hat{z} \) at the boundaries since the magnetic field is along the conductive wall. Including the conditions (5.33) and (5.32) in (5.27) with no current source, an equivalent boundary condition suitable
for $B_z$ is
\[
\frac{\partial B_z}{\partial n} = 0 \tag{5.42}
\]
where $\hat{n}$ is normal to the transverse boundaries as appropriate. At the boundaries along $\hat{x}$, $\nabla_x B_z = 0$, and at the boundaries along $\hat{y}$, $\nabla_y B_z = 0$. With these boundary conditions, a solution for the wave equation for $B_z$ can be obtained.

Since the waveguide along $\hat{z}$ is infinite, the magnetic variation along this direction is assumed to be a simple traveling wave, $B_z = \Psi(x,y)e^{i(k_z z - \omega t)}$, where $\Psi(x,y)$ describes the variation along $\hat{x}$ and $\hat{y}$ and is required to satisfy the scalar wave equation,
\[
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right) \Psi = 0 \tag{5.43}
\]
where
\[
\gamma^2 = \mu \epsilon \omega^2 - k_z^2 \tag{5.44}
\]
already includes the results of the spatial derivatives along $\hat{z}$ and the derivatives in time.

The boundary condition (5.42) is satisfied at the transverse boundaries at $x = 0, L_x$ and $y = 0, L_y$ if
\[
\Psi_{lm} = B_0 \cos \left(\frac{l \pi x}{L_x}\right) \cos \left(\frac{m \pi y}{L_y}\right). \tag{5.45}
\]
Substituting $\Psi_{lm}$ back into wave equation and computing the remaining derivatives,
\[
\left(\frac{l^2 \pi^2}{L_x^2} + \frac{m^2 \pi^2}{L_y^2} - \gamma^2\right) \Psi_{lm} = 0 \tag{5.46}
\]
is always satisfied if
\[
\gamma_{lm}^2 = \pi^2 \left(\frac{l^2}{L_x^2} + \frac{m^2}{L_y^2}\right). \tag{5.47}
\]
Defining a cutoff frequency for mode $l, m$ as $\omega_{lm} = \gamma_{lm}/\sqrt{\mu \epsilon}$, the dispersion relation can be obtained by substituting (5.47) into (5.44),
\[
k_{lm} = \sqrt{\mu \epsilon} \sqrt{\omega^2 - \omega_{lm}^2} \tag{5.48}
\]
with a corresponding phase velocity,
\[
v_{\text{phase}} = \frac{\omega}{k_{lm}} = \left(\frac{\sqrt{\mu \epsilon} \sqrt{1 - \frac{\omega_{lm}^2}{\omega^2}}}{\omega}\right)^{-1}. \tag{5.49}
\]
If $\omega < \omega_{lm}$, the phase velocity becomes imaginary and the wave can not propagate. At the cutoff frequency $\omega = \omega_{lm}$, the phase velocity is infinite and propagation of the wave becomes possible. The infinite phase velocity along $\hat{z}$ at cutoff indicates that the observed field is due to waves propagating perpendicular to $\hat{z}$. These waves produce a standing wave pattern, with a constant field value along $\hat{z}$. The group velocity along $\hat{z}$ is obtained by differentiating (5.48)

$$\frac{\partial \omega}{\partial k_{lm}} = \frac{1}{\sqrt{\mu \epsilon}} \frac{\sqrt{\omega^2 - \omega_{lm}^2}}{2\omega}. \tag{5.50}$$

At cutoff, the group velocity is zero and no energy propagates along $\hat{z}$, reflecting the standing wave nature of the fields along $\hat{x}$, $\hat{y}$. As frequencies increase, waves travel increasingly along $\hat{z}$, and both the group and phase velocity approach the free space value. In a similar manner, the same dispersion relation is obtained for the scalar wave equation describing an acoustic waveguide.

The solution for an electromagnetic waveguide may be modified to describe a rectangular cavity by introducing conductive walls at $z = 0, L_z$. The boundary condition (5.33) requiring that $B_z$ go to zero at these walls is satisfied if

$$B_z = \Psi B_0 \sin \left(\frac{p\pi z}{L_z}\right) \tag{5.51}$$

for $p = 1, 2, 3 \cdots$. The boundary conditions along $\hat{x}$ and $\hat{y}$ haven’t changed, therefore $\Psi$ is still given by (5.45) and

$$B_z = B_0 \cos \left(\frac{l\pi x}{L_x}\right) \cos \left(\frac{m\pi y}{L_y}\right) \sin \left(\frac{p\pi}{d}\right) e^{-i\omega_{lm}pt} \tag{5.52}$$

with

$$\gamma_{lm}^2 = \mu \epsilon \omega_{lm}^2 - \left(\frac{p\pi}{L_z}\right)^2. \tag{5.53}$$

The wavenumber $k_z$ is restricted by the endcaps of the cavity along $\hat{z}$ to $k_z = p\pi/L_z$ and $\gamma_{lm}^2$ is given by (5.47). Using the equations for the transverse field components, (5.40) and
(5.41), the other field components are

\[
B_x = \frac{iB_0 \ell \pi}{\gamma_{lm}^2 L_x L_z} \sin \left( \frac{l \pi x}{L_x} \right) \cos \left( \frac{m \pi y}{L_y} \right) \cos \left( \frac{p \pi z}{L_z} \right) \tag{5.54}
\]

\[
B_y = \frac{iB_0 \gamma_m \pi}{\gamma_{lm}^2 L_y L_z} \cos \left( \frac{l \pi x}{L_x} \right) \sin \left( \frac{m \pi y}{L_y} \right) \cos \left( \frac{p \pi z}{L_z} \right) \tag{5.55}
\]

\[
E_x = \frac{-i \omega B_0 m \pi}{\gamma_{lm}^2 L_x} \cos \left( \frac{l \pi x}{L_x} \right) \sin \left( \frac{m \pi y}{L_y} \right) \sin \left( \frac{p \pi z}{L_z} \right) \tag{5.56}
\]

\[
E_y = \frac{i \omega B_0 \ell \pi}{\gamma_{lm}^2 L_x} \sin \left( \frac{l \pi x}{L_x} \right) \cos \left( \frac{m \pi y}{L_y} \right) \sin \left( \frac{p \pi z}{L_z} \right) \tag{5.57}
\]

Though not specifically listed, each term has a time dependence of \( e^{-i \omega_{lm} p t} \) where

\[
\omega_{lm} p = c \sqrt{\left( \frac{l \pi}{L_x} \right)^2 + \left( \frac{m \pi}{L_y} \right)^2 + \left( \frac{p \pi}{L_z} \right)^2}. \tag{5.58}
\]

Notice that without considering amplitudes, the magnetic induction field has the same wave-form as the velocity field of the closed acoustic drum (5.15-5.17), but the electric field in the TE mode has no acoustic analog. The ratio of the TE mode resonance frequencies to the acoustic rigid box (5.14) is

\[
\frac{\omega_e}{\omega_a} = \frac{c_e}{c_a} \approx 874, 636. \tag{5.59}
\]

Since both acoustic and electromagnetic fields have the same dispersion relation in waveguides and in free space, the wavelengths of these resonant modes are the same. For free space, \( \omega \lambda/(2\pi) = c \) and \( \lambda_e = \lambda_a \). Though only specifically shown for a rectangular cavity, the equivalence of resonant wavelengths for acoustic and electromagnetic cavities holds for arbitrary shapes. This general equivalence is a consequence of the wave equation description of both electromagnetic and acoustic fields and the similarity of the boundary conditions for both systems.

For TM modes, setting \( B_z = 0 \) in (5.37) and (5.38),

\[
E_t = \frac{i}{\mu \epsilon \omega^2 - k_z^2} \left( k_z \nabla_t E_z \right) \tag{5.60}
\]

\[
B_t = \frac{i}{\mu \epsilon \omega^2 - k_z^2} \left( \mu \epsilon \omega \hat{z} \times \nabla_t E_z \right) \tag{5.61}
\]
where \( E_z \) is free (5.34) at \( z = 0, L_z \), and goes to zero at the transverse boundaries (5.32). Using the same techniques for the TE case,

\[
E_z = E_0 \sin \left( \frac{l \pi x}{L_x} \right) \sin \left( \frac{m \pi y}{L_y} \right) \cos \left( \frac{p \pi z}{L_z} \right) e^{-i \omega_{lmn} t}.
\] (5.62)

and the transverse components of the TM modes are

\[
E_x = \frac{i E_0}{\gamma_{lm}^2} \frac{l \pi}{L_x} \cos \left( \frac{l \pi x}{L_x} \right) \sin \left( \frac{m \pi y}{L_y} \right) \sin \left( \frac{p \pi z}{L_z} \right) e^{-i \omega_{lmn} t} \]
\] (5.63)

\[
E_y = \frac{i E_0}{\gamma_{lm}^2} \frac{m \pi}{L_y} \sin \left( \frac{l \pi x}{L_x} \right) \cos \left( \frac{m \pi y}{L_y} \right) \sin \left( \frac{p \pi z}{L_z} \right) e^{-i \omega_{lmn} t} \]
\] (5.64)

\[
B_x = \frac{-i \omega \mu \epsilon E_0}{\gamma_{lm}^2} \frac{m \pi}{L_y} \sin \left( \frac{l \pi x}{L_x} \right) \cos \left( \frac{m \pi y}{L_y} \right) \cos \left( \frac{p \pi z}{L_z} \right) e^{-i \omega_{lmn} t} \]
\] (5.65)

\[
B_y = \frac{i \omega \mu \epsilon E_0}{\gamma_{lm}^2} \frac{l \pi}{L_x} \cos \left( \frac{l \pi x}{L_x} \right) \sin \left( \frac{m \pi y}{L_y} \right) \cos \left( \frac{p \pi z}{L_z} \right) e^{-i \omega_{lmn} t} \]
\] (5.66)

with a time dependence that oscillates at

\[
\omega_{lmn} = c_a \sqrt{\left( \frac{l \pi}{L_x} \right)^2 + \left( \frac{m \pi}{L_y} \right)^2 + \left( \frac{n \pi}{L_z} \right)^2}.
\] (5.67)

Apart from amplitude, the electric field has the same waveform as velocity in a pressure release drum (5.19-5.21) and resonates at the same wavelengths. The magnetic field in the TM mode has no acoustic analog. Since TE modes correspond to rigid acoustic cavities and TM modes to soft acoustic cavities, the general response of an electromagnetic cavity is essentially the combination of two acoustic responses.

Acoustic and electromagnetic resonant cavities are not exactly the same since electromagnetic cavities have additional fields not found in acoustics and the amplitude of the shared waveforms are not the same for both responses. However, the resonant wavelengths of an electromagnetic cavity are given by the combination of two acoustic resonators, giving the systems generally similar behavior. Since electromagnetic cavities support more resonance modes for a given cavity shape than acoustic fields, electromagnetic instruments are capable of producing a denser sound than acoustic instruments.
5.3 Helmholtz Resonator in Musical Instruments

A simple acoustic Helmholtz resonator is a wine bottle. The air in the neck of the bottle acts like a mass and the body acts like a spring. If one tries to push air from the neck into the bottle, the bottle’s pressure increases and pushes the air back out of the bottle. If one tries to remove air from the neck, the pressure in the bottle decreases and pulls the air back in. An ideal Helmholtz resonator produces a single low frequency resonance, creating a sound with a wavelength that is much larger than the bottle itself.

In addition to describing Helmholtz resonance through the use of mechanical analogs such as springs and masses, an electrical description is also possible; the instrument body is a capacitor and the sound hole an inductor. This standpoint is prescriptive; one can describe the behavior of any ideal Helmholtz resonator using circuits. Firth [18] and Schelleng [74] used electrical circuits to model low order modes of a guitar and a violin respectively.

The impedances accounting for the aperture and cavity are valid when waves are much larger than the cavity. Given the relatively small size of the cavity, the wave nature of oscillations between the aperture and cavity are not apparent and the resonance between these elements is not a wave resonance. The impedances presented are called lumped-elements [3] in order to differentiate them from higher frequency wave impedances. Ideally, the pressure increase throughout the cavity is uniform, demonstrating the lack of wave character. A numerical model of an acoustic guitar [17] in Fig. 5.1 indicates some variation in pressure through the guitar body at its breathing mode \((A_0)\). The aperture is approximated as a pressure release surface, so there is no deviation in pressure in the aperture. The pressure immediately around the aperture is close to this null pressure and increases with distance from the sound hole. Apart from the null pressure in the aperture, though there are no null locations in the response of the guitar body. The net pressure within the instrument changes at this resonance, so air must be entering/exiting the instrument through the sound hole.

The breathing mode is associated with the Helmholtz resonance. At the Helmholtz resonance frequency, the null imaginary impedance leads to a minimum in radiated power.
However, by transitioning through a resonance, there will likely be a match to the source impedance near the resonance frequency, leading to a maximum in radiated power.

The difference in pressure for the $A_0$ mode in Fig. 5.1 is a minimum near the aperture and increases with distance away from the sound hole. Equating pressure deviations with differences in potential with respect to the aperture, a similar distribution is expected for an electromagnetic guitar with an aperture, where all conductors of the EM guitar are electrically connected. Near the aperture, the guitar is expected to have a potential similar to the aperture due to proximity, correlated with the minimum difference in pressure in the acoustic guitar. Away from the aperture, the full potential difference between the aperture and guitar should be observed, correlated with the increased pressure difference in the acoustic guitar.

The inductance $L_{ac}$ of an aperture for acoustic waves [3] much larger than the aperture is

$$L_{ac} = \frac{\rho_0 d'}{S} \quad (5.68)$$

where $\rho_0$ is the mass density of gas, $d'$ is the effective thickness of the aperture, and $S$ is the aperture area. The effective thickness of an acoustic aperture includes the actual thickness of the aperture, $d$, plus an end correction to account for the sound waves radiated out of the aperture, $\delta d$, where $d' = d + \delta d$. The end correction for an open tube is $\delta d = 0.85a$, reducing to $\delta d = \pi a/4$ if the tube is surrounded by an infinite plane [3]. The capacitance of an acoustic cavity with volume $V$ [3] is

$$C_{ac} = \frac{\rho_0 c_a^2}{V} \quad (5.69)$$

If the cavity is driven by an external wave, then the impedance of the aperture and cavity are added in series [3]. The resonance frequency of an $LC$ circuit is

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \quad (5.70)$$

therefore the acoustic Helmholtz resonance frequency is

$$f_{a0} = \frac{c_a}{2\pi} \sqrt{\frac{S}{d'V}} \quad (5.71)$$
Figure 4. Vibration modes and natural frequencies of the air cavity of the guitar. The phase of the pressure variations in 0 or 180° due to the absence of viscosity in the air, and so only the vibration amplitudes are represented.

\[ \frac{2}{7} \cdot \frac{59}{10}/ \frac{18}{92/p15} \text{ m and } \frac{343}{15} \text{ m/s) and the definitive mesh built up as shown in Figure 1.} \]

Setting the sound pressure to zero on the top of the neck correction. This having been done, the numerical analysis provided the same vibration pattern and frequency as those shown in Figure 3 for the "1st cavity mode."

3. RESULTS AND DISCUSSION

The calculated vibration patterns and natural frequencies corresponding to the "1st six modes are shown in Figure 4. The modes are categorized as in reference [1] and are of normal character since no viscous effects were present. The blue colour indicates the nodal lines: i.e., the zero pressure zones. Different colours represent the pressure amplitude without differentiating between positive and negative values.

Figure 4 shows that the pressure patterns are not distorted around the sound hole except A0; besides, it is clear that A0 is not a pure Helmholtz mode because the sound pressure in the cavity does not feature a constant distribution. Moreover, the sound pressure in the zone of the sound hole varies perpendicularly. Although the results show an in-phase vibrational behaviour, they also reveal a different value for the sound pressure between the upper and lower part of the cavity. The natural frequency obtained is higher than the experimental values measured by other authors [1, 3, 5], see Table 1, although it should be noted that these measurements correspond to real, not completely rigid, structures; the similitude of the upper frequencies indicates that the guitar box dimensions are very much.
Figure 5.2. Admittance of an acoustic guitar with aperture (sound hole/tone hole) open (top) and closed (bottom). Reprinted with permission from Firth. Physics of the guitar at the Helmholtz and first top-plate resonances. The Journal of the Acoustical Society of America, vol. 61 (2) pp. 588-593 Copyright 1977, American Institute of Physics.

Instead, if the source is driving the cavity directly, like the excitation provided by a string on a guitar, then measurements of an acoustic guitar indicate a parallel distribution of impedance [18].

The admittance of an acoustic guitar with and without an aperture is in Fig. 5.2. With an aperture, the instrument has a series resonance at 89.6 and 178.6 Hz and a parallel resonance near 120 Hz. Without the aperture, the acoustic guitar only has a series resonance at 163 Hz. The interaction between the cavity and aperture alters the low frequency behavior of the instrument, giving it enhanced bass response. Based upon these measurements, an effective circuit description of the instrument is obtained in [18] and is reproduced in Fig. 5.3. The resonances of the plates comprising the acoustic guitar are represented by series RLC circuits, while the interaction between the aperture and capacitor is in parallel.

The proposed electromagnetic Helmholtz resonator is constructed from a parallel plate capacitor with an aperture. The capacitance of an ideal parallel plate capacitor is

\[ C_{cm} = \frac{A \epsilon_0}{L_z} \]  

(5.72)
Figure 5.3. Effective circuit based upon measurements of an acoustic guitar with and without an aperture. Reprinted with permission from Firth. Physics of the guitar at the Helmholtz and first top-plate resonances. The Journal of the Acoustical Society of America, vol. 61 (2) pp. 588-593. Copyright 1977, American Institute of Physics.

for plates of area $A$ and separation $L_z$, where $A >> L_z$. Using the low frequency aperture inductance from (3.49) and the parallel plate capacitance (5.72) in (5.70), the estimated electromagnetic Helmholtz resonance for a series combination of impedances is

$$f_{e0} = \frac{c_e}{2\pi} \sqrt{\frac{L_z S}{A\alpha_m}}. \quad (5.73)$$

This result ignores the capacitive aperture impedance because it is not significant well below aperture resonance. Instead, if the elements are in parallel the resonance frequency will be reduced

$$f^2 = \frac{f_{0}^2}{2} - \frac{R^2}{4\pi^2 L^2} \quad (5.74)$$

where $R$ is the radiation resistance for the magnetic aperture dipole.

Assuming that the electromagnetic and acoustic resonators are built using cavities and apertures with the same dimensions, then the ratio of the two resonance frequencies is

$$\frac{f_{e0}}{f_{a0}} = \frac{c_e}{c_a} \sqrt{\frac{L^2 d'}{\alpha_m}} \quad (5.75)$$

where $V = AL_z$ has been used. Using the free space wave dispersion, $\lambda f = c$, the ratio of radiated wavelengths is

$$\frac{\lambda_{a0}}{\lambda_{e0}} = \sqrt{\frac{L^2 d'}{\alpha_m}}. \quad (5.76)$$
Using a typical circular aperture radius for an acoustic guitar, \( a = .05 \ m \), and the minimum effective aperture thickness \( d' = \pi a/4 \), then the acoustic and electromagnetic wavelengths are equal when

\[
L_z = a \sqrt{\frac{32}{3\pi}} = .092 \ m.
\]  

(5.77)

Typical acoustic instruments have an \( L_z \approx .1 \ m \), or about 4 inches. The aperture can also display characteristics of an increased thickness due to the space within the instrument, as well as the space outside the instrument. The end correction has a maximum effective thickness of approximately \(.85a\) for each region. Including both sides of the aperture, \( d' \approx 1.7a \). Inserted into (5.76), equivalence in radiated wavelength will occur at \( L_z = .078 \ m \approx 3 \) in. This range in thickness is consistent with acoustic guitar dimensions. Thus, it may be possible for electromagnetic and acoustic musical instruments with the same dimensions to have a Helmholtz resonance at the same wavelength for typical musical instrument sizes.

Experiments performed to confirm to postulated electromagnetic Helmholtz resonance did not detect a resonant interaction between the two. However, measurements of an electromagnetic guitar with an aperture did confirm the presence of the aperture real impedance in the system in §4.5. The imaginary aperture impedance was not detected due to the presence of induced charges in the same instrument face as the aperture. Since the results confirm that the derived aperture impedance in §3.2 can be used to predict the coupling between a capacitor with an aperture and it is already known that the electromagnetic Helmholtz resonance exists \([21, 9, 71, 62, 20, 19, 40]\), a suitable resonance for electromagnetic musical instruments is expected. Based upon measurements, two configurations expected to demonstrate the EM Helmholtz resonance have been discussed in §4.5.1. Though it is not known for certain that the aperture couples with a capacitance described by an ideal parallel plate capacitor, the resonance frequency should be calculable using circuit elements.
5.4 Electromagnetic Musical Instrument Construction

Since electromagnetic and acoustic waveguides resonate at the same wavelengths for a given shape and the acoustic and electromagnetic Helmholtz resonance occur at the same wavelengths for typical acoustic instrument sizes, the size and shape of an electromagnetic musical instrument can be copied from acoustic instruments. While acoustic instruments are generally constructed with wood, electromagnetic instruments\(^1\) should use metals and dielectrics.

The capacitance used to determine the electromagnetic Helmholtz resonance in (5.73) is from an ideal parallel plate capacitor, therefore the EM instrument itself should also form a low frequency capacitor. A simple way to achieve this is to electrically isolate at least two sections of the instrument body. In the prototype electromagnetic guitar in Figs. 5.4(a) and (b), the front face of the instrument is electrically disconnected from the back face. The faces of the instrument are made of pure aluminum and the sidewall is a stainless steel foil glued to a mylar sheet for additional rigidity. The sidewall is electrically connected to the back face of the instrument using copper tape with conductive adhesive. Constructed with assistance from Keith Swaim\(^2\).

An external acrylic frame provides physical support for both guitar faces. Acrylic rods provide lateral support, while nylon threaded rods maintain the spacing between the plates. The shape of the sidewall is enforced by rectangular acrylic strips glued to the acrylic frame, manually bent into shape using a heat gun and an instrument plate as a form. Since the guitar faces are attached to one side of an acrylic plate, while the cavity formed by the steel foil sidewall is on the other side, copper foil covers the acrylic in these areas to prevent leakage of the fields within, Fig. 5.4(c). The screws pictured are not used in the final instrument, using copper tape to connect a new sidewall to the instrument, Fig. 5.4(d). To maintain the electrical disconnect between one plate and the other, non-conductive tape (red tape, Fig. 5.4(d)) separates the sidewall from one guitar plate inside the cavity.

---

\(^1\)U.S. Patent Pending 20070017345
\(^2\)Machinist, W.B. Hanson Center for Space Sciences
Though significant effort went into forming the exact same shape for the sidewall on both acrylic plates, there remained deviations. When this sidewall is taped to the frame, these deviations cause deformations in the thin sidewall. Though these deviations will likely affect higher order modes, comparison of measured low frequency behavior of the guitar to a parallel plate capacitor with no sidewall indicates that the guitar sidewall does not significantly affect measured behavior.

The electromagnetic instrument is excited using a high frequency electrical source. To couple the source to the instrument, an SMA connector was attached to a guitar face. The SMA standard is a compact coaxial cable connector suitable for frequencies in the GHz range. Since aluminum is not easily soldered to, a conductive epoxy\(^3\) was used to attach the center conductor of the SMA connector to the instrument face, leaving the ground free. This ground is located very close to the capacitor plate, leading to a possible interaction between the two. A variation was also investigated where the ground of the coaxial cable is applied to one plate and the center conductor is applied to the other plate, requiring a coaxial cable in the instrument body, Fig. 5.5. This excitation mechanism is similar to the excitation of capacitors in low frequency circuits, although the cable inside the instrument will alter the fields. The length of this internal cable must be accounted for when interpreting measurements.

5.5 Obtaining the Sound of an Electromagnetic Musical Instrument

Since electromagnetic fields can not be heard directly, the music produced by an electromagnetic musical instrument must be converted from an electromagnetic wave to an acoustic wave. Though electromagnetic and acoustic instruments resonate at the same wavelengths, these resonances occur at very different frequencies. Since the electromagnetic frequencies are much larger than the acoustic frequencies, the time scales on which the EM instruments operate are much shorter. A note played on an acoustic guitar that lasts a couple seconds

\(^3\)MGChemicals Silver Conductive Epoxy 8331
Figure 5.4. Prototype electromagnetic guitar shaped like an acoustic guitar.
Figure 5.5. Ground of coaxial cable is connected to the front plate, while the center coaxial conductor is terminated in the back plate.

is expected to last for only a couple microseconds ($10^{-6}s$) in the EM guitar. Since these time scales are too short for human interaction, EM instruments can only be played using an intermediate interface that bridges the gap between the two time scales$^4$.

The simplest method for playing an EM instrument overcomes this time difference by only considering digitally recorded sounds. The digital samples on audio CDs are traditionally converted to an analog waveform by a Digital to Analog Converter (DAC) operating at 44.1 KHz. If a DAC is instead operated at $(c_e/c_a)44.1kHz$, then the resultant waveform would be identical to the music it normally produces, but at frequencies a factor $c_e/c_a$ larger than audio frequencies. If this higher frequency music is used to excite an EM instrument via a coaxial cable, then the behavior of the instrument modifies this signal much like an acoustic instrument would modify the same waveform at acoustic frequencies. The behavior of the EM instrument is measured, digitized by an Analog to Digital Converter (ADC), and converted back down to audio frequencies for listening by using a DAC at standard CD

---

$^4$U.S. Patent Pending 20070017344
audio rates. However, to cover the entire range of audible sounds (20 – 20,000 Hz), a DAC that can produce electrical signals between 20 MHz – 20 GHz (40 GSamples/s) with 16 bit resolution is needed, with similar requirements for an ADC. Though speeds are steadily increasing, this equipment is not presently available.

Alternatively, instead of increasing the frequencies of the music exciting the instrument, one can try to reduce the frequencies of the EM instrument. Since the operational frequency range of an instrument is determined by size, a reduction of the actual frequencies of an EM instrument down to audio frequencies requires an instrument about one million times larger than standard acoustic instruments. Since these instruments would be impossibly large, this frequency conversion is done computationally.

A computer audio method that is suitable for this purpose utilizes the convolution of digital music with an impulse response. This method is commonly used to make a piece of music sound like it was recorded in a different acoustic space than the one actually used. For example, to make music recorded in a studio sound like it was played in Carnegie Hall, the impulse response of Carnegie Hall is convolved with the recorded music.

The impulse response of a system is simply the response of that system to a very brief input. For acoustics, the impulse response of a room can be measured by recording the sound produced in a room due to a loud brief stimulus, such as a gunshot. As the sound waves travel outward from the gun, these waves interact with all areas of the room, producing a characteristic response [36] at the measurement location. The first sounds received correspond to waves traveling directly from the gun, followed by waves with only a few reflections, generally representing major features of the overall architecture and nearby objects. These first few reflections give the listener an estimate on the size of the space. After many reflections, signals arrive continuously, close enough in time that they are generally perceived as a continuous decaying signal. This behavior is known as reverberation (reverb), defined as the persistence of a signal that continues after the original sound has ceased [36]. The impulse response of a space directly measures the room’s reverb.
Replacing the gun with a musical instrument, the music heard will involve the sounds produced immediately by the instrument, plus the sounds made by the instrument still in the room due to reverb. The instant a sound is produced by an instrument, it’s received intensity will follow the previously measured impulse response for the gun. Since a musical instrument also has reverb apart from the room’s reverb response, sound continues in the instrument, and these new sounds also follow the same impulse response. As the direct waves for the new sounds reach the listener, sounds emitted just prior are reaching the listener again after a couple reflections. The total sound heard at that instant is the combination of the two. As this process continues, sounds produced by the instrument over a reverb time will be in all different stages of the impulse response (reverb) and the total sound is the sum of these responses.

Using a discrete audio representation, the sound heard at any time $t_m = m\delta t$ is given by the discrete convolution of recorded music, $f(m)$, with an impulse response, $g(m)$,

$$ (f * g)(t_m) = \sum_{p=0}^{M-1} f(m - p) \cdot g(p) $$

(5.78)

where $M$ is the number of samples corresponding to reverb time and $f(m) = 0$ for $m < 0$. Though the impulse response was introduced using an acoustic space, an impulse response can reproduce the behavior of any linear, time invariant system. Measurements of an electromagnetic guitar are repeatable and consistent, therefore the impulse response can be used to describe this system.

To recreate the effect of an electromagnetic musical instrument at acoustic frequencies, the impulse response of the instrument can be measured and digitized. The time stamps associated with these samples can simply be relabeled to reduce the constituent frequencies. Any recorded music convolved with the impulse response of the electromagnetic musical instrument will sound as if it was actually played by that instrument. If the impulse response is measured with precision, then this method exactly reproduces the function of the musical instrument body.
If the impulse response is measured on the instrument itself, then the corresponding impulse contains the total behavior of the musical instrument. However, in most cases, listeners are located away from the musical instrument. Since musical instruments have a complicated radiation pattern that varies with frequency, the sound a given listener hears will be different than the total instrument response. This effect can be recreated simply by measuring the impulse response of the waves radiated by the instrument, rather than the instrument itself. These measurements would require antennas with a very large bandwidth or an array of antennas with a combined bandwidth that covers the range of interest.

An alternative to measuring the impulse response directly involves measuring the frequency response of the system. Using the Fourier transform, the frequency response of an ideal impulse has constant magnitude at all frequencies. Correspondingly, the inverse Fourier transform converts a constant magnitude signal to an impulse response. Therefore, the frequency response of a system excited by a signal with constant magnitude over the range of interest can be converted to the system’s impulse response using the inverse Fourier transform. Readily available network analyzers are designed to measure the frequency response of high frequency electrical systems and some results have already been presented in chapter 4. The obtained impulse responses can be used in a number of audio programs.

5.5.1 Real Time Digital Methods

Despite the difference in time scales between acoustic and electromagnetic instruments, it is possible to obtain the sound of an electromagnetic musical instrument due to music played in real time by a musician\textsuperscript{5}. If the input music is digitized with an Analog to Digital Converter (ADC) at sample rate \( f_a \), then the high frequency DAC creating the signal for the electromagnetic instrument must operate at \( f_a c_e/c_a \). This disparity in sample rate means that the high frequency DAC must wait for approximately 874,636 cycles of its clock before the next audio sample arrives. One second of a musician's time corresponds to approximately

\textsuperscript{5}U.S. Patent Pending 20070214940
10 days of music that could be played in the EM instrument. If a single musicians input was used to directly create a high frequency signal, the produced waveform would have short spikes followed by long periods of silence and would bear no relation to the input audio.

The method to achieve this real time interaction between instrument and musician is initially introduced generally to motivate the structure of the process. For clarity, the specific steps are then covered in detail. The first step to overcoming the significant time difference utilizes input buffers that collect a number of samples into a short segment (excitation signal) and these segments are used to excite the EM instrument in short bursts. Measurements of the system will still have long regions where the instrument does nothing and these null samples must be removed to create an illusion of continuity. The resultant short snippets of audio are joined back together and presented as continuous output. However, since musical instruments have reverb, notes played in the past affect the present sound of the instrument, if those notes were played within the reverb time of the instrument. Therefore, the short signals used to excite the instrument must include at least a reverb time of past audio to place the instrument (and the room it’s in) in the correct physical state before new audio can be introduced. The response of the instrument to the past audio must also be removed from the measured response because it would not be present if the instrument were excited continuously.

The reverb response of an EM instrument also means that the instrument will continue to radiate energy after the input signal has ceased. In general, the end of a given excitation signal will not be the end of the musician’s input, the very next sample measured by the ADC at audio frequencies will be more music. If that additional sample was included, the instrument’s response would be different. Therefore, the measured response of the instrument after input ceases will be incorrect from a standpoint of continuity and must be removed.

In general, the measured response of the instrument will have a delay from the time the instrument was excited. This time ($t_d$) can be obtained by measuring the time between exciting the instrument with an impulse and measuring the response to that input. To
maintain maximum efficiency, this time delay can be used to estimate which samples, after input ceases, correspond to behavior of the instrument while it was being excited, and which are actually reverb response.

Once the measured response of the instrument has both the past audio and reverb response after input ceases removed (treated), the response can be connected to the previously measured and treated signals. Since both signals have been extracted from complicated waveforms produced by two separate excitations, these signals may not match at the boundaries. These repeated discontinuities will cause artifacts in output music and need to be avoided. Since there are still many samples available to the EM system from the musician’s perspective, the input buffers can hold even more past input than the reverb time. This additional past behavior is called common response, as it enables multiple measurements of the instrument to the same input signal. The inclusion of common response is important since a newly measured instrument response to signal $\ell + 1$ contains the previously measured response to signal $\ell$, in addition to new behavior.

The newly measured music can be added to the previously measured samples by using a weighted sum of the two signals over the region of common response. The oldest sample in this region is determined by measured response $\ell$, while the most recent is purely determined by measured response $\ell + 1$. In this manner, any differences in these two signals are spread across multiple samples, limiting discontinuities in output music. Once two signals have been joined, the oldest samples can be output for listening.

Fig. 5.6 is a flow chart graphically listing this process. Input music from a musician starts at the top left, and as samples are measured by the ADC at audio frequencies, the samples are added to the audio queues $X, X + 1, X + 2$, moving from left to right. These audio queues are used to create a high frequency signal suitable for exciting the instrument, represented by queues $Y, Y + 1, Y + 2$. The segment $Y$ sets up past behavior of the instrument, segment $Y + 1$ is the common response region, and segment $Y + 2$ is new music.
Figure 5.6. Flow chart for the real time processing of live music using an electromagnetic musical instrument.
The response of the instrument to a particular input sample is measured after a time delay of $t_d$. Using this knowledge, the measured response of the instrument is divided into segments based upon the input signal it corresponds to. The beginning and end of each input segment is determined in the measured output. The response of the instrument to past behavior and the response after input ceases are discarded. The remaining segments $Y + 1$ and $Y + 2$, the common response and new audio respectively, are aligned with the samples in the output buffer $Z$. The last sample of the common response corresponds to the last sample in the output buffer. In this region, both the output buffer and the measured $Y + 1$ segment contain the response of the instrument to the same input signal. Ideally, these two signals are the same, although differences are expected. A normalized and weighted sum of these signals is used to connect the new audio samples to the output buffer $Z$. The sum of the common response and output buffer $Z$ is denoted $Z'$. Once the new signal is appended to the output buffer, $L_3$ of the oldest samples are removed from the output buffer and presented for audio output. When these $L_3$ have been played, $L_3$ more samples have been input to the system and the cycle repeats. Through this repetitive process, an output signal can be generated that maintains an illusion of continuity for the EM instrument’s response.

The effort of maintaining this illusion of continuity will be worthless if the time delay between input and output is too long. If this happens, the musician will hear a time difference between playing a note and hearing that note, which should be avoided. The time delay of this process is determined by the length of the input buffer $X + 2$, the length of the output buffer $Z$, and the time taken to measure and process the EM instrument audio. Since the time scale of the EM instrument is so fast compared to audio, the measurement of the electromagnetic system takes less time than an audio ADC takes to measure one more sample, and is ignored. Choosing the output buffer to be as long as input buffer $X + 2$, the total delay between presenting an input sample and receiving the output is $2L_3\delta t$.

A total time lag of less than 30 ms is generally not perceived by the ear [36, pg. 51]. For CD audio at a sampling rate of 44.1 KHz, the time between samples is $\delta t \approx 2.26 \times 10^{-5}$s.
If $L_3 = 256$, then the total audio delay is 512 samples (12 ms), a common buffer size in digital audio. This choice in length adds 256 new samples of audio every iteration and has 256 samples of common response in which the new samples can be added to the output buffer. The common response should be no longer than the output buffer, since these extra elements can’t be used to join the newly measured signal to the output. If 512 samples are insufficient, then buffer sizes of 1024 samples can also be used. Additional buffer lengths will extend the delay past 30 ms and should be avoided.

Although a very accurate determination of $t_d$ is desired, it is not strictly necessary. Consider the measured response of an electromagnetic musical instrument to three different input audio segments in Fig. 5.7. The segments $A_1$ (red), $A_2$ (green), $A_3$ (purple) correspond to measurements of the instrument for input segments $Y_1$, $Y_2$ for three successive loops of the presented digital audio method. The final output audio is also illustrated, though it only covers half of $A_1$ and $A_3$ because the output outside this domain is also determined by signals $A_0$ and $A_5$, not shown. However, the output audio for these segments is obtained in the same cyclic process illustrated. Each rectangle is comprised of 512 subrectangles, therefore the gradient in color in the output audio is representative of the rate in transition from one signal to the other.

The output signal is obtained using a linear weighting of signals ($L_3 = 256$),

$$A_{out}(t_0 + m - 1) = A_1(L_3 - 1 + m) \left(1 - \frac{m - 1}{L_3 - 1}\right) + A_2(m)\frac{m - 1}{L_3 - 1} \quad (5.79)$$

for the time where $A_1$, $A_2$ overlap. The values of the elements in each array ($A_1$, $A_2$) are simply scaled by a linear function that increases/decreases from one to zero, or zero to one as time increases. This transitions from measurements of one excitation of the instrument to measurements of a separate excitation of the instrument. The colors of the rainbow were mapped to the range $[0, 1]$ in Mathematica and each signal was assigned a numerical value, $A_1 = 1$, $A_2 = 0.5$, $A_3 = 0$. Note that the sound at the endpoints of the output are determined from the midpoints of signals $A_1$, $A_3$. The endpoints of signals $A_1$, $A_3$ that occur at the midpoint of signal $A_2$ are also not represented in the output. Therefore, the endpoints of
the measured instrument responses never reach the listener and the system may be tolerant to an uncertain determination of $t_d$.

Figure 5.7. Demonstration of the influence of separate measurements of the electromagnetic instrument response in the final output audio.

Figure 5.8. Influence of $m_0$ on output audio. The time scale is the same for the output audio in Fig. 5.7.

Additional tolerance to uncertainty in $t_d$ can be built into the process by using a past audio time greater than reverb. This ensures that measured samples associated with common response do not include a few samples from past audio that are still trying to initialize the instrument and the room the instrument is in. If more than reverb time of past audio is used and if some of this additional past audio is incorrectly labeled as common response, the instrument is already in an accurate state regarding continuity and no harm is done. This will slightly increase the delay between input and output, but will otherwise produce a correct and continuous signal.

An uncertain $t_d$ also leads to uncertainty of measured behavior after input ceases. Though some measurements after input ceases can be used to account for the propagation delay, this procedure increases the risk of incorrectly labeling the reverb response of the
instrument as new audio. Since the reverb response of the EM instrument generally violates continuity from the musician’s perspective, it should be avoided.

The uncertainty of these endpoints can be further removed if samples near it are also not used to create the output signal. To accomplish this, the weighting function transitioning from one measured response to another is modified to only operate on samples at least $m_0$ samples away from the endpoints,

$$A_{\text{out}}(t_0 + m - 1) = A_1(L_3 - 1 + m), \quad 1 \leq m \leq m_0$$

$$A_{\text{out}}(t_0 + m - 1) = A_1(L_3 - 1 + m) f(L_3 - 1 + m) + A_2(m) g(m), \quad m_0 < m \leq L_3 - 1 - m_0$$

$$A_{\text{out}}(t_0 + m - 1) = A_2(m), \quad L_3 - 1 - m_0 < m \leq L_3$$

where

$$f(m) = \left(1 - \frac{m - m_0}{L_3 - 1 - 2m_0}\right)$$

$$g(m) = \frac{m - m_0}{L_3 - 1 - 2m_0}.$$  

Figure 5.8 illustrates the behavior of the weighting for signals $A_1 - A_3$ with a variation in $m_0$. The case $m_0 = 0$ is the output audio signal in Fig. 5.7 and all output audio is constantly transitioning from one measured signal to another. As $m_0$ increases, the fraction of output music that is the result of a weighted sum decreases and the transition from one signal to the other occurs further away from any endpoints, limiting their influence. At $m_0 = 120$, there is a sharp transition from one response to the other at the midpoint of the common response. However, this reduces the number of samples available to transition from one measurement to another, $L_3 - 2m_0$. Since $m_0$ is easily varied, it may be useful to allow a musician to vary this quantity. If repeated measurements of the instrument produced exactly the same result, variations in $m_0$ would have no effect on the instrument sound.

If, in spite of this effort, there are still unacceptable levels of variation between successive excitations of the EM instrument, then the averaged response of the instrument can be used. Since several days of audio can be processed every second in an EM instrument,
the exact same input signal can be used to repeatedly excite the instrument, and the many measured waveforms can be averaged together. This should increase the chances of a successful join between new audio samples and output, although the averaging procedure could alter the instrument’s sound.

This method works with any input music, therefore the sound of an EM instrument body can be used to modify any musical instrument. For acoustic instruments, the sound produced can be measured with a microphone and fed into the EM instrument system. For electric musical instruments, the electromagnetic musical instrument offers a resonant acoustic-like instrument sound.

5.6 Comparison of Acoustic and Electromagnetic Guitars

The peaks in real impedance for the electromagnetic guitar with no aperture scaled down to the equivalent acoustic frequencies are at 180, 220, 335, 440, 490, 630, 780, 815 and 860 Hz in Fig. 5.9. The actual frequencies of these resonances are obtained by multiplying by the factor of electromagnetic and acoustic wave speeds, $c_e/c_a$. The plate modes obtained for a finite element guitar model that imposes simply supported boundary conditions at the edge are in Fig. 5.10. A different numerical model of the air cavity modes of a rigid guitar [16] finds modes at 155, 418, 545, 718 and 771 Hz. The lowest frequency resonance in [18] is at 180 Hz. The peaks obtained for the electromagnetic guitar are generally consistent with observations of acoustic guitars.

5.6.1 Example Sounds using the Electromagnetic Guitar

Measurements of the prototype Electromagnetic Guitar using a network analyzer are converted to impulse responses at audio frequencies using the method outlined in §5.5. The impedance of the instrument itself can be used to generate this response directly, however, for acoustic instruments, the sound heard is due to the combination of strings and instrument. The difference in impedance between the string and instrument determines how much
Figure 5.9. Real impedance of copper guitar with no aperture. The actual frequency range of operation is scaled down to the equivalent acoustic frequency by matching the free space wavelengths of acoustic and electromagnetic waves.

Figure 5.10. Resonant modes of an acoustic guitar obtain with FEM. Guitar plates are assumed to be simply supported to the guitar sidewall. Reprinted with permission from Elejabarrieta et al. Coupled modes of the resonance box of the guitar. The Journal of the Acoustical Society of America, vol. 111 (5) pp. 2283-2292. Copyright 2002, American Institute of Physics.
of the power in the string’s oscillations is coupled into the instrument, and as shown in §4.1, the same impedance relationships are found for electromagnetic systems. Thus, this effect is included when calculating the EM instrument’s impulse response. Measurements of the power radiated by the instrument have not been made, so the power transmitted into the instrument is used as a proxy. However, since power can travel through the instrument and back towards the source via the ground plate, the power transmitted into the system is not the same as the power radiated by the instrument. Thus, the sounds obtained should be considered approximations and additional experiments are hoped for.

The power transmitted into the guitar is given by

\[
P = \text{Re} \left[ \frac{4Z_L Z_0^*}{|Z_0 + Z_L|^2} \right]
\]  

(5.85)

where \(Z_0\) is the coaxial cable impedance and \(Z_L\) is the guitar impedance. The power transmitted into the copper guitar with no aperture for various coaxial cable impedances is in Fig. 5.11. There is generally little power transmitted into the instrument, though for a source cable with impedance \(Z_0 = 1500 \, \Omega\) has a peak near 300 Hz that extends past 500 Hz. The power above 1 KHz is not shown for clarity.

The measurements of the guitar stop at 6 GHz, which corresponds to the same wavelength as an acoustic wave near 7 KHz. The lack of measurements above this frequency leads to artifacts in the resultant music. Thus, a low pass filter is applied to reduce audible frequencies above 1 KHz. The filter must be applied below the end of measurements at 6 KHz so that output can gradually decrease towards zero. A third order Butterworth filter is used, with magnitude

\[
H = \frac{1}{|(s + 1)(s^2 + s + 1)|^2}
\]  

(5.86)

where \(s = j\omega/\omega_c\), and \(\omega_c\) is the cut-off frequency. At cutoff, the low pass filter reduces output by \(-3\) dB, or roughly half the output volume. The low pass filter is applied to the transmitted voltage and current separately. The product of these terms when calculating the transmitted power leads to a reduction in power given by the magnitude of the low pass filter.
A guitar recording (original.aif) in the root directory of the included audio files is convolved with different impulse responses for the prototype guitar with a copper plate, assuming a source impedance of 1500 Ω. The results are in the directory Waveguide 1.0. The files are labeled based upon the number of samples used from the total impulse response calculated. The audio file 100.aif is the result of convolution of original.aif with the first 100 samples of the calculated impulse response. Consistent with the low pass filter applied, the high frequencies of the guitar are reduced. The bass and mid range frequencies of the guitar become more apparent and are generally pleasant. Increasing the number of samples of the derived impulse response to 500 (500.aif), the sound becomes a bit tinny. The increased transmitted power at 800 Hz and possibly resonances above 1 KHz not shown are amplifying an aspect of the original recording that isn’t particularly pleasant. Results are presented in stages, with a maximum impulse length of 20,000 samples, and an instrument reverb response that approaches 1/2 second. Though the resonance at 800 Hz still produces a tinny sound, the reverb response of the instrument heard in the background is harmonious, most audible in 20000.aif. Thus, the negative aspects of the music produced by convolution with the electromagnetic guitar appears to be a matter of tuning the instrument, rather than an indication the instrument can not be used to produce music. Mixes of the original guitar recording combined with the music produced by the electromagnetic guitar are in the subdirectory Mix and each file is labeled by the number of impulse response samples used.

The low pass filter applied at 1 KHz reduces many peaks in the instrument’s response above this frequency, ignoring a significant portion of the instrument’s response. However, the sounds amplified for the electric guitar in this frequency range are not very pleasant, demonstrated by the audio file in the subdirectory 5KHz Cutoff, where the low frequency cutoff has been increased to 5 KHz.

Since the measured electromagnetic frequencies can be labeled with arbitrary acoustic frequencies, rather than enforce strict equality between acoustic and electromagnetic wavelengths, any conversion factor may be used. Thus, consider the power transmitted into the
EM guitar where the equivalent acoustic frequencies are determined by $f_a = 0.44c_a/c_e f_e$. Audio files using the derived impulse response (low pass cut-off 1 KHz) are in directory Waveguide 0.44. The reduction in frequency improves the sound of the instrument and more impulse response samples can be used before the sound becomes tinny. Some of the audio files in the Mix subdirectory sound improved compared to the original.

The response of the instrument for frequencies $f_a = 0.35c_a/c_e f_e$ is in Fig. 5.12. The resonances above 500 Hz are three dimensional resonances, and admit as much power as the peak near 100 Hz, illustrating how these modes can amplify unwanted high frequency sounds from electric instruments. The resonances themselves may have a frequency distribution that could produce pleasant sounds if located at a lower frequency. However, measurements are limited to 6 GHz, which corresponds to an approximate acoustic frequency of 2 KHz in this case, very close to the cutoff frequency of the low pass filter. Thus, these higher modes cannot be fully investigated here.

The prototype guitar presented here did not include the Helmholtz resonance, thus it does not reproduce all of the behaviors of an acoustic guitar. Despite this lack of behavior, the included music samples demonstrate that the electromagnetic guitar can be used to modify music and produce pleasant results. The lack of low frequency power in Fig. 5.11 could be attributed to the lack of Helmholtz resonance, since this resonance is used in acoustic guitars to increase the bass response. As calculated earlier, the electromagnetic Helmholtz resonance is expected near 100 Hz, consistent with the resonance of an acoustic instrument. Assuming a parallel resonance, the real impedance in Fig. 5.9 will be much larger near 100 Hz and may lift the impedance of the peak near 180 Hz to be more consistent with the peak at 300 Hz.

The large number of waveguide modes above 1 KHz (using $f_a = f_e c_e/c_a$) may be reduced by increasing losses in the instrument. Since power loss scales with frequency, higher order modes are preferentially reduced. Additionally, the separation between the plates could be reduced, driving the three dimensional resonances higher. This would provide a greater
separation between two dimensional resonances below 1 KHz and the resonances above 1 KHz. A low pass filter would still need to be employed, but at a higher cutoff, the filter would have less impact on input guitar music while still reducing unwanted behavior.

In addition to using the expected waveguide resonances, the unexpected low frequency resonances are used to modify music as well. Since these resonances begin near 10 MHz, much lower than 6 GHz, these resonances can be scaled to the full range of audio frequencies. Consider acoustic frequencies given by $f_a = 3.1 f_e c_a / c_e$ in Fig. 5.13, and a source impedance of 625 Ω. The variation in power is less than 9 dB over a frequency range covering three orders of magnitude. A low pass filter is still employed, with a cut-off above 15 kHz, to prevent aliasing of frequencies above 22 KHz, the limit for CD audio. Audio files obtained using the derived impulse response are in the directory Low Frequency Modes and labeled by the sample length of the impulse response. The music produced suffers none of the problems found with the waveguide modes. The high frequency sounds of the input guitar are naturally reduced by the resonances and the bass and mid frequencies are boosted (100.aif excepted). The audio files in the Mix subdirectory are improved compared to the original. In particular, the Mix audio file 10000.aif and the original recording sound like different guitars. These results confirm that resonant electromagnetic behavior can be used to modify music in a positive way.

Using varied source impedances and different scaling factors between electromagnetic and acoustic frequencies, the impulse response of the low frequency modes has been used to modify all of the instruments in a performance by The Smiling Knights. For comparison, the original recording is included. The EM guitar is used to increase the bass of the drums and the keyboard, as well as alter the tone of the two electric guitars. The song is not normally performed as an instrumental, though vocals are not included here for simplicity.

---

6Guitar, Vocals - Eric Streightoss; Drums, Keyboard - Josh Brown; Recorded by Russell Stoneback
Figure 5.11. Power transmitted into electromagnetic guitar at the equivalent acoustic frequencies for source impedance $Z_0 = 50 \, \Omega$ (blue), $Z_0 = 1500 \, \Omega$ (green).

Figure 5.12. Power transmitted into electromagnetic guitar at the equivalent frequency $f_a = 0.35c_a/c_e f_e$ for $Z_0 = 50 \, \Omega$ (blue), $Z_0 = 1500 \, \Omega$ (green).
Figure 5.13. Power transmitted into electromagnetic guitar at the equivalent frequency $f_a = 3.1 c_a / c_r f_e$ for $Z_0 = 50 \Omega$ (blue), $Z_0 = 625 \Omega$ (green) configured like Fig. 4.10.
CHAPTER 6
EFFECTIVE APERTURE IN THE POLAR IONOSPHERE

In previous chapters, it is shown that an aperture in a conductor may be modeled by considering a plane conductor without an aperture and adding equivalent electric and magnetic dipole fields to represent the effects of the aperture. This superposition technique is particularly useful in considering the interaction of radiation fields on each side of the aperture, since very simple boundary conditions across the plane conductor can be imposed. In pursuing this approach we find that the equivalent aperture electric field is applied across the aperture in much the same fashion as the cross-polar cap potential is applied across the high latitude ionosphere. The equivalent aperture magnetic field is similar to that produced by the Birkeland current system at high latitudes that delivers the cross-polar potential. This correspondence between the high latitude ionosphere electric fields and field-aligned currents and those required to model a circular aperture in a plane conductor, prompt this investigation of the electromagnetic interactions between the magnetosphere and the ionosphere in which the ionosphere is treated as a conducting plane with an aperture in it defined by the region of open magnetic field lines at high latitudes. Dr. Roderick A Heelis contributed language to this chapter clarifying the presentation for researchers studying the geospace environment.

6.1 General Resonator Model

Consider a infinite perfect plane conductor with a circular aperture dividing space in half, with an incident plane wave in region 1 radiating into region 2 through the aperture. In the low frequency limit, the aperture behaves like an electric and magnetic dipole, radiating power equally into both regions. This problem may be solved by appealing to an equivalent situation introduced by Bethe [2] and corrected by Bouwkamp [37]. The aperture is replaced
Figure 6.1. Real part of fields produced in and around aperture excited by a normally incident plane wave, where the aperture is replaced by a conductor containing magnetic currents. The reflection of a normally incident plane wave with magnetic field $H_0$ from a plane conductor leads to a doubled magnetic field (Gray) and no electric field near the conductor surface. The fields produced by the effective aperture currents (black) are generally constant in the aperture, but the fields created outside the aperture vary strongly as a function of distance.

with a perfect conductor containing magnetic currents that recreates the same radiated dipole fields produced by the original aperture, Fig. 6.1. By replacing the aperture with a perfect conductor, the incident wave reflects from a plane conductor with no aperture, leading to a doubling of the incident tangential magnetic field at the conductor surface and no tangential electric field. The introduced magnetic currents recreate the presence of the aperture and the generated fields add linearly to the fields obtained with no aperture.

In the source region, the aperture acts against the incident field, reducing the total magnetic field to account for the power transmitted through the aperture. From the boundary conditions in the aperture, the magnetic field in the aperture is determined by the incident wave, while the magnitude of the electric field varies with frequency to account for changes in radiated power. The effective dipole behavior of the aperture also induces image currents around the aperture and these fields vary as a function of position around the aperture. In chapter 3, the conductor replacing the aperture also has an impedance equivalent to the effective aperture dipoles. The reflection and transmission of incident waves upon the aperture can then be calculated using standard relations at the interface between media.
Since the same fields are produced by the magnetic currents and the aperture, the use of fictional magnetic currents to describe the aperture is mathematically convenient and produces accurate results. The equivalence of a suitable current distribution to the behavior of an aperture also indicates that aperture behavior may be found in systems where no physical aperture is present.

The magnetic currents ($\vec{M}$) in the equivalent aperture conductor used to mathematically replace a physical aperture satisfy

$$\vec{M} = \vec{E} \times \hat{n}_{12}$$  \hspace{1cm} (6.1)

where $\vec{E}$ is the field produced by the aperture and $\hat{n}_{12}$ is normal to the aperture pointing from region 1 to region 2. In the polar ionosphere, currents produced by the Hall conductivity satisfy

$$J_{Hall} = \sigma_H \vec{E} \times \hat{z}$$  \hspace{1cm} (6.2)

where $\vec{E}$ is the electric field in the polar region and $\hat{z}$ points along the geomagnetic field, identifying Hall currents as a functional replacement for the equivalent aperture magnetic currents.

In this model the conductor containing the aperture is the ionospheric slab characterized by a constant Pedersen conductance. Our previous studies show how an aperture in this conductor can be modeled by introducing a fictitious magnetic current (i.e. a current of fixed length with no closure current) in the aperture. Hall currents flowing from midnight to noon in the polar region serves this function. Thus, the aperture is defined by the polar cap in the ionosphere.

The dipole configuration of the geomagnetic field yields open field lines in the polar regions and closed field lines equatorward. Since magnetic fields around the Earth tend to have a large direct conductivity, it is supposed that the closed magnetic field lines make the ionospheric conducting slab an equipotential surface outside the aperture (polar cap),
confining observable large scale magnetic currents to the polar region. Clearly, this disregards the effects of charged particles in the inner magnetosphere.

For waves from the high latitude magnetosphere, \( \hat{n}_{12} \) points along the magnetic field in the northern hemisphere, but is oriented opposite the magnetic field in the southern hemisphere. Thus, the northern and southern polar regions are expected to act out of phase.

Since the basic properties of the Earth’s ionosphere and magnetic field appear to satisfy the requirements used to describe electromagnetic apertures, the polar regions will be treated as effective apertures. The remaining ionosphere is considered a uniform conductor, along with the surface of the Earth. Thus, the proposed model treats the Earth as pair of concentric conductive spheres, where the outer sphere has a circular aperture in both the northern and southern polar regions.

The interaction between the effective polar aperture and the cavity surrounding the Earth is accounted for by developing an equivalent electrical circuit, similar in nature to electric circuits used to describe acoustic Helmholtz resonators. The behavior of waves incident upon the polar regions can then be computed using standard relations for waves at the interface between media with different impedances.

### 6.1.1 Aperture Impedance

The dipole impedances of an aperture (Ch. 3) of radius \( a \) and area \( S \) are

\[
Z_{2e} = 2 \left( \frac{Sb^2k_2^4g(k_2)^2}{12\pi Z_2} + \frac{j\epsilon_2\omega Sb^2}{\alpha_e} \right)^{-1} \tag{6.3}
\]

for the electric dipole, and

\[
Z_{2h} = \frac{1}{2} \left( \frac{2S k_2^2g(k_2)^2}{3\pi Z_2} + \frac{2S}{j\omega\mu_2\alpha_m} \right)^{-1} \tag{6.4}
\]

for the magnetic dipole, where \( \epsilon_2, \mu_2 \) are the material properties of the region not containing the source. \( Z_2 \) is the wave impedance in region 2, \( \omega \) is angular frequency, \( k_2 = \omega\sqrt{\epsilon_2\mu_2} \) is the wave number, and \( b \) is the length of the aperture dipoles. The function \( g(k_2) \) accounts...
for the frequency dependence of the image charges in the perfect conductor induced by the aperture dipoles [32],

\[ g(k) = \left( 1 - \frac{3 \cos 2k_2 h}{(2k_2 h)^2} + \frac{3 \sin 2k_2 h}{(2k_2 h)^3} \right) \]  

(6.5)

where \( h \) is the effective distance of the equivalent aperture current from the plane conductor. For a circular aperture (Ch. 3), the electric and magnetic dipoles produced by the aperture have an effective length for determining impedance

\[ b = \frac{\pi a}{4} \]  

(6.6)

and it is assumed that these dipole moments are distributed over a minimum distance of \( h = 2a \) away from the plane conductor. The electric (\( \alpha_e \)) and magnetic (\( \alpha_m \)) aperture polarizabilities for a circular aperture [38] are

\[ \alpha_e = \frac{4a^3}{3} \]  

(6.7)

\[ \alpha_m = \frac{8a^3}{3} \]  

(6.8)

The derived aperture impedances are treated as circuit elements.

The boundary conditions in the aperture require that the observed impedance is the same on both sides of the aperture. The introduced magnetic currents within the aperture indicate that the total magnetic dipole impedance is given by the series combination of impedances on both sides of the aperture, while the electric impedance reflects a parallel combination (Ch. 3). The total impedance for each dipole is divided into equal impedances for each region assuming these rules (6.3, 6.4). Since an incident electromagnetic wave has electric and magnetic fields in phase, both aperture dipoles can be excited simultaneously from the incident wave. The individual dipole impedances are combined in parallel to reflect this coupling and the observed aperture impedance is

\[ Z_{p0} = Z_{2e}||Z_{2h} = \frac{Z_{2e}Z_{2h}}{Z_{2e} + Z_{2h}}. \]  

(6.9)

Note that the impedance of the aperture is solely determined by the region not containing the source (region 2) and does not vary with source orientation. If the material
properties of region 2 vary as a function of position, it is presumed that an average of the properties surrounding the aperture should be used. Since the aperture impedance is partly determined by a associated dipole length $b$, a suggested minimum volume to compute a material average extends at least length $b$ away from the aperture. Since image currents surrounding the aperture contribute to the radiated power, the radius of the minimum volume should be greater than the aperture radius. However, the minimum volume sufficient to average material properties is unknown.

The aperture impedance values have been confirmed by numerical results, where the impedance of the aperture is determined by

$$Z = \frac{V}{I}$$

(6.10)

where $V, I$ are voltage and current in the aperture, determined by appropriate path integrals of the aperture electric and magnetic fields (Ch. 3).

6.1.2 Proposed Resonator

Suppose that the effective polar aperture couples with the concentric spherical capacitor formed by the Earth and the ionosphere. The capacitance of the Earth Ionosphere Capacitor (EIC) is ideally

$$C_E = 4\pi\epsilon_0 \frac{R_e R_I}{R_I - R_e}$$

(6.11)

where $R_e$ is the radius of the Earth and $R_I$ is the effective global radius of the ionosphere. The average separation between the Earth and the bottomside ionosphere is taken to be 60 km.

Although it is presumed that the EIC interacts with both the northern/southern polar apertures, because these apertures are out of phase, the resultant fields generally cancel within the EIC. Therefore, the interaction between fields in the EIC and both polar apertures is ignored. Confined to areas near the polar cap, the fields observed might be approximated by a system with only one aperture, thus only one aperture is included in the presented model.
In the low frequency limit, incident Alfvén waves upon the polar region from space will excite the aperture and the fields produced by the aperture excite the EIC. The potential difference created between the Earth and the ionosphere by the polar aperture requires currents flowing parallel to the aperture into the surrounding ionosphere. Since these currents are correlated with the incident Alfvén waves, it is presumed that the interaction between aperture and EIC is mediated by fast magnetoacoustic waves traveling perpendicular to the background polar magnetic field. The impedance of the fast magnetoacoustic mode is given by the combination of impedances for magnetostatic waves \(Z_s\) and Alfvén \(Z_a\) waves,

\[
Z_{fc} = \sqrt{Z_s^2 + Z_a^2}.
\]  

The impedance of the EIC observed by the effective polar aperture is approximated as

\[
Z_{EIC} = Z_{fc} + \frac{1}{j\omega C_E}.
\]  

where the real magnetoacoustic impedance \(Z_{fc}\) is placed in series with the impedance of the EIC.

The separation between the Earth’s surface and ionosphere is small compared to the size of the polar aperture, therefore the Earth’s surface is located within the space that normally functions as part of the effective aperture dipole, Fig. 6.2. Since Earth’s surface is conductive, its close proximity is expected to cancel the tangential electric fields produced by the aperture in the EIC. Because the aperture impedance depends upon the tangential electric field, this cancelation removes the impedance contribution of the aperture dipoles in the EIC from the total aperture impedance. However, it is presumed that the impedance of the dipole for the region above the aperture is unchanged by the Earth.

The boundary conditions in the aperture require that the impedance must be the same on both sides of the aperture. However, as presented, one side of the aperture has zero impedance, while the other side has the normal aperture impedance. To satisfy the boundary conditions, some of the impedance from one side of the aperture must be transferred to the other side. For magnetic dipoles, coupled in series across the aperture (Ch. 3), the remaining
aperture impedance is divided in two using a series combination of circuit elements, leading to aperture impedance $\frac{1}{2}Z_0$ in each region. If the electric dipole is excited, the parallel distribution of electric dipoles leads to aperture impedance $2Z_0$ in each region. While an aperture alone has the same impedance regardless of source field orientation, it is proposed that an aperture backed by a close conductor breaks this degeneracy.

For an incident plane wave, tangential magnetic fields excite the magnetic dipole and normal electric fields excite the electric dipole. For an arbitrary source, the distribution of power in the incident wave that interacts with the magnetic or electric aperture dipole is determined by calculating the symmetric product of the incident fields with the respective electric/magnetic dipole current distribution (Ch. 3),

$$< A, B > = \frac{\int A \cdot B \, dS}{S}. \quad (6.14)$$

The portion of the incident magnetic field $\mathbf{H}$ that excites the aperture magnetic dipole is $< 2\mathbf{H}, \hat{\gamma}_{1,2} >$, where $\hat{\gamma}_{1,2}$ are a pair of constant orthogonal unit vectors in the plane of the aperture. The portion of the incident electric field that excites the aperture electric dipole is $< \mathbf{E}, \hat{z} >$. 

Figure 6.2. Aperture backed by a conductor where the conductor is located within a distance much smaller than the aperture radius. It is proposed that the impedance of the aperture facing the conductor is shorted, while the impedance of the opposite aperture face is unchanged. To satisfy boundary conditions, the remaining aperture impedance is divided into equal impedances for each aperture face.
Consider the case for the coupling between the aperture magnetic dipole and the EIC. Based upon the series coupling of the aperture magnetic dipole, the aperture impedance is placed in series with the local impedance of the EIC. Since the voltage between the plates in the EIC is the same as the voltage created by the effective aperture between those plates, the total local impedance of the EIC is a parallel combination of aperture and EIC impedance. The total impedance for magnetic dipole coupling is

\[ Z_{\text{ph}}|_{\mu,\epsilon} = \frac{1}{2} Z_{p0}|_{\mu,\epsilon} + \left( \frac{1}{2} Z_{p0}|_{\mu,\epsilon} \right) \parallel Z_{\text{EIC}} \]  

(6.15)

where \( \mu, \epsilon \) characterize the region not containing the wave source.

Consider the case for coupling between the aperture electric dipole and the EIC. Given the parallel distribution of impedance for electric dipoles and the normal orientation of the electric dipole, the coupling between the EIC and the aperture is taken to be in parallel. The total impedance is

\[ Z_{\text{pe}}|_{\mu,\epsilon} = (2 Z_{p0}|_{\mu,\epsilon}) \parallel Z_{\text{EIC}}. \]  

(6.16)

Using the impedances for the different interactions between the effective polar aperture and the EIC, the power transmitted into the ionosphere from the magnetosphere can be calculated using standard relations for the transfer of power at an impedance mismatch. Assuming a source Alfvén wave impedance of \( Z_a \), and a polar region impedance \( Z_a + Z_{pw} \) where \( w = h, e \) denotes magnetic/electric dipole coupling, the fractional power transmitted into the polar region for parallel polarization is

\[ P_{w\text{trans}} = \text{Re} \left( \frac{4(Z_a + Z_{pw})Z_a^* \cos^2 i}{|Z_a \cos i + (Z_a + Z_{pw}) \cos r|^2} \right) \]  

(6.17)

and for perpendicular polarization

\[ P_{w\text{trans}} = \text{Re} \left( \frac{4(Z_a + Z_{pw})Z_a^* \cos^2 i}{|Z_a \cos r + (Z_a + Z_{pw}) \cos i|^2} \right) \]  

(6.18)

where \( i \) is the incident angle, \( r \) is the refracted angle, and \( P_{\text{inc}} \) is the incident power over the polar cap. The fractional power reflected is

\[ P_{\text{refl}} = 1 - P_{\text{trans}}. \]  

(6.19)
For the case of an actual aperture, the power transmitted into the aperture is further reduced by the material properties in both regions (Ch. 3) in the static limit

\[ P_{ap} = \frac{\mu_1^2}{(\mu_1 + \mu_2)^2} \frac{Z_1 S}{2} | < 2\mathbf{H}, \hat{\gamma}_{1,2} > |^2 P_{trans}^h + \frac{\epsilon_2^2}{(\epsilon_1 + \epsilon_2)^2} \frac{S}{Z_1 2} | < \mathbf{E}, \hat{z} > |^2 P_{trans}^e \]  

(6.20)

where only the tangential magnetic and normal electric fields transmit power into the aperture. However, since the ionosphere does not contain an actual aperture, the same static boundary conditions are not enforced and it is presumed that (6.20) does not apply to the transmitted power into the ionosphere. However, it is also presumed that the power radiated by the proposed polar aperture is the same as an actual aperture.

For a circular aperture in a plane conductor, the power radiated into region 1 as a dipole (Ch. 3) is

\[ P_1 = \frac{2g(k_2)}{3g(k_2) - 2} P_{ap} \]  

(6.21)

and the power radiated into region 2 is

\[ P_2 = \frac{g(k_2)}{3g(k_2) - 2} P_{ap}. \]  

(6.22)

For a wave source in space, the large relative permittivity in the ionosphere (\( \epsilon_1 \approx 140,000 \)) compared to the permittivity of the neutral atmosphere in the EIC (\( \epsilon_2 = 1 \)) implies that almost none of the power transmitted into the proposed resonator is radiated away by the aperture as a dipole field. Though it is uncertain where the excess in power transmitted into the polar ionosphere compared to the power radiated away is deposited, the excess may be transferred into the EIC or into the plasma in the ionosphere.

For waves in the EIC near the polar region, the fraction of power transmitted from the Earth into space is described by the interaction of a wave with impedance \( Z_{pw} \) incident upon the ionosphere with average Alfvén wave impedance \( Z_a \),

\[ P_{trans} = \frac{4Z_a(Z_{pw})^*}{|Z_a + Z_{pw}|^2} P_{inc} \]  

(6.23)
for normal incidence. Since the aperture properties are now determined by the ionosphere, the capacitance of the aperture is increased significantly, becoming larger than the capacitance of the EIC. Thus, the resonance of the system is principally determined by the aperture properties. For magnetic dipole coupling, the impedance $Z_{ph}$ approximately reduces to $Z_{p0}$.

For the electric dipole, the impedance $Z_{pe}$ is approximated by $2Z_{p0}$.

The phase velocity of waves created by the proposed resonator may be estimated using

$$V_{phase} = Re\left(\frac{Z_{dc} + \sum_w Z_{pw}}{\mu_0}\right)$$

where $Z_{dc}$ is the DC impedance in the ionosphere, and $\sum_w Z_{pw}$ is the sum of the appropriate resonance impedances. In general, there will be multiple contributions from the aperture, based upon whether the coupling is via the magnetic or electric dipole and the location of the wave source.

### 6.1.3 Typical Model Parameters

A average Alfvén wave impedance between the ionosphere and 1500 km altitude is $Z_a = 1\Omega$ [51] and a typical static wave impedance is $Z_s = .1\Omega$. A low frequency estimation of the electric permittivity in the ionosphere is

$$\epsilon = \frac{\mu_0}{Z_a^2} = \frac{1}{\mu_0 V_a^2}$$

assuming the magnetic permeability is given by vacuum and $V_a$ is the Alfvén phase velocity. The polar aperture is assumed to be 120 $km$ above the surface of the Earth with a radius covering 15 degrees of latitude

$$a = (R_e + 120 \, km) \sin \pi/12.$$  

(6.26)

Use of the Earth’s polar cap radius in (6.6) yields a minimum effective aperture dipole length of $b = 1300 \, km$. 

6.2 Correspondence with Measurements

Consider measurements performed by the FAST satellite during the main phase of a major geomagnetic substorm on Oct 22, 1999 presented in [14]. FAST detected transverse electric and magnetic perturbations on magnetic fields between 66.7° – 67.5° invariant latitude, which map to the plasmasheet side of the plasma sheet boundary layer (PSBL). The measured transverse fields were used to compute the frequency dependence of power and phase velocity of waves traveling along the field using the Fast Fourier Transform (FFT). The measured waveform was separated into 8 band limited waveforms with overlapping frequency domains covering 0.005 – 4 Hz. The phase velocity and power are calculated on a point by point basis for the waveforms in each frequency sub-band, whereby only points with sufficient flux are used for phase velocity. The range in phase velocity is given by the spread of 75% of the points with sufficient flux and the median value is marked. The results from [14] are reproduced in Fig. 6.3.

Assuming a wave source in the magnetosphere, there are two predicted relationships for power incident upon the polar region. Power in an electric field normal to the aperture has one coupling with the proposed aperture (6.17, \( w = e \)), while power in a transverse magnetic field has a different relationship (6.17, \( w = h \)). In general, the total response of the proposed aperture will have a mixture of the two responses. Observations performed by FAST [14] reproduced in Fig. 6.3 matches the general behavior expected from a combination of the effective aperture electric and magnetic dipoles, consistent with the presence of significant electromagnetic flux from the magnetosphere headed towards the Earth.

The predicted power transmitted from Earth to space assuming (6.23) is illustrated in Fig. 6.4. The large permittivity in the ionosphere lowers the dipole resonance of the aperture to 0.11 Hz for magnetic coupling and 0.12 Hz for electric coupling. These peaks are observed in the measured phase velocity [14] reproduced in Fig. 6.3. If both of these resonances are excited simultaneously, a beat frequency of 0.01 Hz may be detected, similar to the period observed for modulated PC1 waves [58].
Figure 6.3. Measurements using FAST satellite on Oct. 22, 1999 [14] during the main phase of a geomagnetic storm (*) along with predicted values for transmitted power and Alfvén phase velocity for aperture magnetic (solid) and electric (dashed) dipoles. (a) Power transmitted into Earth (b) Alfvén phase velocity, measured using $\delta E/\delta B$. The FAST measurements are reproduced with permission from Dombeck et al. Alfvén waves and Poynting flux observed simultaneously by Polar and FAST in the plasma sheet boundary layer. Journal of Geophysical Research (2005) vol. 110 (A12) pp. 8
The Alfvén phase velocity is constructed using the impedance of the aperture for radiation from the magnetosphere into the EIC and for radiation from the EIC into the aperture. For waves traveling from Earth out into space, the aperture impedance is determined by the electric permittivity in the ionosphere, obtained using (6.25). For electric/magnetic dipole coupling,

\[
V_{aw} = \frac{Z_{pw}|_{\mu_0,\varepsilon_0} + Z_{pw}|_{\mu_0,\varepsilon} + Z_{dc}}{\mu_0},
\]

(6.27)

The observations performed in [14] indicate a DC impedance \( Z_{dc} \approx 0.3\Omega \).

For electric dipole coupling in Fig. 6.3(b), the peak near 0.4 Hz is associated with waves from space and represents a resonance between the aperture and the EIC. The peak near 0.1 Hz is the self resonance of the aperture. While the peak in phase velocity for the self resonance near 0.1 Hz matches well with measurements, the predicted peak centered near 0.4 Hz does not. Since the power transmitted from the magnetosphere to the Earth depends upon the impedance mismatch between the two, the observed minimum in transmitted power should have a corresponding peak in measured phase velocity. However, the proposed aperture behavior predicts that essentially none of the power transmitted into the polar aperture from space via the electric dipole will be radiated by the polar aperture. Further, waves reflected by the polar aperture will not display aperture characteristics. Thus, the increase in Alfvén wave speed may not be detected outside the aperture. Note, however, the peak in predicted wave speed matches well with the Alfvén wave speed assuming 100% \( H^+ \) [14]. The resonance between the aperture and EIC for electric dipole coupling varies between 0.3 – 0.5 Hz for variations in the separation between the Earth and Ionosphere of 50 – 90 km.

For magnetic dipole coupling, the predicted Alfvén speed matches the observed wave speed over the range of measurements. The maximum in wave speed at 0.11 Hz is near the observed peak in velocity. The somewhat larger spread in measured velocity observed at 1 Hz is suggestive of a peak, observed in the predicted velocity at 0.9 Hz. Note that the peak predicted in wave speed closely matches the wave speed assuming 100% \( O^+ \) ions [14].
Consider measurements [68] of a pulsation event on March 7, 2001 at Finnish stations using incoherent scattered radar and ground based magnetometers. The event was observed simultaneously at Kilpisjarvi (KIL, L=6.0) and Ivalo (IVA, L=5.6). The power in the magnetic field recorded on the ground shows two clear resonances at constant frequencies. The low frequency resonance is distributed between 0.05 – 0.17 Hz [68], consistent with the predicted self resonance of the aperture at 0.12 Hz. A resonance near 1.3 Hz appears at KIL a brief period after the beginning of the event. This observed peak is consistent with the predicted peak in transmitted power centered upon 1.25 Hz for magnetic dipole coupling for waves from space. While there is some scattered power observed in between 0.1 – 2 Hz, the resonances at approximately 0.1 Hz and 1.3 Hz are observed for over 6 hours.

The radar was used to measure ion/electron densities in the ionosphere during the pulsation event. The measured plasma density was used to calculate relative maximums in the transmission coefficient of Ionospheric Alfvén Resonator (IAR), obtaining $f_0 \approx 0.13$ Hz, $f_1 \approx 0.39$ Hz, $f_2 \approx 0.68$ Hz, and $f_3 \approx 0.98$ Hz [68]. The presented aperture resonator has relative maximums in transmitted power at 0.12 Hz and 1.25 Hz, with relative minimums at 0.36 Hz, 0.68 Hz. Because the polar aperture is located below the IAR, power reflected from the aperture could couple into the IAR. The correspondence of the lowest three frequencies...
of the IAR with the predicted aperture resonance frequencies suggests that the structure of the IAR may be influenced by the effective aperture behavior in the polar region. The standing waves produced by reflection from the Earth and ionosphere may move plasma such that the resultant plasma distribution supports the standing wave.

Measurements made by Polar and a Finnish magnetometer at Sodankylä (L=5.1) detected a multiband Pc1 event with a pearl structure on April 25, 1997 [58]. The authors note that the pearl structure of a wave near 1.3 Hz arises due to a modulation from low frequency waves around 0.01 Hz, detected near the magnetic equator. The waves detected at the equator had a radial electric field [58], consistent with energy in the EIC. The capacitive operation of the EIC should produce both a normal electric field and a tangential magnetic field to the ionosphere, exciting both self resonances of the aperture, resulting in a beat frequency near 0.01 Hz. The waves detected near 1.3 Hz are consistent with the peak in magnetic dipole coupling, thus the beat frequency produced in the aperture is expected to modify this resonance as well.

6.2.1 Numerical Model

To illustrate the correspondence of the fields produced by an aperture and measurements in the polar region, a numerical model of an aperture in a TEM waveguide backed by a finite grounded plane conductor was investigated using the finite element method. The model evolved from an investigation of an aperture in a plane conductor alone (3). A TEM waveguide is used to direct a normally incident wave upon the aperture in a plane conductor. To maintain numerical accuracy, a curved geometry is used to transition from the plane conductor to the sidewalls in the aperture of thickness $d$. The aperture radiates dipole fields into an approximate half space, truncated using perfectly matched layers (PML), a computational method that absorbs incident radiation while producing little reflection. A square perfect conductor is centered upon the aperture, located distance $l$ away, and parallel to the aperture plane.
Because both conductors are grounded due to program limitations, the plate backing the aperture does not function as idealized parallel plate capacitor with the plane conductor containing the aperture. While the proposed model assumes a resonant interaction well below the first waveguide resonance of the capacitor, the reduced capacitance of the numerical system yields a resonance close to the lowest waveguide resonance. Though the numerical model does not account for all interactions between the proposed effective polar aperture and EIC, it may still be useful to determine general properties of the proposed resonator.

Consider illustrations of the tangential electric and magnetic vector fields in a planar slice through the middle of the aperture near resonance with the backing conductor, along with the plasma convection pattern due to $\mathbf{E} \times \mathbf{B}$ drift in Fig. 6.5. This drift is obtained by assuming a static magnetic field directed along $-\hat{z}$ with unit magnitude. The magnetic field follows the magnetic current, consistent with Hall currents observed in the polar cap as required by (6.2), where the Hall conductivity is negative.

The aperture electric field has a generally similar vector orientation to the polar cap electric field, however, the two differ in relative magnitudes. The aperture electric field increases near the aperture edge, while the field in the polar cap tends towards zero. However, the pictured electric field is normal to the sidewall at the aperture edge and is therefore discontinuous across the aperture boundary. In the polar cap, there is no sidewall to the effective aperture, therefore the electric field must be continuous across the aperture boundary. When combined with the continuity of potential across the aperture boundary, the electric field should tend towards zero across the aperture boundary.

Since the effective conductor replacing the aperture is not connected to a charge source, the potential distribution is odd about $\hat{y}$ in the aperture. Assuming the input wave is confined to the aperture, one side of the aperture will be at a higher potential than the surrounding conductor, while the other is at a lower potential. Thus, a radial electric field outside the aperture is expected, with odd symmetry about $\hat{y}$, and components along $\hat{x}$ are opposite the aperture electric field, consistent with fields observed outside the polar cap.
Figure 6.5. Real tangential vector fields in an aperture backed by a conductor near resonance with conductor: (a) electric field (b) magnetic field (c) associated plasma velocity. The colors of the rainbow are linearly scaled using Mathematica to illustrate the relative magnitudes of each quantity; the smallest values are purple, while the largest values are red.
Figure 6.6. Plasma convection pattern near aperture self resonance, just above maximum in radiated power. The colors correspond to a linear mapping of the colors of a rainbow in Mathematica, where purple corresponds to a magnitude of 0 and red is the maximum attained value.

Figure 6.6 is the polar convection pattern near the self resonance of the aperture, at a frequency just above the maximum in transmitted power. Note that the convection pattern reverses direction near the aperture boundary. Since the wavelength is smaller than the aperture size, this reversal boundary occurs before the aperture edge. At the peak in transmission, this reversal occurs at the aperture boundary. Note the similarity to the plasma convection pattern without corotation inside the polar cap. The model used near the aperture self resonance does not include a backing conductor.

The deflections of magnetic fields tangential to the polar cap are traditionally assumed to arise due to the presence of field aligned currents, generally known as Birkeland currents. In the presented model, the magnetic field in the polar cap arises due to the effective aperture behavior of the polar cap. These magnetic fields drive electric fields in the aperture and the normal electric field distribution inside and just outside the aperture in the source region is in Fig. 6.7. The overall field distribution of the aperture is highly similar to the Birkeland
current distribution inferred assuming only electric currents. Inside the aperture, the normal electric field has odd symmetry about $\hat{y}$. Just outside the aperture, the direction of the electric field is reversed and concentrated at the aperture edge. The magnitude of the normal electric fields in the aperture increase with decreasing plate separation. Because the Earth is located closer to the proposed polar aperture than modeled here, an increase in the normal electric fields is expected.

The normal electric fields outside the aperture arise due to gradients in the $\hat{y}$ component of the magnetic field along $\hat{x}$. In the source region, the reflection of the incident wave upon the plane conductor leads to a magnetic field of $2H_0$ at the surface, and $H_0$ inside the aperture, where $H_0$ is the magnitude of the incident magnetic field. In the transition from the value of the magnetic field along the plane conductor to the value inside the aperture, normal electric fields are observed due to incomplete cancellation of the electric fields induced by the time dependence of the magnetic field. In the region not containing the source, the normal electric fields outside the aperture have the same orientation as the fields inside the aperture. This occurs since there is no magnetic field in this region other than what is introduced by the aperture.

The aperture impedance may be obtained by considering only the fields scattered by the aperture and integrating the electric field along $\hat{x}$ across the aperture and by integrating the magnetic field along a closed loop along $\hat{y}$. The numerical model presented utilized the
total field, incident wave plus the aperture field, so the current in the aperture is computed by integrating the total magnetic field in the aperture across \( \hat{y} \) and doubling the result to include the scattered magnetic field of equal magnitude and opposite orientation on the other side of the aperture. The theoretical aperture impedance presented strictly describes the average impedance of the aperture, while the numerical results presented here use the impedance of the aperture near the edge. Though the impedance in the aperture is generally constant across the area, the impedance near the edge of the aperture is approximately 10% larger than average. The impedance in the space between the plates is determined using the current obtained from the aperture and by integrating the normal electric fields along a closed loop along \( \hat{x}, \hat{z} \). Note that the current obtained by integrating the magnetic fields in the waveguide outside the aperture is the same as the current in the aperture.

Consider the impedance measured within the aperture for a plate backing the aperture with edge length 10 times the aperture radius, with plate separation \( l \) of half an aperture radius, along with the impedance of the space between the plates, Fig. 6.8. For the imaginary impedance in Fig. 6.8(b) the red line is the measured impedance in the aperture, the blue line is the impedance in the capacitor just outside the aperture, and the green line is the impedance of an aperture alone, with no backing capacitor. The purple line is the impedance obtained by adding \( Z_{p0}/2 \) to the measured impedance of the capacitor just outside the aperture. This method closely follows the impedance measured within the aperture below resonance, while generating slightly low impedances above resonance. These results generally support the presented configuration for magnetic dipole coupling. Note that the same method applied to the impedance near the edge of the capacitor (not shown) yields an impedance that is too low, indicating that the aperture interacts with the impedance of the local area. The space between the two plates is inductive at low frequencies because only the magnetic dipole in the aperture radiates power for normally incident waves.

For the real impedance in Fig. 6.8(a), the blue line is the impedance of the capacitor just outside the aperture, the red line is the real impedance of the aperture, and the green line
is the real impedance of an aperture alone. Both the aperture and capacitor real impedances are less than the real impedance of an aperture alone. The real impedance of the capacitor is slightly larger than the impedance of the aperture. At resonance, the aperture backed by a conductor has a real impedance approximately 600 times greater than the real impedance of the aperture alone and the aperture admits $\approx 20\%$ of the power incident upon it Fig. 6.8(c).

The blue line is the measured time average Poynting flux in the aperture and the black line is the predicted power in the aperture using the measured impedance of the aperture and (6.22). The predicted power is slightly higher than the measured flux since the aperture impedance used is slightly higher than the average impedance in the aperture.

6.3 Discussion and Conclusion

The equivalence of open magnetic fields in the polar ionosphere to an aperture in a conductor appears particularly effective in predicting measured behavior for oscillatory waves. Further, the electric and magnetic fields found in an aperture correspond to the electric and magnetic field distributions found in the ionosphere. However, the polar cap potential and the Birkeland current distribution are idealized as static field distributions, while the results presented so far are oscillatory. Though the voltage applied to the polar cap by the solar wind does certainly vary, the polar cap electric field does not oscillate between dawn to dusk and dusk to dawn orientations for a given IMF orientation.

Consider a co-rotating frame with the Earth’s magnetic field and the potential applied across two flux tubes symmetric about the magnetic pole. As these fields rotate with the Earth, the generally constant polar cap potential appears to oscillate with a 1 day period. Because of this time dependence due to rotation, the same field distributions found for an aperture excited by an oscillatory source should also be found in the ionosphere, even though those fields appear static to an observer at a fixed point at magnetic local time.

It is noted in [14] that the percentage of earthward power measured by FAST in Fig. 6.3 reflects properties of both standing and traveling waves and is not a consistent
Figure 6.8. Numerically determined (a) real impedance (b) imaginary impedance (c) radiated power
phase relationship. However, the standing waves produced by reflection at the ionosphere are directly determined by the impedance in the ionosphere, thus providing a direct measurement of the ionospheric impedance. The proposed aperture impedance will not be observed in waves that don’t produce a standing wave pattern, consistent with observations.

The correspondence in frequency between the lowest modes of the Ionospheric Alfvén Resonator (IAR) and the peak in reflected power at 0.36 Hz from the ionosphere, in addition to 0.12 Hz power from the Earth, suggests a correspondence between the two. Measurements indicate that the IAR is a quarter wave resonator, operating below an altitude 1200 – 1500 km [68]. This range is consistent with the effective impedance length of the effective polar aperture of 1300 km. Since the effective aperture will generally always be reflecting power at 0.12 and 0.36 Hz, the resultant standing waves could drive a plasma distribution that supports these frequencies. The Alfvén phase velocity is proportional to the ratio of magnetic field and plasma density,

\[
V_a = \frac{B}{\sqrt{\mu_0 \rho}}
\]

where \( \rho \) is the plasma density. Assuming the Earth’s magnetic field is fixed, the phase velocity in the ionosphere changes due to variations in plasma density. The effect of the aperture on density is supported by the correspondence in phase velocity for the predicted electric and magnetic dipoles to the limits in measured phase velocity by FAST assuming 100% \( H^+, O^+ \). Since the effect of the aperture can only move plasma already in the ionosphere, it would appear in measurements as a transport process.

The electromagnetic fields associated with an aperture in a plane conductor are consistent with observed field distributions in the polar ionosphere. A model is constructed based upon coupling of the aperture with the capacitor formed between the Earth and ionosphere. Predictions are consistent with measurements of the IAR and frequencies observed by ground-based magnetometers. Based upon the correspondence of predictions to measurements presented, the postulated effective aperture behavior of the polar ionosphere is confirmed.
CHAPTER 7
EFFECTIVE APERTURE BEHAVIOR ON THE SUN

Like the Earth, during solar minimum the Sun has a generally dipole magnetic field, with open field lines in the polar region and closed field lines equatorward. Hall currents are possible from the photosphere outward [65], enabling the magnetic currents required for effective aperture behavior. However, effective aperture behavior does not require the global field of the Sun be dipolar; localized regions with a magnetic field normal to the Sun should be sufficient. Thus, regions with open/normal field lines on the Sun will be treated as effective apertures, using the models derived for the Earth’s polar ionosphere.

Consider a model description of the Sun accounting for the outer three layers: the photosphere, chromosphere, and corona. The photosphere is the visible surface of the Sun, where photons are finally free to travel out into space and has an averaged temperature near 5800 K. Traveling inward, the gas increases in density until it becomes opaque. Just above the photosphere is the chromosphere, characterized by emissions of Hydrogen in the red visible spectrum. These observations indicate that Hydrogen is being ionized in the chromosphere. The temperature of the Sun has a minimum at the photosphere near 4200 K, and increases steadily through the chromosphere reaching a temperature on the order of 10,000 K. Approximately 2000 km above the photosphere, there is a steep increase in temperature, known as the transition region. This large temperature gradient separates the chromosphere and corona and temperatures in the corona can reach several million Kelvin. Due to the large temperature in the corona, atoms can be highly ionized. The steep gradient in temperature in the transition region implies that certain ionization states may be confined to a thin layer. Observations of the Sun at different wavelengths can be used to investigate particular layers of the solar atmosphere.
Figure 7.1. A view of the Sun in extreme ultraviolet from FE XII on March 19, 2009. Image from SOHO/EIT (ESA and NASA).

Figure 7.2. A view of the Sun’s corona in white light using the Mark IV K-coronameter at the Mauna Loa Solar Observatory on January 20, 2009.
A view of the southern coronal hole is visible in Fig. 7.1. The light is in the extreme ultraviolet and corresponds to Fe XII at 195 Å. The polar regions are dimmer than the surrounding area because the density of plasma is reduced in the polar region. The magnetic field points towards/away from the Sun in the polar corona, allowing plasma to flow freely away, reducing the plasma density, and apparent brightness. Equatorward of the polar cap, the magnetic fields are closed and resist movement of plasma away from the Sun. These field lines have more plasma due to this relative confinement, thus appearing brighter. The effective apertures in the Sun’s polar regions are assumed to be circular apertures, each with a radius of 15 degrees latitude.

Another look at the corona is in Fig. 7.2, taken in white light, where the Sun itself is blocked by a disk. This allows the telescope to see the comparatively dim glow of the corona surrounding the Sun. The plasma is expanding outward, producing a stream of charged particles known as the slow solar wind. A faster solar wind flows from the dimmer polar regions, not observed due to the lower plasma density.

7.1 Solar Model

Although the properties of the Sun certainly vary as a function of radial distance, each region is simply approximated here by an average wave impedance. The impedance of the photosphere (\(Z_{\text{ph}}\)) is obtained assuming a typical Alfvén wave speed of 3 km/s and using (6.24). Measured wave speeds in the chromosphere (\(Z_{\text{ch}}\)) vary between 50 – 200 km/s \[67\], averaging to 125 km/s. For the corona (\(Z_{\text{co}}\)), the average wave speed is averaged over a spherical shell with thickness \(b\), from the photosphere to \(1.2R_{\text{sun}}\). The Alfvén wave speed at \(1.2R_{\text{sun}}\) is on the order of 5Mm/s \[66, 81\], leading to an average wave speed of 2.5Mm/s.

The impedance of the EIC used in the Earth model is replaced by the impedance of the Sun. Unlike the Earth, the Sun doesn’t have two conductive shells separated by a non-conductive layer, therefore the capacitive impedance is ignored, while the real impedance of the fast magnetoacoustic mode is kept. It is still presumed that the effective aperture
behavior is that of an aperture backed by a conductor. Using the results presented by [65], Hall diffusion dominates between 600 – 1100 km above the surface (where Sun becomes opaque). Between approximately 1500 – 2000 km, ambipolar diffusion becomes larger than Hall, although Hall is only half in magnitude. The Pedersen conductivity is several orders of magnitude higher than Hall conductivity in this regime. The drop in neutral density above 2500 km limits the ambipolar diffusion here, while Hall effects remain possible, but reduced. As magnetic fields increase, these layers lower and Hall diffusion becomes important in the photosphere, with ambipolar in the chromosphere. The distribution of Hall diffusion through the Sun is treated as an effective aperture, backed by a conductor approximated by the Pedersen conductivity in the ambipolar region.

The rapid increase in temperature between the chromosphere and corona appears as a large shift in wave impedance, preventing waves from reaching the corona. In the simple wave impedance model presented, using the impedances of the chromosphere $Z_{ch}$ and corona $Z_{co}$, 20% of incident power is transmitted into the corona, more than found numerically. This is expected based upon the simplicity of the model. The effective aperture behavior in the chromosphere will increase the impedance of waves incident upon the transition region, increasing the amount of power transmitted into the corona. Since Hall currents are distributed throughout the chromosphere [65], it is proposed that waves below the transition region have an impedance given by the sum of the underlying wave impedance and the resonator impedance. The transmission of power across the transition region is determined by the interface between $Z_{ch} + Z_{pw}$, and $Z_{co}$.

The magnetoacoustic wave impedance $Z_{fc}$ is determined using properties of the presumed effective aperture region between 600 – 1100 km. Using an Alfvén speed between 40 – 50 km/s [65], an acoustic speed on the order of 10 km/s for $Z_{fc}$, and using an average phase velocity of 2.75 $Mm/s$ for the corona, predictions are in Fig. 7.3. A peak in transmitted power is observed at a 5 min. period for magnetic coupling, consistent with Alfvén waves observed in the corona [81].
The power transferred to the corona through the effective polar aperture can be simply estimated. Using an estimated power density in the chromosphere between $4 - 7 \ kW/m^2$ [67], and the fact that 70% of incident power is transmitted at 5 min. from chromosphere to corona, multiplied by the fractional area covered by the polar coronal hole (3.5%), yields an average power density of $100 - 170 \ W/m^2$ over the coronal surface. This power is sufficient to heat the corona and drive the solar wind during solar quiet [67]. As more active regions form on the sun with magnetic fields normal to the surface, the increase in surface area will deliver more energy to the corona. Since the presented model overestimates the amount of power between chromosphere and corona without aperture, the power transmitted into the corona was also calculated using only the aperture impedance for the source wave. Though the peak reduces slightly, more than 65% of the power is transmitted for 5 min. waves, sufficient to deliver $90 - 160 \ W/m^2$.

Fig. 7.4 illustrates the Alfvén wave speed corresponding to the aperture impedances $Z_{pw}$. Magnetic coupling yields a peak at 760 km/s, while electric coupling yields 50 km/s. The predicted Alfvén speed for the magnetic dipole is consistent with the fast solar wind. The fast solar wind travels normally away from the polar region, which is also consistent with the normal radiation of the aperture magnetic dipole. The aperture electric dipole radiates in the plane of the polar region. However, using (6.20) and the values assumed for the chromosphere and corona, only 3% of the power transmitted into the aperture electric dipole will be radiated. The remaining power may be deposited into the plasma, but this is not known. If the excess power is deposited into the plasma and if the plasma is accelerated in the aperture plane, heated plasma may be transferred to closed field lines. The continual influx of energy would eventually drive an expansion of this plasma outward. This description is generally consistent with observations of the corona in Fig. 7.2 during solar quiet.
Figure 7.3. Power transmitted from chromosphere to corona for electric (solid) and magnetic (dashed) dipole coupling.

Figure 7.4. Alfvén phase velocity due to the effective polar aperture for electric (solid) and magnetic (dashed) dipole coupling.
7.2 Sunspots

The magnetic structure of sunspots is generally similar to the field structure in the polar region. In the umbra, the magnetic field is generally normal to the solar surface. As one moves radially outward to the penumbra, the magnetic fields deviate significantly from the normal and are presumed closed. The umbra is taken to represent the size and shape of an effective aperture in the photosphere, while the penumbra is associated with the image currents induced by the aperture.

Consider a sunspot with an umbral radius of 5,000 km. Given the larger magnetic field, it is assumed that Hall currents are significant in the photosphere. Low frequency Alfvén waves involve motions of the ions, electrons, and neutrals, therefore the incident wave is assumed to be a slow magnetoacoustic wave, where the behavior of the aperture converts this wave into an Alfvén wave. These low frequency Alfvén waves can travel without dissipation, removing energy from the sunspot. The wave incident upon the sunspot chromosphere boundary has assumed impedance $Z_{ph} + Z_{pw}$. Using the assumed average chromospheric properties to determine the aperture impedance, a peak in transmitted power at 3 min. occurs, Fig. 7.6(a), with a phase velocity due to the aperture impedance of 55 km/s for magnetic coupling and 40 km/s for electric dipole coupling. Including the local wave speed in the sunspot, the observed phase velocity may be higher. Observations indicate upward traveling 3 min. oscillations confined to the umbra, with a phase velocity between 50 − 80 km/s and are observed in the transition region above the umbra.

Observations also find running penumbral waves in sunspots, oscillations traveling in the plane of the Sun. These waves start in the umbra and travel out through the penumbra onto the Sun. As these waves move from the umbra outward, the frequency of the waves decreases, ending near 1.5 mHz, or an 8 min. period at the Evershed flow. Given the horizontal orientation of magnetic field lines in the Evershed flow area, these magnetic field lines have average properties determined by the lower chromosphere. This plasma has a lower Alfvén phase velocity than the plasma along umbral field lines. It is expected that the
Figure 7.5. Transmitted power and Alfvén phase velocity from a sunspot assuming aperture impedance using wave speed in chromosphere for electric (solid) and magnetic (dashed) dipole coupling.
aperture impedance in this area changes due to this lower average phase velocity. Assuming a 50 km/s phase velocity [45] when determining the aperture impedance, oscillations near 8 min. are predicted. Magnetic coupling has a phase velocity of 20 km/s and 30 km/s for electric coupling. If the average phase velocity for aperture impedance is 80 km/s, the peak in Alfvén velocity occurs near 5 min., with a phase velocity of 40 km/s for both electric and magnetic coupling. The observed changes in frequency as a wave travels across a sunspot may reflect the changes in the average plasma properties of the magnetic field lines in the sunspot.

Consider an ideal source of power for the temperature in the photosphere, dominated by a single frequency. Further, suppose that all power reflected from the photosphere-chromosphere interface is perfectly converted into heat. Using these assumptions, a simple prediction for the temperature in the umbra can be made. The power radiated by a perfect black body is given by

\[ P = \sigma T^4 \]  \hspace{1cm} (7.1)

where \( \sigma \) is a constant. Since 90\% of the incident power is reflected from the photosphere-chromosphere boundary without the sunspot, the incident power can be computed assuming a constant temperature of 5800K.

\[ P_{inc} = \frac{\sigma(5800)^4}{.9} \]  \hspace{1cm} (7.2)

Only 15\% is reflected with the sunspot for magnetic dipole coupling at 3 min., leading to a minimum umbral temperature

\[ T = \left( \frac{.15P_{inc}}{\sigma} \right) \frac{4}{3} \approx 3750 \text{ K} \]  \hspace{1cm} (7.3)

and 4150 K for electric dipole coupling. Observations indicate the photospheric 3 min. power is at a minimum in the umbra. However, above the umbra in the chromosphere, there is a maximum in 3 min. power [44]; 3 min. power is also observed in the transition region above the umbra [6, 5]. These observations may reflect the conversion of acoustic power in the sunspot to an Alfvén wave, which is then observed in the chromosphere and transition.
Figure 7.6. Transmitted power and Alfvén phase velocity from a sunspot assuming aperture impedance using average wave speed of 50 km/s for electric (solid) and magnetic (dashed) dipole coupling.
region. Assuming a power source at 5 min., a temperature of 5000 K is obtained for magnetic coupling and 4200 K for electric coupling. For the effective aperture behavior with a peak at 8 min. for Alfvén wave oscillations, a source of power at 5 min. yields 5150 K for electric dipole coupling and 5350 K for magnetic coupling.

The association of a sunspot with an aperture may also motivate the magnetic field structure in the penumbra. It is found that the penumbra consists of interwoven alternating bright/dark magnetic fields with the visibly dark lines more horizontal than the comparably brighter lines [80]. The magnetic dipole produced by an aperture leads to horizontal magnetic fields outside of the aperture, Fig. 6.1. These fields in the source region are in phase with the incident field, while the fields on the opposite side are out of phase.

Consider a sunspot magnetic field pointing out of the Sun. Since the sunspot will have a outward radial magnetic component in the umbra, it is expected that the magnetic field outside the sunspot associated with the source region will also have an outward radial component. On the chromosphere side of the umbra, the fields outside the spot will have an inward radial component. Added to the background sunspot field, the magnetic field associated with the source region will be more horizontal, while fields associated with the chromospheric side of the aperture will be closer to normal. The interwoven structure observed in the penumbra would allow for the oppositely directed image currents expected on both sides of the aperture to be distributed in an area around the sunspot.

If the magnetic fields in the sunspot point into the sun, the aperture acts out of phase, and the same field distribution found outside the sunspot for an outward pointing magnetic field is expected. While flux pumping has been proposed to maintain the field configuration in the penumbra, the author notes that it does not indicate how the structure initially forms [80].
7.3 Conclusion

The similarity of the magnetic structure in the Earth’s polar regions to sunspots and coronal holes on the Sun motivates the use of the proposed aperture models on the Sun as well. The presented model predicts a peak in transmitted power across the transition region at a 5 min. period with an averaged power density sufficient to heat the quiet corona, and drive the solar wind. Two phase velocities are expected from the effective aperture behavior, with minimum speeds at 760 and 50 km/s. The faster phase speed is consistent with the fast solar wind. Applied to sunspots, it is postulated that acoustic waves are converted to Alfvén wave energy. This conversion is postulated to be driven by Hall currents excited by acoustic oscillations of gas in the sunspot. The Hall currents lead to a dipole field in the umbra, radiating Alfvén waves and removing energy from the system. Assuming a 5 min. power source, a variation in temperature across the sunspot is predicted, ranging between an umbral temperature of 4200 K and penumbral temperatures near 5300 K. The period of these waves also varies, a 3 min. peak is predicted in the umbra which increases towards 8 min. in the outer penumbra, consistent with observations.

The successful application of aperture behavior to three different scale sizes and plasma environments between the Earth and Sun supports the postulated equivalence to aperture behavior. Because the requirements for effective aperture behavior are generally satisfied by objects with a generally dipolar magnetic field in a plasma environment, common throughout the universe, effective aperture behavior is expected for these objects as well.
REFERENCES


[58] K Mursula, T Bräysy, R Rasinkangas, and P Tanskanen. A modulated multiband 
PC1 event observed by polar/EFI around the plasmapause. *Advances in Space Research*, 


[60] O Nordhaus and J Pelzl. Frequency dependence of resonant photoacoustic cells: The 


[62] S Ohnuki, T Hinata, and T Hosono. Resonant characteristics of parallel plate waveguide 

[63] E Okon and R Harrington. The polarizabilities of electrically small apertures of arbitrary 

[64] H Ostner and E Biebl. Planar slot antennas backed by a ground plane. *Antennas and 


[66] E Pekunlu, Z Bozkurt, M Afsar, and E Soydugan. Alfvén waves in the inner polar 
2002.

[67] B De Pontieu, S. W Mcintosh, M Carlsson, V. H Hansteen, T. D Tarbell, C. J Schrijver, 
A. M Title, R. A Shine, S Tsuneta, Y Katsukawa, K Ichimoto, Y Suematsu, T Shimizu, 
and S Nagata. Chromospheric alfvénic waves strong enough to power the solar wind. 

[68] K Prikner, K Mursula, T Bosingher, F Feygin, and T Raita. The effective altitude 
range of the ionospheric alfvén resonator studied by high-altitude eiscat measurements. 

[69] Schroeder M. R. Normal frequency and excitation statistics in rooms: Model experiments 

[70] M Rahim, Z Low, P Soh, and A Asrokin. Aperture coupled microstrip antenna with 
different feed sizes and aperture positions. *RF and Microwave Conference*, pages 31–35, 
Jan 2006.


VITA

Born in Thousand Oaks, CA on December 1st, 1979, Russell Stoneback moved with his parents, Lew and Elizabeth Stoneback, to Austin, TX on Thanksgiving Day 1991. He attended Westwood High school in Austin and was a member of the TX-861st Air Force Junior ROTC for three years. A missed financial deadline for the second semester at the University of Texas at Austin lead to a spring and summer spent maintaining lawns for the family business, work that eventually expanded to include general remodeling. Newly obtained construction skills, an emerging interest in making music, and recently obtained physics knowledge led to a desire to build a better guitar. After graduating in May 2003, he helped build his parent's house and worked for a brief time at Ander Laboratory. He entered graduate school in August, 2004 at the University of Texas at Dallas.

Permanent address: 7760 McCallum Blvd, Apt. 16107 Dallas, TX 75252