

An Investment Problem

Suppose an investor has \$100 on Monday. At the start of *every* day of the week (Monday through Friday), the investor has the following investment opportunity available:

If he invests x dollars on that day and matches that initial investment with $x/2$ dollars the next day, then he will receive a total return of $2x$ dollars on the third day.

Thus, with a total investment of $1.5 \times x$ dollars, the investor receives $2x$ dollars in two days, a gain of $0.5 \times x$ dollars. The investor wishes to determine an investment schedule that maximizes his total cash on Saturday.

To facilitate the formulation of a linear program, the investor decides to make the following simplifying assumptions:

1. If an initial investment is not matched on the subsequent day, the initial investment is lost.
2. Any return that is due on any given day can be reinvested immediately.
3. Cash carried forward from one day to the next does not accrue interest.
4. Borrowing money is not allowed.

To understand things better, let us consider the following “naive” strategy: Begin with an investment of $100 \times (2/3)$ dollars on Monday, while putting aside $100 \times (1/3)$ dollars in anticipation of the necessary second installment on the next day (Assumption 1). On Tuesday, the investor executes the second installment and, consequently, he won’t have any remaining cash to initiate a new investment. On Wednesday, the investor receives a total return of $2 \times 100 \times (2/3) = 100 \times (4/3)$ dollars. This completes an investment cycle, during which the total amount invested in two installments (\$100) grew by a factor of $4/3$. Suppose further that the investor immediately reinvests the yield he receives on Wednesday (Assumption 2) in the same manner as in the just-completed investment cycle. Then, a similar analysis shows that he will receive a total yield of $100 \times (4/3) \times (4/3) = 177.7778$ dollars on Friday. At that point, since any new investment that starts on Friday won’t mature until Sunday (a day too late), the investor simply carries this second yield into Saturday, resulting in a final cash position of 177.7778 dollars.

How good is this naive strategy? While we managed to complete two full investment cycles, it seems uncomfortable to watch cash sit idle from Friday to Saturday. This suggests that we might be able to do better. But how? Note that under the divisibility assumption, the set of possible strategies is a continuum, and hence it cannot even be enumerated. Thus, the

task is challenging. To find out what is the best strategy, we now turn to an LP formulation of this problem.

The Decision Variables

Since a new investment cycle can, potentially, be initiated at the start of each day, the investor needs to determine the magnitudes of these first installments. Therefore, let

x_j = the amount of new investment at the beginning of Day j , $j = 1, 2, 3, 4$,

where “Day 1” corresponds to Monday, “Day 2” to Tuesday, and so on. Note that we did not introduce “ x_5 ” since it is unwise to initiate a new investment cycle on Friday. We also assume that these decisions are to be made immediately *after* receiving yields (if any) from prior investments.

In principle, it seems sufficient (and it is) to work with these four decision variables alone: On Monday, we would invest x_1 dollars and carry a cash saving of $100 - x_1$ forward to Tuesday. On Tuesday, after executing a second installment of $x_1/2$ dollars, we would have $100 - x_1 - x_1/2$ dollars available for an allocation of a new investment and a new cash saving. A little bit of reflection should convince oneself that the picture would become rather complex as we move further into future days. In particular, expressing the amount of cash saving at the end of each day as a function of the entire *history* of past decisions quickly becomes a difficult task.

From the previous production-planning example, we learned that it is sometimes desirable to introduce additional sets of decision variables for the purpose of simplifying the formulation task. Now, imagine yourself being at the start of a given day and ask: What actions do I need to take at this point? The answer is:

1. Cough up half of the amount of new investment (if any) that started in the previous day.
2. Initiate a new investment.
3. Carry the remaining cash (if any) forward to the next day.

Observe that these actions cannot be committed unless we know how much saving is being carried forward from the previous day; and that this information depends on the entire prior investment history in a complicated way. To circumvent this difficulty, it therefore seems desirable to define a new set of variables to represent the daily savings. Formally, let

s_j = the amount of saving carried forward from Day j to Day $j + 1$, $j = 1, 2, 3, 4, 5$.

Note that we can think of the initial capital as a saving from Day 0 to Day 1; that is, let $s_0 \equiv 100$.

In summary, we have defined a total of 9 decision variables, four x_j 's and five s_j 's.

The Objective Function

On Saturday (Day 6), there are two income streams; one is the yield from the investment cycle that started on Thursday and the other is the saving from Friday. The first contribution equals $2x_4$ and the second, s_5 . It is important to realize that yields from all earlier investments are “implicitly captured” into these two terms. Thus, our objective is to:

$$\text{Maximize } 2x_4 + s_5 .$$

Note the simplicity of this objective function. It is a consequence of our “clever” choice of the decision variables. In general, it is a good practice to conceptualize the formulation of the decision variables and the objective function jointly.

The Constraints

In the alternative formulation of the production-planning problem, the production levels and the ending inventory levels were linked together via constraints like $y_j = y_{j-1} + x_j - d_j$ to ensure that they logically correspond to their intended interpretations. This can be viewed as “material” balancing. Here again, we need to ensure proper linkage between the daily new investments and the daily savings.

The basic idea is to balance the cash flow at the beginning of each day. For Day 1, we have $s_0 = 100$ dollars available; and this amount is apportioned into two parts: a new investment and a saving. Thus,

$$s_0 = x_1 + s_1 .$$

For Day 2, we have s_1 dollars available; and this is apportioned into three parts: $x_1/2$, x_2 , and s_2 . Thus,

$$s_1 = 0.5x_1 + x_2 + s_2 .$$

At the beginning of Day 3, we receive a yield of $2x_1$, which is immediately available for reinvestment. It is therefore added into s_2 ; and the total amount is then divided into three parts as in Day 2. This leads to

$$2x_1 + s_2 = 0.5x_2 + x_3 + s_3 .$$

Continuation of this argument yields

$$2x_2 + s_3 = 0.5x_3 + x_4 + s_4$$

for Day 4 and, finally,

$$2x_3 + s_4 = 0.5x_4 + s_5$$

for Day 5. Note that the term “ x_5 ” does not appear in the last equation. (What if we didn’t exclude this variable?)

Clearly, the x_j ’s must be nonnegative. To ensure that we never over spend, the s_j ’s are required to be nonnegative as well.

LP Formulation

In summary, we have arrived at the following formulation:

$$\begin{aligned} \text{Maximize} \quad & 2x_4 + s_5 \\ \text{Subject to:} \quad & \\ & s_0 = x_1 + s_1 \\ & s_1 = 0.5x_1 + x_2 + s_2 \\ & 2x_1 + s_2 = 0.5x_2 + x_3 + s_3 \\ & 2x_2 + s_3 = 0.5x_3 + x_4 + s_4 \\ & 2x_3 + s_4 = 0.5x_4 + s_5 \\ & x_j \geq 0 \text{ for } j = 1, 2, 3, 4 \text{ and } s_j \geq 0 \text{ for } j = 1, 2, 3, 4, 5. \end{aligned}$$

This is a linear program with 9 decision variables, 5 functional constraints, and 9 nonnegativity constraints.

How good is the naive strategy proposed earlier? We will determine the optimal solution to this problem a little bit later, using LINDO.

Discussion

It may be instructive to attempt to formulate this problem using the x_j ’s only. Give it a try; it would be quite messy.

If we did not realize that x_5 was unnecessary, then the fifth constraint would have come out as $2x_3 + s_4 = 0.5x_4 + x_5 + s_5$. Consider an investment strategy that prescribes, say, $x_5 = 5$ and $s_5 = 10$. We will argue that this strategy cannot be optimal. Observe that the variable x_5 appears only in the fifth constraint. Therefore, if we simply reset x_5 to 0 and s_5 to 15 (maintaining the sum $x_5 + s_5$ at 15), then the resulting new strategy will have a better objective function value (by how much?). This shows that it is never optimal to assign a positive value to x_5 .

We assumed that there is no reinvestment delay (Assumption 2). This assumption can be easily relaxed. Suppose instead that there is a reinvestment delay of one day. This implies,

for example, that the yield $2x_1$ derived from the new investment on Day 1 won't be available until Day 4. Therefore, we should delete the term $2x_1$ from the left-hand side of the third constraint and transfer this term to that of the fourth constraint. This results in revised constraints $s_2 = 0.5x_2 + x_3 + s_3$ and $2x_1 + s_3 = 0.5x_3 + x_4 + s_4$. After making a similar revision in the fifth constraint, we arrive at the following new formulation:

$$\begin{array}{ll}
\text{Maximize} & 2x_4 + s_5 \\
\text{Subject to:} & \\
& s_0 = x_1 + s_1 \\
& s_1 = 0.5x_1 + x_2 + s_2 \\
& s_2 = 0.5x_2 + x_3 + s_3 \\
& 2x_1 + s_3 = 0.5x_3 + x_4 + s_4 \\
& 2x_2 + s_4 = 0.5x_4 + s_5 \\
& x_j \geq 0 \text{ for } j = 1, 2, 3, 4 \text{ and } s_j \geq 0 \text{ for } j = 1, 2, 3, 4, 5.
\end{array}$$

Note that it is not necessary to revise the objective function to $2x_3 + s_5$.

Assumption 3 can also be relaxed. Suppose the daily interest rate is 1%. Then, our (original) formulation revises to:

$$\begin{array}{ll}
\text{Maximize} & 2x_4 + 1.01s_5 \\
\text{Subject to:} & \\
& s_0 = x_1 + s_1 \\
& 1.01s_1 = 0.5x_1 + x_2 + s_2 \\
& 2x_1 + 1.01s_2 = 0.5x_2 + x_3 + s_3 \\
& 2x_2 + 1.01s_3 = 0.5x_3 + x_4 + s_4 \\
& 2x_3 + 1.01s_4 = 0.5x_4 + s_5 \\
& x_j \geq 0 \text{ for } j = 1, 2, 3, 4 \text{ and } s_j \geq 0 \text{ for } j = 1, 2, 3, 4, 5,
\end{array}$$

where every daily saving grows by a factor of 1.01 overnight.

If borrowing is allowed (see Assumption 4), we can simply remove the nonnegativity requirements for the s_j 's. For realism, we should, however, introduce some limits on loans, and incorporate that into the formulation in the form of additional constraints. Otherwise, one would attempt to borrow an infinite amount of money.