COMPARING ALTERNATING RENEWAL PROCESSES

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ABSTRACT

Sufficient conditions are given for stochastic comparison of two alternating renewal processes based on the concept of uniformization. The result is used to compare component and system performance processes in maintained reliability systems.

1. INTRODUCTION AND SUMMARY

Comparison of stochastic processes has been a rapidly growing area of research. In this paper, we will study alternating renewal processes (ARP) $X = \{X(t), t \geq 0\}$ where the state space $S = \{0, 1\}$ and the holding times of the process in state 1 and 0 are independent random variables having distribution functions $F$ and $G$. Throughout this paper, we assume $F$ and $G$ are absolutely continuous with failure rate functions $r(t)$ and $q(t)$, respectively. We shall denote such a process by $(X, r(t), q(t))$. Similar notations will be used throughout.

Let $X = \{X(t), t \in T\}$ and $Y = \{Y(t), t \in T\}$ be two stochastic processes. We say $X$ is stochastically larger than $Y$, denoted by $X \succeq Y$, iff $E f(X) \geq E f(Y)$ for all nondecreasing functionals $f$ for which the expectations exist. If $X$ and $Y$ have the same distribution, then we write $X = Y$. In a recent paper, Sonderman [8] presented a set of sufficient conditions such that stochastic comparison between two semi-Markov processes can be made. By specializing his conditions to the case of alternating renewal processes, Sonderman (Theorem 5.1 of [8]) obtained the following result.

THEOREM 1 (Sonderman): Let $(X', r_i(t), q_i(t)), i = 1, 2,$ be two alternating renewal processes. Assume that time 0 is a renewal point for both processes and

(a) $X^1(0) \leq X^2(0),$

(b) $r_1(u) \geq r_2(v),$


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(c) \( q_1(u) \leq q_2(v) \),
for all \( u, v \geq 0 \), then there exist two ARP's \( \hat{X}^1 \) and \( \hat{X}^2 \) defined on the same probability space \( \Omega \) such that \( \hat{X}^1 = X^i, i = 1, 2 \), and \( \hat{X}^1 \leq \hat{X}^2 \) everywhere in \( \Omega \).

The purpose of this note is to show that conditions (b) and (c) in Theorem 1 can be weakened to

(b') \( r_1(u) \geq r_2(v) \) whenever \( u \leq v \),

(c') \( q_1(v) \leq q_2(u) \) whenever \( u \leq v \).

The proof of this result and two immediate corollaries will be presented in Section 2. Section 3 contains some remarks on the main results.

2. PATHWISE COMPARISON OF ALTERNATING RENEWAL PROCESSES

We shall start by describing a construction due to Sonderman [8] which reproduces an alternating renewal process \( (X, r(t), q(t)) \) based on a Poisson process. In order to do that, the following technical assumption on \( r(t) \) and \( q(t) \) is needed.

ASSUMPTION: The alternating renewal process \( (X, r(t), q(t)) \) is assumed to be uniformizable, i.e., there exists a real number \( \lambda < \infty \) such that \( \sup_{t \geq 0} \{r(t), q(t)\} \leq \lambda \). \( \lambda \) is called the uniformization rate.

As discussed in Sonderman [8, pp. 113-115], this condition can be relaxed to the case where failure rates are uniformly bounded over finite intervals. Let \( \lambda \) be the uniformization rate of \( X \), the construction can be separated into two steps. First, a Poisson process with rate \( \lambda \) generates a sequence of potential transition epochs \( \{t_i, i \geq 0\} \), where \( t_0 \equiv 0 \). Then a discrete time stochastic process is constructed on \( \{t_i, i \geq 0\} \), determining whether each potential transition epoch is a genuine transition and, if so, the new state of the process. Specifically, let \( \{(S_n, J_n), n \geq 0\} \) be a sequence of ordered pairs of integer-valued random variables, where \( S_n \) has the value 1 or 0 representing the state of the process immediately after \( t_n \). We assume a genuine transition occurs at \( t = 0 \), i.e., \( J_0 = 0 \). The initial state \( S_0 = X(0) \) could either be given or have an initial probability distribution. The transition probabilities of \( \{(S_n, J_n), n \geq 0\} \) are defined as:

\[
\begin{align*}
P(S_n = 0 | S_{n-1} = 1, J_{n-1} = m, t_i, i \geq 0) &= r(t_n - t_m) / \lambda \\
P(S_n = 1 | S_{n-1} = 0, J_{n-1} = m, t_i, i \geq 0) &= q(t_n - t_m) / \lambda \\
P(S_n = S_{n-1}, J_n = J_{n-1} | S_{n-1}, J_{n-1}, t_i, i \geq 0) &= 1 - P(J_n = n | S_{n-1}, J_{n-1}, t_i, i \geq 0) \text{ for } 0 \leq m < n.
\end{align*}
\]

Finally, define a new process \( \hat{X} = \{\hat{X}(t), t \geq 0\} \) by

\[
\hat{X}(t) = S_n \quad \text{if} \quad t_n \leq t < t_{n+1}.
\]

Then it follows from Theorem 2.1 of Sonderman [8] that \( \hat{X} = X \).

We will need the following lemma from Arjas and Lehtonen ([1], Lemma 3). See also Theorem 3.1 of [8].
LEMMA 1: Let \( X = \{X_n, \ n \geq 0\}, \ Y = \{Y_n, \ n \geq 0\}, \) and \( Z = \{Z_n, \ n \geq 0\} \) be three discrete time stochastic processes. Suppose that

(a) \( (X_0 | Z_n = z_n, \ n \geq 0) \overset{st}{\leq} (Y_0 | Z_n = z_n, \ n \geq 0) \)

and (b) \( (X_j | X_0 = x_0, \ldots, X_{j-1} = x_{j-1}, Z_n = z_n, \ n \geq 0) \overset{st}{\leq} (Y_j | Y_0 = y_0, \ldots, Y_{j-1} = y_{j-1}, Z_n = z_n, \ n \geq 0) \)

whenever \( x_i \leq y_i, \ 0 \leq i \leq j - 1, \) for all \( j \geq 1. \) Then there exist two stochastic processes \( \hat{X} = \{\hat{X}_n, \ n \geq 0\} \) and \( \hat{Y} = \{\hat{Y}_n, \ n \geq 0\} \) defined on the same probability space such that \( \hat{X} \overset{st}{\leq} X, \hat{Y} \overset{st}{\leq} Y, \) and \( \hat{X} \leq \hat{Y} \) everywhere, hence, \( X \overset{st}{\leq} Y. \)

We are now ready to state and prove the main theorem of this paper.

THEOREM 2: Let \( (X^i, r_i(t), q_i(t)), \ i = 1, 2, \) be two uniformizable alternating renewal processes. Assume that time 0 is a renewal point for both processes and

(a) \( X^1(0) \overset{st}{\leq} X^2(0), \)

(b') \( r_1(u) \geq r_2(v) \) whenever \( u \leq v, \)

(c') \( q_1(v) \leq q_2(u) \) whenever \( u \leq v. \)

then there exist two new processes \( \hat{X}^1 \) and \( \hat{X}^2 \) defined on the same probability space \( \Omega \) such that \( \hat{X}^1 \overset{st}{=} X^1, \hat{X}^2 \overset{st}{=} X^2 \) and \( \hat{X}^1 \overset{st}{\leq} \hat{X}^2 \) everywhere in \( \Omega, \) hence \( X^1 \overset{st}{\leq} X^2. \)

PROOF: The proof is a modification of the one used by Sonderman [8] to prove his Theorem 3.2. Since both processes are Poisson-uniformizable, let \( \lambda \geq 2 \sup_{t \geq 0} \{r_1(t), q_2(t)\}. \) The basic idea of the proof is to generate potential transition epochs for both processes by the same Poisson process. Let \( \{t_n, \ n \geq 0\} \) be a sequence of events generated by a Poisson process with rate \( \lambda. \) In view of Lemma 1, we need only to show that the two discrete time stochastic processes \( \{S^i_0, \ n \geq 0\} \) and \( \{S^2_0, \ n \geq 0\} \) constructed according to (1) and (2) from \( X^1 \) and \( X^2, \) respectively, satisfy the following stochastic order relationships:

\[
(S^1_i | S^1_0 = s^1_0, \ldots, S^1_{j-1} = s^1_{j-1}, t_n, \ n \geq 0) \overset{st}{\leq} (S^2_i | S^2_0 = s^2_0, \ldots, S^2_{j-1} = s^2_{j-1}, t_n, \ n \geq 0)
\]

whenever \( s^1_i \leq s^2_i, \ 0 \leq i \leq j - 1 \) for all \( j \geq 1, \) or equivalently,

\[
P(S^1_i = 1 | S^1_0 = s^1_0, \ldots, S^1_{j-1} = s^1_{j-1}, t_n, \ n \geq 0) \overset{st}{\leq} P(S^2_i = 1 | S^2_0 = s^2_0, \ldots, S^2_{j-1} = s^2_{j-1}, t_n, \ n \geq 0)
\]

whenever \( s^1_i \leq s^2_i, \ 0 \leq i \leq j - 1 \) for all \( j \geq 1. \)

Suppose \( (s^1_0, \ldots, s^1_{j-1}) \overset{st}{\leq} (s^2_0, \ldots, s^2_{j-1}), \) and let \( J^1_{j-1} = k^1 \) and \( J^2_{j-1} = k^2, \) where \( 0 \leq k^1 \leq j - 1 \) and \( 0 \leq k^2 \leq j - 1. \)
CASE 1: Suppose $s_{j-1} = 1$, hence, $s_j = 1$.

In this case, $k^1 \geq k^2$ and $t_j - t_{k^1} \leq t_j - t_{k^2}$. Then by (1) and condition (b'),

left hand side of (3) = $1 - r_1(t_j - t_{k^1})/\lambda \leq 1 - r_2(t_j - t_{k^2})/\lambda =$ right-hand side of (3).

CASE 2: Suppose $s_{j-1} = 0$ and $s_j = 1$.

l.h.s. of (3) = $q_1(t_j - t_{k^1})/\lambda \leq \frac{1}{2} \leq 1 - r_2(t_j - t_{k^2})/\lambda =$ r.h.s. of (3)

CASE 3: Suppose $s_{j-1} = s_j = 0$.

In this case, $k^1 \leq k^2$ and $t_j - t_{k^1} \geq t_j - t_{k^2}$. Then from (1) and condition (c'), we have

l.h.s. of (3) = $q_1(t_j - t_{k^1})/\lambda \leq q_2(t_j - t_{k^2})/\lambda =$ r.h.s. of (3).

The conclusion of the theorem now follows from Lemma 1 since

$S_0^1 = X^1(0) \leq X^2(0) = S_0^2$.

Q.E.D.

The following corollaries are immediate.

COROLLARY 1: Conditions (a), (b'), and (c') in Theorem 2 can be replaced by

(i) $X^1(0) \leq X^2(0)$,

(ii) $r_1(t)$ or $r_2(t)$ is nonincreasing in $t$,

(iii) $q_1(t)$ or $q_2(t)$ is nonincreasing in $t$,

(iv) $r_1(t) \geq r_2(t)$ and $q_1(t) \leq q_2(t)$ for all $t \geq 0$.

PROOF: Suppose $u \leq v$. If $r_1(t)$ is nonincreasing, then $r_1(u) \geq r_1(v) \geq r_2(v)$. If $r_2(t)$ is nonincreasing, then $r_1(u) \geq r_2(u) \geq r_2(v)$. Hence, in either case, condition (b') of Theorem 2 is satisfied. Condition (c') can be checked in similar fashion.

Q.E.D.

COROLLARY 2: Let $(X, r(t), q(t))$ be a uniformizable alternating renewal process. Then there exist two alternating renewal processes $(Y, r_y(t), q_y(t))$ and $(Z, r_z(t), q_z(t))$, where

$r_z(t) = \sup_{0 \leq s < t} r(s), \quad q_z(t) = \inf_{0 \leq s \leq t} q(s),

r_y(t) = \inf_{0 \leq s \leq t} r(s), \quad q_y(t) = \sup_{t \leq s < \infty} q(s),

such that $X$ is bounded stochastically from below by $Z$ and from above by $Y$. 
PROOF: Clearly the functions \( r_Z(t), q_Z(t), r_Y(t), \) and \( q_Y(t) \) are non-increasing in \( t \). Therefore, the conclusion is a direct consequence of Corollary 1. Q.E.D.

3. COMMENTS AND ADDITIONS

(1) In Theorem 2, the assumption that time 0 is a renewal point for both processes can be relaxed. It is sufficient to assume that at time 0, if both processes are in state 1, then \( X^2 \) has been in state 1 longer than \( X^1 \), and if both processes are in state 0, then \( X^1 \) has been in state 0 longer than \( X^2 \).

(2) In a loose sense, the processes \( Z \) and \( Y \) in Corollary 2 may be viewed as the greatest lower bound and least upper bound, respectively, for process \( X \) within the class of alternating renewal processes whose holding times in both states are DFR (decreasing failure rate).

(3) An alternating renewal process may be used to model the performance of a repairable component in a maintained reliability system (see [3] or Chapter 6 of [2]). The successive operating (or repair) times of a repairable component are assumed to be independent and identically distributed random variables. All components operate independently of one another. Let \( X(t) \) be the state of a component at time \( t \), where

\[
X(t) = \begin{cases} 
1 & \text{if the component is up at time } t, \\
0 & \text{otherwise,}
\end{cases}
\]

then \( X = \{X(t), t \geq 0\} \) is an alternating renewal process. Therefore, Theorem 2 may be used to compare the performance of two maintained reliability systems consisting of \( n \) repairable components. Specifically, let \( \phi \) be a coherent structure function (see [2]) and \( X_i = \{X_i(t), t \geq 0\} \) be the performance process of the \( i \)th component in \( j \)th systems, where \( i = 1, 2, \ldots, n, j = 1, 2 \). Define \( X^{\prime}_j(t) = (X^{\prime}_1(t), \ldots, X^{\prime}_j(t)) \), \( j = 1, 2 \). By forming the product of probability spaces for individual components, the following result follows directly from Theorem 2.

PROPOSITION 1: Suppose that

(i) \( X^{\prime}_i(0) \leq X^{\prime}_j(0) \) for all \( i = 1, \ldots, n \).

(ii) All component performance processes are uniformizable and the failure rates satisfy the conditions of Theorem 2.

Then there exist two stochastic processes \( \hat{\phi}^1 \) and \( \hat{\phi}^2 \) defined on the same probability space \( \Omega \) such that \( \hat{\phi}^1_{\text{st}} = \{\phi(X^{\prime}_1(t)), t \geq 0\}, \hat{\phi}^2_{\text{st}} = \{\phi(X^{\prime}_2(t)), t \geq 0\}, \) and \( \hat{\phi}^1 \leq \hat{\phi}^2 \) everywhere in \( \Omega \). Hence, \( \{\phi(X^{\prime}_1(t)), t \geq 0\} \leq \{\phi(X^{\prime}_2(t)), t \geq 0\} \).

(4) It is interesting to point out that an example of Miller [5, example (ii), p. 308] shows that increasing the failure rate of downtime distribution of a component does not necessarily increase (stochastically) the time to first system failure or system availability. Our result (see Corollary 1) shows that for systems whose repairable components have DFR uptime and downtime distributions, decreasing the failure rates of uptime distributions and increasing the failure rates of downtime distributions do improve the system performance.
Theorem 2 may be used to establish bounds for performance measures of maintained reliability systems. For example, one can bound the performance process of a repairable component by that of a component whose uptime and downtime distributions are exponential (This is a special case of Corollary 1 here or Theorem 5.1 of [8]). Maintained systems with exponential uptime and downtime distributions has been discussed in Brown [4, Ross [6 and 7]. However, the bounds obtained in this fashion are usually quite loose. Finally, we present the following example to illustrate the ideas involved:

**EXAMPLE:** Consider a two-component parallel system. Let $F(G)$ be the uptime (downtime) distribution of component 1 and $\lambda(\mu)$ be the constant failure (repair) rate for component 2. Assume the system starts operation with both components new. Suppose we are interested in the expected time until first system failure, $E(T_0)$. By conditioning on the state of the second component when component 1 fails for the first time, it is not difficult to see that

$$E(T_0) = \int_0^\infty t dF(t) + \left[ \int_0^\infty P_{11}(t)dF(t) \right] \cdot \left[ E(\min[D, U]) + \left( \int_0^\infty e^{-\lambda y} dG(y) \right) E(T_0) \right]$$

where $D(U)$ is a random variable having distribution $G$ (exponential distribution with parameter $\lambda$) and $P_{11}(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda + \mu) t}$. After some simplification, we have

$$E(T_0) = \frac{\int_0^\infty t dF(t) + \left[ \int_0^\infty P_{11}(t)dF(t) \right] \cdot \left[ \int_0^\infty e^{-\lambda y}(1 - G(y))dy \right]}{1 - \left[ \int_0^\infty P_{11}(t)dF(t) \right] \cdot \left[ \int_0^\infty e^{-\lambda y} dG(y) \right]} \equiv h(F, G; \lambda, \mu).$$

Therefore, we can find bounds for $E(T_0)$ for a two-component parallel system whose first component has the same performance process as above and the second component performance process is uniformizable with failure rate function $\lambda(t)$ and repair rate function $\mu(t)$, $t \geq 0$. Specifically, let $\lambda = \sup_{t \geq 0} \{\lambda(t)\}$, $\overline{\lambda} = \inf_{t \geq 0} \{\lambda(t)\}$, $\lambda = \sup_{t \geq 0} \{\mu(t)\}$, and $\mu = \inf_{t \geq 0} \{\mu(t)\}$, then

$$h(F, G; \lambda, \mu) \leq E(T_0) \leq h(F, G; \lambda, \mu).$$

**REFERENCES**