# MULTIVARIATE SYMMETRY AND ASYMMETRY

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**Abstract. Univariate symmetry has interesting and diverse forms of generalization to the multivariate case. Here several leading concepts of multivariate symmetry—spherical, elliptical, central, and angular—are examined and various closely related notions discussed. Methods for testing the hypothesis of symmetry and approaches for measuring the direction and magnitude of skewness are reviewed.**

**Keywords and Phrases. multivariate, symmetry, skewness, asymmetry.**

**Blind Entry.**

**AMS Subject Classification. Primary: 62H99; Secondary: 62G99**

## CONCEPTS OF SYMMETRY

The idea of "symmetry" has served from ancient times as a conceptual reference point in art and mathematics and in their diverse applications. In aesthetics, it is a principle of order, in mathematics an artifact of geometric structure, in philosophy an abstraction of balance and harmony and perfection, in poetry an intuitive essence of nature and divinity. Weyl [52] has created a delightful and wide-ranging treatment of "symmetry," from bilateral symmetry in Greek sculpture to Kant's metaphysical pondering of the problem of left and right to the description of crystalline structure in nature by modern group theory.

Here we focus on the notion of symmetry as it relates to multivariate probability distributions in statistical science. Even in this specialized context, there are many variations on the theme. One can seek to define useful classes of distributions that extend the multivariate normal distribution, or one can formulate multivariate generalizations of particular univariate distributions such as the exponential. One can define symmetry in terms of structural properties of the distribution function, or of the characteristic function, or of the density function. One may impose invariance of the distribution of a random vector with respect to specified groups of transformations. A useful introduction to these and other approaches is provided by Fang et al. [24]. Other general sources are References 43 and 23.

A number of widely used examples of multivariate symmetry conveniently may be expressed in terms of invariance of the distribution of a "centered" random vector $\boldsymbol{X} - \boldsymbol{\theta}$ in $\mathbb{R}^d$ under a suitable family of transformations. In increasing order of generality, these are *spherical*, *elliptical*, *central*, and *angular* symmetry, all of which reduce to the usual notion of symmetry in the univariate case. Below, we provide some perspectives on these and closely related notions of multivariate symmetry.

**Spherical Symmetry**

A random vector $\boldsymbol{X}$ has a distribution *spherically symmetric* about $\boldsymbol{\theta}$ if rotation of $\boldsymbol{X}$ about $\boldsymbol{\theta}$ does not alter the distribution:

$$\boldsymbol{X} - \boldsymbol{\theta} \stackrel{\mathrm{d}}{=} \boldsymbol{A}(\boldsymbol{X} - \boldsymbol{\theta}) \tag{1}$$

for all orthogonal $d \times d$ matrices $\boldsymbol{A}$, where "$\stackrel{\mathrm{d}}{=}$" denotes "equal in distribution." In this case, $\boldsymbol{X}$ has a characteristic function of the form $\mathrm{e}^{i\boldsymbol{t}'\boldsymbol{\theta}}\, h(\boldsymbol{t}'\boldsymbol{t})$, $\boldsymbol{t} \in \mathbb{R}^d$, for some scalar function $h(\cdot)$, and a density, if it exists, of the form $g((\boldsymbol{x}-\boldsymbol{\theta})'(\boldsymbol{x}-\boldsymbol{\theta}))$, $\boldsymbol{x} \in \mathbb{R}^d$, for some nonnegative scalar function $g(\cdot)$.

Among spherically symmetric distributions are not only multivariate normal distributions with covariance matrices of form $\sigma^2 \boldsymbol{I}_d$ but also, for example, certain cases of standard multivariate $t$ and logistic distributions (see Ref. 34, p. 34 and 573). In particular, the standard $d$-variate $t$-distribution with $m$ degrees of freedom, denoted $T(m, \mathbf{0}, \boldsymbol{I}_d)$, is defined as the distribution of $m^{1/2}\boldsymbol{Z}/s$, with $\boldsymbol{Z}$ standard $d$-variate normal and $s$ independently distributed as chi-square with $m$ degrees of freedom.

An important result from Thomas [50] is that in the univariate general linear model $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, the usual $t$ and $F$ statistics for inference on $\boldsymbol{\beta}$ have unchanged null distributions across all spherically symmetric distributions for the sample $\boldsymbol{\epsilon}_i$'s. See Reference 19 for general discussion.

There are interesting and useful characterizations of spherical symmetry. One (see Ref. 20) is that $\|\boldsymbol{X}-\boldsymbol{\theta}\|$ and the corresponding random unit vector $(\boldsymbol{X}-\boldsymbol{\theta})/\|\boldsymbol{X}-\boldsymbol{\theta}\|$ be independent, where $\|\cdot\|$ stands for Euclidean norm, and that $(\boldsymbol{X}-\boldsymbol{\theta})/\|\boldsymbol{X}-\boldsymbol{\theta}\|$ be distributed uniformly over $\mathbb{S}^{d-1}$, the unit spherical shell of $\mathbb{R}^d$. (Here $\boldsymbol{x}/\parallel \boldsymbol{x} \parallel = 0$ if $\parallel \boldsymbol{x} \parallel = 0$.) Another is that the projections of $\boldsymbol{X}-\boldsymbol{\theta}$ onto lines through the origin have identical univariate distributions.

A characterization in terms of probabilities of half-spaces, and covering certain broader versions of symmetry as well, is provided by Beran and Millar [14]. For

$$\mathcal{A} = \{\text{all orthogonal transformations on } \mathbb{R}^d\}$$

noindent and $\mathcal{A}_0$ any compact subgroup of $\mathcal{A}$, define $\boldsymbol{X}$ to be $\mathcal{A}_0$-*symmetric* about $\boldsymbol{\theta}$ if Equation 1 holds for all $A \in \mathcal{A}_0$. Define half-spaces on $\mathbb{R}^d$ by

$$H(\boldsymbol{s}, t) = \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{s}, \boldsymbol{x} \rangle \leqslant t\},\ \boldsymbol{s} \in \mathbb{S}^{d-1},\ t \in \mathbb{R},$$

with $\langle \cdot, \cdot \rangle$ the inner product on $\mathbb{R}^d$. Then, a distribution is $\mathcal{A}_0$-symmetric if

$$P(H(\boldsymbol{s}-\boldsymbol{\theta}, t)) = P(H(A(\boldsymbol{s}-\boldsymbol{\theta}), t)) \text{ for all } A \in \mathcal{A}_0.$$

Among other applications of this result, the asymptotics of efficient nonparametric estimators of $\mathcal{A}_0$-symmetric distributions are obtained.

As noted above, spherical symmetry may be described easily by the form of the characteristic function. More general notions of symmetry may be defined similarly: a random vector $\boldsymbol{X}$ is $\alpha$-*symmetric* about the origin, $\alpha > 0$, if its characteristic function is of the form $h(|t_1|^\alpha, \ldots, |t_d|^\alpha)$, $\boldsymbol{t} \in \mathbb{R}^d$. This provides a natural way to extend symmetry for univariate stable laws to the multivariate case. The 2-symmetric distributions are the spherically symmetric ones. For any $\alpha$, this reduces in one dimension to the usual symmetry. See Reference 17 for useful development.

**Elliptical Symmetry**

A random vector $\boldsymbol{X}$ has an *elliptically symmetric* (or ellipsoidally symmetric) distribution with parameters $\boldsymbol{\theta}$ and $\boldsymbol{\Sigma}$ if it is an affinely equivalent to that of a spherically symmetric random vector $\boldsymbol{Y}$:

$$\boldsymbol{X} \stackrel{\mathrm{d}}{=} \boldsymbol{A}'\boldsymbol{Y} + \boldsymbol{\theta}, \tag{2}$$

where $\boldsymbol{A}_{k\times d}$ satisfies $\boldsymbol{A}'\boldsymbol{A} = \boldsymbol{\Sigma}$ with $\mathrm{rank}(\boldsymbol{\Sigma}) = k \leqslant d$. The associated characteristic function has the form $\mathrm{e}^{i\boldsymbol{t}'\boldsymbol{\theta}}\, h(\boldsymbol{t}'\boldsymbol{\Sigma}\boldsymbol{t})$ for some scalar function $h(\cdot)$, and the density, if it exists, has the form $|\boldsymbol{\Sigma}|^{-1/2} g((\boldsymbol{x}-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\theta}))$ for some nonnegative scalar function $g(\cdot)$. In the case of $d \times d$ nonsingular $\boldsymbol{A}$, the density may be written as $|\boldsymbol{A}|^{-1} g_0(\boldsymbol{A}^{-1}(\boldsymbol{x}-\boldsymbol{\theta}))$, for a density $g_0$ spherically symmetric about the origin. Thus, for elliptically symmetric distributions, the contours of equal density are elliptical in shape. The family of elliptically symmetric distributions is readily seen to be closed under affine transformations and conditioning. For robustness and nonparametric studies, one way to relax the assumption of multivariate normality while still retaining some specific structure is via the class of elliptically symmetric distributions.

For $\boldsymbol{Y}$ $m$-variate standard normal $N(\boldsymbol{0}, \boldsymbol{I}_m)$, $\boldsymbol{\theta} \in \mathbb{R}^d$, and $\boldsymbol{A} : m\times d$, the relation (2) defines $\boldsymbol{X}$ to be $d$-variate normal $N(\boldsymbol{\theta}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma} = \boldsymbol{A}'\boldsymbol{A}$. Similarly, for $\boldsymbol{Y}$ distributed as $T(m, \boldsymbol{0}, \boldsymbol{I}_d)$, the relation (2) defines $\boldsymbol{X}$ to be a multivariate $t$-distribution with parameters $\boldsymbol{\theta}$ and $\boldsymbol{\Sigma} = \boldsymbol{A}'\boldsymbol{A}$ and $m$ degrees of freedom. Various special subclasses of elliptically symmetric distributions play special roles. For example, the *multiuniform* distributions arise in the statistical analysis of directional data (see Refs. 51 and 49).

For detailed discussion of elliptically symmetric distributions, including complex-valued variates, see Reference 24. Applications to correlational studies are reviewed in Reference 21 and to minimax estimation, stochastic processes, pattern recognition, fiducial inference, and probability inequalities in Reference 19. For a brief overview, *see* ELLIPTICALLY CONTOURED DISTRIBUTIONS.

We may also define spherical and elliptical symmetry for *matrix-valued* random variates. As in Reference 19, an $n \times m$ random matrix $\boldsymbol{X}$ is spherically symmetric in distribution if

$$\boldsymbol{X} \stackrel{\mathrm{d}}{=} \boldsymbol{A}\boldsymbol{X}\boldsymbol{B}$$

for all orthogonal $n \times n$ matrices $\boldsymbol{A}$ and orthogonal $m \times m$ matrices $\boldsymbol{B}$. Elliptically symmetric versions are obtained by affine transformations. A context of application is the multivariate general linear model $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\theta} + \boldsymbol{E}$ under the assumption that $\boldsymbol{E}(n \times m)$ contains $n$ independent and identically distributed spherically symmetric $m$-vectors. For a unified theoretical treatment of characterizations, properties, and inference for random vectors and matrices with elliptically contoured distributions, see Reference 28.

Certain asymmetric distributions closely related to the spherical or elliptically symmetric types have been formulated by multiplying a normal, $t$-, or other density by a suitable skewing factor. These so-called *skew-normal*, *skew-t*, and *skew-elliptical* distributions retain some broad similarity with the multivariate normal distribution, for example unimodality and roughly elliptical contours, but provide greater flexibility in modeling than strictly symmetric extensions. For example, features such as "heavy tails" can be parameterized. This approach began with Azzalini [4, 5] in the univariate case and its extension to the multivariate case is in active current development (see Refs. 1, 2, 6, 7, 8, 16, and 31, for example).

### Central and Sign Symmetry

In general nonparametric multivariate location inference, a broader relaxation of normality assumptions than elliptical symmetry is desired. A random vector $\boldsymbol{X}$ has a distribution *centrally symmetric* (or "reflectively" or "diagonally" or "simply" or "antipodally" symmetric) about $\boldsymbol{\theta}$ if

$$\boldsymbol{X} - \boldsymbol{\theta} \stackrel{\mathrm{d}}{=} \boldsymbol{\theta} - \boldsymbol{X}. \tag{3}$$

The density, if it exists, satisfies $f(\boldsymbol{\theta} - \boldsymbol{x}) = f(\boldsymbol{x} - \boldsymbol{\theta})$, so that Equation 3 represents the most direct nonparametric extension of univariate symmetry. This is equivalent to all of the projections of $\boldsymbol{X} - \boldsymbol{\theta}$ onto lines through the origin having symmetric univariate distributions. To illustrate the difference between spherical and central symmetry, we note that the uniform distribution on a $d$-cube of form $[-c, c]^d$ is centrally, but not spherically, symmetric. In fact, this distribution is *sign-symmetric* about $\boldsymbol{\theta}$: $\boldsymbol{X} - \boldsymbol{\theta} = (X_1 - \theta_1, \ldots, X_d - \theta_d)' \stackrel{\mathrm{d}}{=} (\pm(X_1 - \theta_1), \ldots, \pm(X_d - \theta_d))'$ for all choices of $+, -$.

Note that central symmetry corresponds to $\mathcal{A}_0$-symmetry with $\mathcal{A}_0$ consisting of just the identity transformation and its negative, and the above-mentioned result of Reference 14 yields that $\boldsymbol{X}$ is *centrally symmetric* about $\boldsymbol{\theta}$ if and only if $P(\boldsymbol{X}-\boldsymbol{\theta} \in H)$ $= P(\boldsymbol{X} - \boldsymbol{\theta} \in -H)$ for each closed half-space $H \subset \mathbb{R}^d$. An equivalent alternative criterion is that $\boldsymbol{u}'(\boldsymbol{X} - \boldsymbol{\theta}) \stackrel{\mathrm{d}}{=} \boldsymbol{u}'(\boldsymbol{\theta} - \boldsymbol{X})$ for each unit vector $\boldsymbol{u}$ in $\mathbb{R}^d$. See Reference 55 for discussion and application of these criteria.

Sign symmetry, on the other hand, corresponds to $\mathcal{A}_0$-symmetry with $\mathcal{A}_0$ consisting of the $2^d$ transformations defined by $d$-vectors with $+$ or $-$ at each coordinate. As noted in Reference 14, although this group and the one corresponding to central symmetry cannot be compared by set theoretic inclusion, sign-symmetric distributions lie between centrally and spherically symmetric distributions.

In the other direction, a *relaxation* of central symmetry is given by the notion of "degree of symmetry" attributable to Blough [15]. Let $\boldsymbol{B}_k$ be the $d \times d$ matrix defined by

$$\boldsymbol{B}_k = [b_{ij}]_k = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j \neq k, \\ -1 & \text{if } i = j = k. \end{cases}$$

Then, a random vector $\boldsymbol{X}$ is *symmetric of degree* $m$ if there exists a vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m, 0, \ldots, 0)'$ in $\mathbb{R}^d$ and an orthogonal transformation $\boldsymbol{T}$ such that

$$\boldsymbol{T}(\boldsymbol{X} - \boldsymbol{\theta}) \stackrel{\mathrm{d}}{=} \boldsymbol{B}_1 \boldsymbol{B}_2 \cdots \boldsymbol{B}_m (\boldsymbol{T}(\boldsymbol{X} - \boldsymbol{\theta})). \tag{4}$$

Symmetry of degree $m$ thus means symmetry of the distribution of $\boldsymbol{X}$ about $m$ mutually orthogonal $(d-1)$-dimensional hyperplanes and hence about their $(d-m)$-dimensional intersection. That is, if $\boldsymbol{X}$ is symmetric of degree $m$, it possesses $m$ mutually orthogonal directions of symmetry. In particular, symmetry of degree 1 means that the distribution of $\boldsymbol{X}$ is symmetric about a $(d-1)$-dimensional hyperplane, and symmetry of degree $d$ is equivalent to central symmetry about $\boldsymbol{\theta}$. As shown in Reference 15, determination of the degree of symmetry can be approached by projection pursuit methods.

**Angular and Half-Space Symmetry**

Another broadening of central symmetry was introduced by Liu [36, 37]. A random vector $\boldsymbol{X}$ has a distribution *angularly symmetric* about $\boldsymbol{\theta}$ if

$$\frac{\boldsymbol{X}-\boldsymbol{\theta}}{\parallel \boldsymbol{X}-\boldsymbol{\theta} \parallel} \stackrel{\mathrm{d}}{=} \frac{\boldsymbol{\theta}-\boldsymbol{X}}{\parallel \boldsymbol{X}-\boldsymbol{\theta} \parallel}, \tag{5}$$

or, equivalently, if $(\boldsymbol{X}-\boldsymbol{\theta})/ \parallel \boldsymbol{X}-\boldsymbol{\theta} \parallel$ has centrally symmetric distribution.

Key features of angular symmetry are as follows (see Ref. 55 for detailed discussion). *(i)* The point $\boldsymbol{\theta}$ of angular symmetry, if it exists, is unique unless the distribution is concentrated on a line and its probability distribution on that line has more than one median. *(ii)* If $\boldsymbol{\theta}$ is a point of angular symmetry, then any hyperplane passing through $\boldsymbol{\theta}$ divides $\mathbb{R}^d$ into two open half-spaces with equal probabilities, which equal 1/2 if the distribution is continuous. The converse is also true. *(iii)* If $\boldsymbol{\theta}$ is a point of angular symmetry, then $\boldsymbol{\theta}$ agrees with the median of the conditional distribution of $\boldsymbol{X}$ on any axis through $\boldsymbol{\theta}$. (Here however, the converse fails to hold.)

For symmetric distributions, it is especially desirable that a location measure agrees with the point of symmetry (at least within linear transformation). In this regard, it is preferable to employ as broad a notion of symmetry as possible. In particular, it is desirable that any reasonable notion of "multidimensional median" should agree with the point of symmetry in the case of a symmetric distribution. A new nonparametric notion of multivariate symmetry that provides precisely the broadest possible manifestation of this criterion is introduced in Reference 55. A random vector $\boldsymbol{X}$ has a distribution *half-space symmetric* about $\boldsymbol{\theta}$ if

$$P(\boldsymbol{X} \in H) \geqslant 1/2, \quad \text{each closed half-space } H \text{ with } \boldsymbol{\theta} \text{ on the boundary.}$$

Since every half-space containing $\boldsymbol{\theta}$ contains a closed half-space with $\boldsymbol{\theta}$ on its boundary, it is equivalent to say simply "$P(\boldsymbol{X} \in H) \geqslant 1/2$ for any half-space $H$ containing $\boldsymbol{\theta}$." Clearly, it also is equivalent that any hyperplane passing through $\boldsymbol{\theta}$ must divide $\mathbf{R}^d$ into two closed half-spaces each of which has probability at least 1/2. Although half-space symmetry reduces to angular symmetry except for certain discrete distributions with positive probability on the center of half-space symmetry, it provides a relevant generalization, however, because the true underlying distributions for the phenomena we observe in practice are invariably discrete, and, further, it is reasonable to permit the center of an approximating discrete half-space symmetric distribution to carry some probability mass. For detailed discussion of characterizations of angular and half-space symmetry and their interrelations, see Reference 55.

## TESTING FOR SYMMETRY

The problem of testing the hypothesis of symmetry of a multivariate distribution has been approached from various points of view and the topic remains in active development. We briefly review some approaches.

For testing *spherical* symmetry, Kariya and Eaton [32] and Gupta and Kabe [27] develop UMP tests against various classes of alternatives, based on the distributions of standardized linear and quadratic forms in the given multivariate random vector. Extending a Cramér–von Mises type test [48] for circular symmetry in the bivariate case, Baringhaus [10] develops rotationally invariant test statistics that are distribution-free under the null hypothesis. These authors use the aforementioned characterization of spherical symmetry in terms of the distance $\|\boldsymbol{X}-\boldsymbol{\theta}\|$ and direction $(\boldsymbol{X}-\boldsymbol{\theta})/\|\boldsymbol{X}-\boldsymbol{\theta}\|$ from $\boldsymbol{\theta}$. Alternatively, one may base a test on the sample value of a measure of asymmetry, as for example in Reference 33 using a particular measure they define based on the so-called spatial version of multivariate quantiles. Bootstrap tests have been proposed [46]. Zhu and Neuhaus [44, 54] adapt the Monte Carlo approach of Barnard [11] for testing a hypothesis using, for any chosen criterion, reference datasets obtained by simulation under the null hypothesis; see also Reference 22. Zhu et al. [53] introduce a projection pursuit* approach and test the equivalent hypothesis that all the one-dimensional projections are identically distributed. Graphical methods are introduced by Li et al. [35], who propose QQ-plots associated with various statistics invariant under orthogonal rotations.

For testing *elliptical* symmetry, Beran [13] draws upon the representation of the density as $|\boldsymbol{A}|^{-1}g_0(\boldsymbol{A}^{-1}(\boldsymbol{x}-\boldsymbol{\theta}))$ with $g_0$ a density spherically symmetric about the origin, utilizes sample estimates $\hat{\boldsymbol{\theta}}_n$ and $\hat{\boldsymbol{A}}_n$, and develops a test statistic in terms of the residuals $\hat{\boldsymbol{A}}_n^{-1}(X_i-\hat{\boldsymbol{\theta}}_n)$, based on their corresponding directions from the origin and the ranks of their distances from the origin. See References 22 and 54 for Monte Carlo methods, Reference 25 for a projection pursuit method based on skewness and kurtosis measures, Reference 35 for graphical methods, and Reference 41 for a method based on averaging spherical harmonics over scaled residuals.

Finally, without elaboration, we mention several sources on testing for central symmetry, angular symmetry, or still other notions of symmetry: Reference 15 using projection pursuit and multivariate location regions, References 26 and 29 using projection pursuit and the empirical characteristic function, References 22 and 54 using Monte Carlo, and References 38 and 47 using graphical methods based on statistical depth functions and multivariate quantile functions.

## MEASURING SKEWNESS AND ASYMMETRY

When symmetry as a property fails to hold for a distribution, it is of interest to characterize the "skewness," that is, the nature or direction of the departure from symmetry, and to measure the asymmetry in a quantitative sense. One approach is to *model* the skewness parametrically, for example with the skew-elliptical distributions described above. More broadly, from a nonparametric perspective, here we review several ways to *measure* skewness and asymmetry.

In general, a skewness measure should be location- and scale-free and reduce to 0 in the case of a symmetric distribution. Classical *univariate* examples are $E(X-\mu)^3/\sigma^3$ and $(\mu-\nu)/\sigma$, for a distribution with mean $\mu$, median $\nu$, and variance $\sigma^2$. The latter is simply a difference of two location measures divided by a scale measure, and one can replace any of $\mu$, $\nu$, and $\sigma$ by alternative measures to produce quite attractive competitors. Such measures characterize skewness by a sign indicating *direction* and a magnitude measuring *asymmetry*. Along with such measures, associated notions of *orderings* of distributions according to their skewness have been introduced. For recent reviews of skewness concepts and measures in the univariate case, see Reference 12 and SKEWNESS–CONCEPTS AND MEASURES.

Extension of the above notion of a skewness measure to the *multivariate* case should in principle yield a *vector*, in order to be able to characterize skewness both by a direction and by an asymmetry measure. Of course, one must specify a notion of multivariate symmetry relative to which skewness represents a deviation. In the present development, we require that a quantitative measure of skewness reduce to the null vector in the case of *central* symmetry.

Despite the natural appeal of a *vector* notion of multivariate skewness, the classical treatment of the multivariate case has tended to focus upon numerical measures of asymmetry, developing many different versions that generalize the univariate case, but leaving largely unattended the treatment of directional measures of skewness and of the ordering of distributions by skewness. *See* MULTIVARIATE SKEWNESS AND KURTOSIS and Reference 34, section 44.20 for useful overviews with detailed discussion.

A few examples of scalar- and vector-valued measures will illustrate the variety of possibilities. Mardia [42] introduces

$$E\left\{[(\boldsymbol{X}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{Y}-\boldsymbol{\mu})]^3\right\},$$

for $\boldsymbol{X}$ and $\boldsymbol{Y}$ independent and identically distributed with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Malkovich and Afifi [40] extend the classical measure (squared) to a supremum over all univariate projections of $\boldsymbol{X}$:

$$\sup_{\boldsymbol{u}\in\mathbb{S}^{d-1}} \frac{[E\{(\boldsymbol{u}'\boldsymbol{X}-\boldsymbol{u}'\boldsymbol{\mu})^3\}]^2}{[\mathrm{Var}\{\boldsymbol{u}'\boldsymbol{X}\}]^3}.$$

Isogai [30] introduces

$$(\boldsymbol{\mu}-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-\boldsymbol{\theta}),$$

along with more general varieties, with $\boldsymbol{\theta}$ the mode of the distribution, and Oja [45] proposes

$$\frac{E[\Delta(\boldsymbol{X}_1,\ldots,\boldsymbol{X}_{d-1},\boldsymbol{\mu}_{\mathbf{1}},\boldsymbol{\mu}_{\mathbf{2}})]}{(E[\Delta(\boldsymbol{X}_1,\ldots,\boldsymbol{X}_d,\boldsymbol{\mu}_{\mathbf{2}})^2])^{1/2}},$$

where $\Delta(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{d+1})$ denotes the volume of the simplex in $\mathbb{R}^d$ determined by the $d{+}1$ points $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{d+1}$ and $\boldsymbol{\mu}_\alpha$ denotes the location measure defined as the minimizer of $E[\Delta(\boldsymbol{X}_1,\ldots,\boldsymbol{X}_d,\boldsymbol{\lambda})^\alpha]$ with respect to $\lambda$.

Opening up notions of *vector-valued skewness measures*, Avérous and Meste [3] introduce two vector-valued functionals oriented to the spatial median, along with corresponding definitions of quantitative skewness, directional qualitative skewness, and directional ordering of multivariate distributions. In particular, one of their functionals is given by

$$\boldsymbol{s}_F(r) = \frac{\boldsymbol{\ell}_F(r) - \boldsymbol{M}_F}{1/f(\boldsymbol{M}_F)},\ 0 < r < 1,$$

where $f$ is the density of the distribution $F$, $\boldsymbol{M}_F$ is the "spatial median" (that minimizes $E\{\|\boldsymbol{X} - \boldsymbol{c} - \|\boldsymbol{X}\|\}$ with respect to $\boldsymbol{c}$), and $\boldsymbol{\ell}_F(r)$ is a "median balls" location functional, where the "median balls" generalize the univariate "interquantile intervals"

$$[F^{-1}(\tfrac{1}{2} - \tfrac{r}{2}),\ F^{-1}(\tfrac{1}{2} + \tfrac{r}{2})],\ 0 < r < 1.$$

For each $r = r_0$, $\boldsymbol{s}_F(r_0)$ represents a quantitative *vector-valued* skewness measure, indicating an overall direction of skewness. We obtain associated quantitative *real-valued* measures of the skewness of $F$ in any particular direction $\boldsymbol{h}$ from the median $\boldsymbol{M}_F$ by taking scalar products with the vector measures: $\langle \boldsymbol{s}_F(r), \boldsymbol{h}\rangle$, $0 < r < 1$. Of course, one also may take $\langle \boldsymbol{s}_F(r), \boldsymbol{h}\rangle$, $0 < r < 1$, as a *functional* real-valued measure of skewness in the direction $\boldsymbol{h}$ from $\boldsymbol{M}_F$. This provides a basis for straightforward *qualitative* notions of skewness. A distribution $F$ is called *weakly skew* in the direction $\boldsymbol{h}$ from $\boldsymbol{M}_F$ if $\langle \boldsymbol{s}_F(r), \boldsymbol{h}\rangle$ is nonnegative for each $r$, *strongly skew* if $\langle \boldsymbol{s}_F(r), \boldsymbol{h}\rangle$ is increasing in $r$. A related *ordering* of distributions "$F$ is less weakly skew than $G$ in the direction $\boldsymbol{h}$ from $\boldsymbol{M}_F$" is defined by

$$F \prec_{\boldsymbol{h}} G \quad \Leftrightarrow \quad \langle \boldsymbol{s}_G(r) - \boldsymbol{s}_F(r), \boldsymbol{h}\rangle \geqslant 0 \text{ for each } r.$$

See Reference 3 for elaboration and Reference 47 for an analogous treatment defining and utilizing "spatial" location and dispersion functionals in place of $\boldsymbol{\ell}_F(\cdot)$ and $1/f(\boldsymbol{M}_F)$ respectively. See also Reference 9 for an alternative vector-valued approach related to the measure of Reference 40.

The foregoing vector-valued skewness functional yields a corresponding real-valued *asymmetry functional*, $\|\boldsymbol{s}_F(r)\|$, $0 < r < 1$, from which may be obtained real-valued *indices of asymmetry* $A_F = \sup_{0<r<1} \|\boldsymbol{s}_F(r)\|$, extending an index suggested by MacGillivray [39] in the univariate case, and $\int_0^1 \|s_F(r)\|\, dr$. Such indices may be used to order distributions, for example "$F$ is less asymmetric than $G$" if $A_F \leqslant A_G$. Alternative asymmetry functionals are given in References 33 and 18.

## Acknowledgements

The support of NSF Grant DMS-0103698 is gratefully acknowledged.

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