Chapter 11
Application to Economics

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The first model treated in Section 11.1.1 is a finite horizon fixed-end-point model with a stationary population. The problem is to maximize the present value of the utility of consumption for the society, as well as to accumulate a specified capital stock by the end of the horizon.

The second model incorporates an exogenously and exponentially growing population in the infinite horizon setting. A technique known as the method of phase diagrams is used to analyze the model.

For related discussion and extensions of these models, see Arrow and Kurz (1970), Burmeister and Dobell (1970), Intriligator (1971), and Arrow et al. (2007, 2010).
Consider a one-sector economy in which the stock of capital, denoted by $K(t)$, is the only factor of production. Let $F(K)$ be the output rate of the economy when $K$ is the capital stock.

Assume $F(0) = 0$, $F(K) > 0$, $F'(K) > 0$, and $F''(K) < 0$, for $K > 0$. These conditions imply the diminishing marginal productivity of capital as well as the strict concavity of $F(K)$ in $K$.

A part of this output is consumed and the remainder is reinvested for further accumulation of capital stock. Let $C(t)$ be the amount of output allocated to consumption, and let $I(t) = F[K(t)] - C(t)$ be the amount invested. Let $\delta$ be the constant rate of depreciation of capital. Then, the capital stock equation is

$$\dot{K} = F(K) - C - \delta K, \quad K(0) = K_0. \quad (11.1)$$
Let $U(C)$ be the society’s utility of consumption, where we assume $U'(0) = \infty$, $U'(C) > 0$, and $U''(C) < 0$, for $C \geq 0$. These conditions ensure that $U(C)$ is strictly concave in $C$. Let $\rho$ denote the social discount rate and $T$ denote the finite horizon. Then, a government which is elected for a term of $T$ years could consider the following problem:

$$\max \left\{ J = \int_0^T e^{-\rho t} U[C(t)] dt \right\}$$  \hspace{1cm} (11.2)

subject to (11.1) and the fixed-end-point condition

$$K(T) = K_T,$$ \hspace{1cm} (11.3)

where $K_T$ is a given positive constant. It may be noted that replacing (11.3) by $K(T) \geq K_T$ would give the same solution.
Solution by the Maximum Principle

- Form the current-value Hamiltonian as

\[ H = U(C) + \lambda[F(K) - C - \delta K]. \]  

(11.4)

The adjoint equation is

\[ \dot{\lambda} = \rho \lambda - \frac{\partial H}{\partial K} = (\rho + \delta) \lambda - \lambda \frac{\partial F}{\partial K}, \quad \lambda(T) = \alpha, \]  

(11.5)

where \( \alpha \) is a constant to be determined.

The optimal control is given by

\[ \frac{\partial H}{\partial C} = U'(C) - \lambda = 0. \]  

(11.6)

Since \( U'(0) = \infty \), the solution of this condition always gives \( C(t) > 0 \).
The economic interpretation of the Hamiltonian is straightforward. It consists of two terms: the first one gives the utility of current consumption and the second one gives the net investment evaluated by price $\lambda$, which, from (11.6), reflects the marginal utility of consumption.
Conditions For Optimality

For the economic system to be run optimally, the solution must satisfy the following three conditions:

(a) The static efficiency condition (11.6) which maximizes the value of the Hamiltonian at each instant of time myopically, provided that \( \lambda(t) \) is known.

(b) The dynamic efficiency condition (11.5) which forces the price \( \lambda \) of capital to change over time in such a way that the capital stock always yields a net rate of return, which is equal to the social discount rate \( \rho \). That is,

\[
d\lambda + \frac{\partial H}{\partial K} dt = \rho \lambda dt.
\]

(c) The long-run foresight condition, which establishes the terminal price \( \lambda(T) \) of capital in such a way that exactly the terminal capital stock \( K_T \) is obtained at \( T \).
Equations (11.1), (11.3), (11.5), and (11.6) form a two-point boundary value problem which can be solved numerically.

A qualitative analysis of this system can also be carried out by the phase diagram method of Chapter 7; see also Burmeister and Dobell (1970). We do not give details here since a similar analysis will be given for the infinite horizon version of this model treated in Sections 11.1.3 and 11.1.4.

In Exercise 11.1, you are asked to solve a simple version of the model in which the TPBVP can be solved analytically.
Introduction of a Growing Labor Force

- We now introduce labor as a new factor (treated the same as population, for simplicity), which grows exponentially at a fixed rate \( g, \ 0 < g < \rho \).
- Let \( L(t) \) denote the amount of labor at time \( t \). Since it is growing exponentially at rate \( g \), we have
  \[
  L(t) = L(0)e^{gt}.
  \] (11.7)
- Let \( F(K, L) \) be the production function which is assumed to be strictly increasing and concave in both factors of production so that \( F_K > 0, \ F_L > 0, \ F_{KK} < 0, \) and \( F_{LL} < 0 \) for \( K \geq 0, \ L \geq 0 \). Furthermore, it is homogeneous of degree one so that \( F(mK, mL) = mF(K, L) \) for \( m \geq 0 \). We define \( k = K/L \) and the per capita production function \( f(k) \) as
  \[
  f(k) = \frac{F(K, L)}{L} = F\left(\frac{K}{L}, 1\right) = F(k, 1). \] (11.8)
To derive the state equation for $k$, we note that

$$\dot{K} = \dot{k}L + k\dot{L} = \dot{k}L + kgL.$$

Substituting for $\dot{K}$ from (11.1) and defining per capita consumption $c = C/L$, we get

$$\dot{k} = f(k) - c - \gamma k, \quad k(0) = k_0,$$

(11.9)

where $\gamma = g + \delta$. 

\[ \]
The Objective Function

- Let \( u(c) \) be the utility of per capita consumption \( c \), where \( u \) is assumed to satisfy

\[
\begin{align*}
  u'(c) &> 0 \text{ and } u''(c) < 0 \text{ for } c \geq 0 \text{ and } u'(0) = \infty. \tag{11.10}
\end{align*}
\]

As in Section 11.1.2, the last condition in (11.10) rules out zero consumption.

- According to the position known as total utilitarianism, the society’s discounted total utility is \( \int_0^\infty e^{-\rho t}L(t)u(c(t))dt \), which we aim to maximize. In view of (11.7), this is equivalent to maximizing

\[
J = \int_0^\infty e^{-rt}u(c)dt, \tag{11.11}
\]

where \( r = \rho - g > 0 \). Note also that \( r + \gamma = \rho + \delta \).
The current-value Hamiltonian is

\[ H = u(c) + \lambda[f(k) - c - \gamma k]. \] (11.12)

The adjoint equation is

\[ \dot{\lambda} = r\lambda - \frac{\partial H}{\partial k} = (r + \gamma)\lambda - f'(k)\lambda = (\rho + \delta)\lambda - f'(k)\lambda. \] (11.13)

To obtain the optimal control, we differentiate (11.12) with respect to \( c \), set it to zero, and solve

\[ u'(c) = \lambda. \] (11.14)
Solution By The Maximum Principle Continued

- Let $c = h(\lambda) = u^{-1}(\lambda)$ denote the solution of (11.14). In Exercise 11.3, you are asked to show that $h'(\lambda) < 0$. This can be easily shown by inverting the graph of $u'(c)$ vs. $c$. Alternatively, you can rewrite (11.14) as $u'(h(\lambda)) = \lambda$ and then take its derivative with respect to $\lambda$.

- To show that the maximum principle is sufficient for optimality, it is enough to show that the derived Hamiltonian

$$H^0(k, \lambda) = u(h(\lambda)) + \lambda[f(k) - h(\lambda) - \gamma k] \tag{11.15}$$

is concave in $k$ for any $\lambda$ satisfying (11.14). The concavity follows immediately from the facts that $\lambda$ is positive from (11.10) and (11.14) and $f(k)$ is concave because of the assumptions on $F(K, L)$. 
In Figure 11.1, we have drawn a phase diagram for the two equations

\[
\begin{align*}
\dot{k} &= f(k) - h(\lambda) - \gamma k = 0, \\
\dot{\lambda} &= (r + \gamma)\lambda - f'(k)\lambda = 0,
\end{align*}
\]

(11.16) (11.17)

obtained from (11.9), (11.13), and (11.14).

In Exercise 11.2, you are asked to show that the graphs of \(\dot{k} = 0\) and \(\dot{\lambda} = 0\) are like the dotted curves in Figure 11.1. Given the nature of these graphs, known as isoclines, it is clear that they have a unique point of intersection denoted as \((\bar{k}, \bar{\lambda})\). In other words, \((\bar{k}, \bar{\lambda})\) is the unique solution of the equations

\[
\begin{align*}
f'(\bar{k}) - h(\bar{\lambda}) - \gamma \bar{k} &= 0 \quad \text{and} \quad (r + \gamma) - f'(\bar{k}) = 0.
\end{align*}
\]

(11.18)
Figure 11.1: Phase Diagram for the Optimal Growth Model
The two isoclines divide the plane into four regions, I, II, III, and IV, as marked in Figure 11.1. To the left of the vertical line $\dot{\lambda} = 0$, we have $k < \bar{k}$ and therefore $r + \gamma < f'(k)$ in view of $f''(k) < 0$. Thus, $\dot{\lambda} < 0$ from (11.13).

Therefore, $\lambda$ is decreasing, which is indicated by the downward pointing arrows in Regions I and IV. On the other hand, to the right of the vertical line, in Regions II and III, the arrows are pointed upward because $\lambda$ is increasing. In Exercise 11.3, you are asked to show that the horizontal arrows, which indicate the direction of change in $k$, point to the right above the $\dot{k} = 0$ isocline, i.e., in Regions I and II, and they point to the left in Regions III and IV which are below the $\dot{k} = 0$ isocline.
• The point \((\bar{k}, \bar{\lambda})\) represents the optimal long-run stationary equilibrium. The values of \(\bar{k}\) and \(\bar{\lambda}\) are obtained in Exercise 11.2. The next important thing is to show that there is a unique path starting from any initial capital stock \(k_0\), which satisfies the maximum principle and converges to the steady state \((\bar{k}, \bar{\lambda})\).

• Clearly such a path cannot start in Regions II and IV, because the directions of the arrows in these areas point away from \((\bar{k}, \bar{\lambda})\).

• For \(k_0 < \bar{k}\), the value of \(\lambda_0\) (if any) must be selected so that \((k_0, \lambda_0)\) is in Region I. For \(k_0 > \bar{k}\), on the other hand, the point \((k_0, \lambda_0)\) must be chosen to be in Region III.

• We analyze the case \(k_0 < \bar{k}\) only, and show that there exists a unique \(\lambda_0\) associated with the given \(k_0\), and that the optimal path, shown as the solid curve in Region I of Figure 11.1, starts from \((k_0, \lambda_0)\) and converges to \((\bar{k}, \bar{\lambda})\). It should be obvious that this path also represents the locus of such \((k_0, \lambda_0)\) for \(k_0 \in [0, \bar{k}]\). The analysis of the case \(k_0 > \bar{k}\) is left as Exercise 11.4.
In Region I, \( \dot{k}(t) > 0 \) and \( k(t) \) is an increasing function of \( t \) as indicated by the horizontal right-directed arrow in Figure 11.1. Therefore, we can replace the independent variable \( t \) by \( k \), and then use (11.16) and (11.17) to obtain

\[
\lambda'(k) = \frac{d\lambda}{dk} = \frac{d\lambda}{dt} \left/ \frac{dk}{dt} \right. = \frac{[f'(k) - (r + \gamma)]\lambda}{h(\lambda) + \gamma k - f(k)}.
\]

(11.19)

Thus, our task of showing that there exists an optimal path starting from any initial \( k_0 < \bar{k} \) is equivalent to showing that there exists a solution of the differential equation (11.19) on the interval \( [0, \bar{k}] \), beginning with the boundary condition \( \lambda(\bar{k}) = \bar{\lambda} \). For this, we must obtain the value \( \lambda'(\bar{k}) \). Since both the numerator and the denominator in (11.19) vanish at \( k = \bar{k} \), we need to derive \( \lambda'(\bar{k}) \) by a perturbation argument. To do so, we use (11.19) and (11.18) to obtain

\[
\lambda'(k) = \frac{[r + \gamma - f'(k)]\lambda}{f(k) - \gamma k - h(\lambda)} = \frac{[f'(\bar{k}) - f'(k)]\lambda}{f(k) - f(\bar{k}) - \gamma k + \gamma \bar{k} - h(\lambda) + h(\bar{\lambda})}.
\]
Uniqueness of the Convergence Path

- We use L’Hôpital’s rule to take the limit as $k \to \bar{k}$ and obtain

$$
\lambda'(\bar{k}) = \frac{-f''(\bar{k}) \lambda}{f'(\bar{k}) - \gamma - h'(\lambda) \lambda'(\bar{k})} = \frac{-f''(\bar{k}) \lambda}{f'(\bar{k}) - \gamma - \lambda'(\bar{k})/u''(h(\lambda))},
$$

(11.20)

or

$$
- \frac{(\lambda'(\bar{k}))^2}{u''(h(\bar{\lambda}))} + \lambda'(\bar{k}) [f'(\bar{k}) - \gamma] + \bar{\lambda} f''(\bar{k}) = 0.
$$

(11.21)

Note that the first equality in (11.20) is obtained by differentiating the numerator and the denominator with respect to $k$, and then replacing $k$ by $\bar{k}$. The second equality in (11.20) uses the relation $h'(\lambda) = 1/u''(h(\lambda))$ obtained by differentiating $u'(c) = u'(h(\lambda)) = \lambda$ of (11.14) with respect to $\lambda$ at $\lambda = \bar{\lambda}$. 
It is easy to see that (11.21) has one positive solution and one negative solution. We take the negative solution for $\lambda'(\bar{k})$ because of the following consideration. With the negative solution, we can prove that the differential equation (11.19) has a smooth solution, such that $\lambda'(k) < 0$. For this, let

$$\pi(k) = f(k) - k\gamma - h(\lambda(k)).$$

Since $k < \bar{k}$, we have $r + \gamma - f'(k) < 0$. Then from (11.19), since $\lambda'(\bar{k}) < 0$, we have $\lambda(\bar{k} - \varepsilon) > \lambda(\bar{k})$. Also since $\bar{\lambda} > 0$ and $f''(\bar{k}) < 0$, equation (11.20) with $\lambda'(\bar{k})$ implies

$$\pi'(\bar{k}) = f'(\bar{k}) - \gamma - \frac{\lambda'(\bar{k})}{u''(h(\bar{\lambda}))} < 0,$$

and thus,

$$\pi(\bar{k} - \varepsilon) = f(\bar{k} - \varepsilon) - \gamma(\bar{k} - \varepsilon) - h(\lambda(\bar{k} - \varepsilon)) > 0.$$
Therefore, the derivative at $\bar{k} - \varepsilon$ is well defined and $\lambda'(\bar{k} - \varepsilon) < 0$. We can proceed as long as

$$\pi'(k) = f'(k) - \gamma - \frac{\lambda'(k)}{u''(h(\lambda(k)))} < 0. \quad (11.22)$$

This implies that $f(k) - k\gamma - h(\lambda) > 0$, and also since $r + \gamma - f'(k)$ remains negative for $k < \bar{k}$, we have $\lambda'(k) < 0$. 
Existence of a Convergence Path Continued

Suppose now that there is a point $\tilde{k} < \bar{k}$ with $\pi(\tilde{k}) = 0$. Then, since $\pi(\tilde{k} + \varepsilon) > 0$, we have $\pi'(\tilde{k}) \geq 0$. But at $\tilde{k}$, $\pi(\tilde{k}) = 0$ in (11.19) implies $\lambda'(\tilde{k}) = -\infty$, and then from (11.22), we have $\pi'(\tilde{k}) = -\infty$, which is a contradiction with $\pi'(\tilde{k}) \geq 0$. Thus, we can proceed on the whole interval $[0, \bar{k}]$. This indicates that the path $\lambda(k)$ (shown as the solid line in Region I of Figure 11.1) remains above the curve

$$\dot{k} = f(k) - k\gamma - h(\lambda) = 0,$$

shown as the dotted line in Figure 11.1 when $k < \bar{k}$. Thus, we can set $\lambda_0 = \lambda(k_0)$ for $0 \leq k_0 \leq \bar{k}$ and have the optimal path starting from $(k_0, \lambda_0)$ and converging to $(\bar{k}, \bar{\lambda})$. 
Existence of a Convergence Path Continued

- Similar arguments hold when the initial capital stock \( k_0 > \bar{k} \), in order to show that the optimal path (shown as the solid line in Region III of Figure 11.1) exists in this case. You have already been asked to carry out this analysis in Exercise 11.4.

- We should mention that the conclusions derived in this subsection could have been reached by invoking the Global Saddle Point Theorem stated in Appendix D.7, but we have chosen instead to carry out a detailed analysis for illustrating the use of the phase diagram method. The next time we use the phase diagram method will be in Section 11.3.3, and there we shall rely on the Global Saddle Point Theorem.
(Global Saddle Point Theorem). Let \((\bar{x}, \bar{\lambda})\) be a unique saddle point of the canonical system (D.73) of the differential equations and let \(x_0\) be a given initial state for which the vertical line \(x = x_0\) (see Figure D.1 intersects both isoclines \(\dot{x} = f^*(x, \lambda) = 0\) and \(\dot{\lambda} = \psi^*(x, \lambda) = 0\). Assume further that the region bounded by the isoclines and the line \(x = x_0\) has a triangular shape as in Figure D.1 (i.e., the isoclines themselves do not intersect in the open interval between \(x_0\) and \(\bar{x}\)). Then, there exists a unique saddle point path starting for \(x = x_0\) and leading to the saddle point \((\bar{x}, \bar{\lambda})\).
Figure D.1: Phase diagram for system \((D.73)\)
Certain infectious epidemic diseases are seasonal in nature. Examples are the common cold, the flu, and certain children’s diseases. When it is beneficial to do so, control measures are taken to alleviate the effects of these diseases.

Here we discuss a simple control model due to Sethi (1974c) for analyzing an epidemic problem. Related problems have been treated by Sethi and Staats (1978), Sethi (1978d), and Francis (1997). See Wickwire (1977) for a good survey of optimal control theory applied to the control of pest infestations and epidemics, and Swan (1984) for applications to biomedicine.
Let $N$ be the total fixed population. Let $x(t)$ be the number of infectives at time $t$ so that the remaining $N - x(t)$ is the number of susceptibles. To keep the model simple, assume that no immunity is acquired so that when infected people are cured, they become susceptible again. The state equation governing the dynamics of the epidemic spread in the population is

$$\dot{x} = \beta x(N - x) - vx, \quad x(0) = x_0,$$

(11.23)

where $\beta$ is a positive constant termed *infectivity* of the disease, and $v$ is a control variable reflecting the level of medical program effort. Note that $x(t)$ is in $[0, N]$ for all $t > 0$ if $x_0$ is in that interval.
The objective of the control problem is to minimize the present value of the cost stream up to a horizon time $T$, which marks the end of the season for that disease. Let $h$ denote the unit social cost per infective, let $m$ denote the cost of control per unit level of program effort, and let $Q$ denote the capability of the health care delivery system providing an upper bound on $v$.

The optimal control problem is:

$$\max \left\{ J = \int_{0}^{T} -(hx + mv)e^{-\rho t} \, dt \right\}$$  \hspace{1cm} (11.24)

subject to (11.23), the terminal constraint that

$$x(T) = x_T,$$  \hspace{1cm} (11.25)

and the control constraint

$$0 \leq v \leq Q.$$
Solution by Green’s Theorem

- Rewriting (11.23) as

\[ vdt = \left[ \beta x(N - x)dt - dx \right]/x \]

and substituting into (11.24) yields the line integral

\[ J_\Gamma = \int_{\Gamma} - \left\{ [hx + m\beta(N - x)]e^{-\rho t} dt \frac{m}{x} e^{-\rho t} dx \right\}, \quad (11.26) \]

where \( \Gamma \) is a path from \( x_0 \) to \( x_T \) in the \((t, x)\)-space.
Let $\Gamma_1$ and $\Gamma_2$ be two such paths from $x_0$ to $x_T$, and let $R$ be the region enclosed by $\Gamma_1$ and $\Gamma_2$. By Green’s theorem, we can write

$$J_{\Gamma_1 - \Gamma_2} = J_{\Gamma_1} - J_{\Gamma_2} = \iint_R \left[ \frac{m\rho}{x} - h + m\beta \right] e^{-\rho t} dt \, dx. \quad (11.27)$$

To obtain the singular control we set the integrand of (11.27) equal to zero, as we did in Section 7.2.2. This yields

$$x = \frac{\rho}{h/m - \beta} = \frac{\rho}{\theta}, \quad (11.28)$$

where $\theta = h/m - \beta$. 
Define the singular state $x^s$ as follows:

$$x^s = \begin{cases} \frac{\rho}{\theta} & \text{if } 0 < \frac{\rho}{\theta} < N, \\ N & \text{otherwise.} \end{cases} \quad (11.29)$$

The corresponding singular control level

$$v^s = \beta(N - x^s) = \begin{cases} \beta(N - \frac{\rho}{\theta}) & \text{if } 0 < \frac{\rho}{\theta} < N, \\ 0 & \text{otherwise.} \end{cases} \quad (11.30)$$

We will show that $x^s$ is the turnpike level of infectives. It is instructive to interpret (11.29) and (11.30) for the various cases. If $\frac{\rho}{\theta} > 0$, then $\theta > 0$ so that $h/m > \beta$. Here the smaller the ratio $h/m$, the larger the turnpike level $x^s$, and therefore, the smaller the medical program effort should be. In other words, the smaller the social cost per infective and/or the larger the treatment cost per infective, the smaller the medical program effort should be.
When $\rho/\theta < 0$, you are asked to show in Exercise 11.9 that $x^s = N$ in the case $h/m < \beta$, which means the ratio of the social cost to the treatment cost is smaller than the infectivity coefficient. Therefore, in this case when there is no terminal constraint, the optimal trajectory involves no treatment effort. An example of this case is the common cold where the social cost is low and treatment cost is high.
The optimal control for the fortuitous case when $x_T = x^s$ is

$$v^*(x(t)) = \begin{cases} 
Q & \text{if } x(t) > x^s, \\
v^s & \text{if } x(t) = x^s, \\
0 & \text{if } x(t) < x^s.
\end{cases} \quad (11.31)$$

When $x_T \neq x^s$, there are two cases to consider. For simplicity of exposition we assume $x_0 > x^s$ and $T$ and $Q$ to be large.
Solution by Green’s Theorem Continued

- **Case 1:** $x_T > x^s$. The optimal trajectory is shown in Figure 11.2. In Exercise 11.8 you are asked to show its optimality by using Green’s theorem.

- **Case 2:** $x_T < x^s$. The optimal trajectory is shown in Figure 11.3. It can be shown that $x$ goes asymptotically to $N - Q/\beta$ if $v = Q$. The level is marked in Figure 11.3.

  The optimal control shown in Figures 11.2 and 11.3 assumes $0 < x^s < N$. It also assumes that $T$ is large so that the trajectory will spend some time on the turnpike and $Q$ is large so that $x^s \geq N - Q/\beta$.

  The graphs are drawn for $x_0 > x^s$ and $x^s < N/2$; for all other cases see Sethi (1974c).
Figure 11.2: Optimal Trajectory when $x_T > x^s$
Figure 11.3: Optimal Trajectory when $x_T < x^s$
In this section we will describe a simple pollution control model due to Keeler, Spence, and Zeckhauser (1971). We will describe this model in terms of an economic system in which labor is the only primary factor of production, which is allocated between food production and DDT production. It is assumed that all of the food produced is used for consumption. On the other hand, all of the DDT produced is used as a secondary factor of production which, along with labor, determines the food output. However, when used, DDT causes pollution, which can only be reduced by natural decay. The objective of the society is to maximize the total present value of the utility of food less the disutility of pollution due to the use of DDT.
We introduce the following notation:

\[ L = \text{the total labor force, assumed to be constant for simplicity,} \]

\[ v = \text{the amount of labor used for DDT production,} \]

\[ L - v = \text{the amount of labor used for food production,} \]

\[ P = \text{the stock of DDT pollution at time} \ t, \]

\[ a(v) = \text{the rate of DDT output;} a(0) = 0, \ a' > 0, \ a'' < 0, \text{ for} \ v \geq 0, \]

\[ \delta = \text{the natural exponential decay rate of DDT pollution,} \]
Notation

\[ C(v) = f[L - v, a(v)] = \text{the rate of food output to be consumed;} \]
\[ C'(v) \text{ is concave, } C(0) > 0, \ C(L) = 0; \ C(v) \text{ attains a} \]
\[ \text{unique maximum at } v = V > 0; \text{ see Figure 11.4.} \]
\[ \text{Note that a sufficient condition for } C(v) \text{ to be strictly} \]
\[ \text{concave is } f_{12} \geq 0 \text{ along with the usual concavity and} \]
\[ \text{monotonicity conditions on } f \text{ (see Exercise 11.10),} \]
\[ u(C) = \text{the utility function of consuming the food output } C \geq 0; \]
\[ u'(0) = \infty, \ u'(C) > 0, \ u''(C) < 0, \]
\[ h(P) = \text{the disutility function of pollution stock } P \geq 0; \]
\[ h'(0) = 0, \ h'(P) > 0, \ h''(P) > 0. \]
Figure 11.4: Food Output Function

Figure 11.4 Food Output Function

\[ C(v) \]

\[ v \]

\[ 0 \]

\[ V \]

\[ L \]
The optimal control problem is:

$$\max \left\{ J = \int_0^\infty e^{-\rho t} [u(C(v)) - h(P)] dt \right\} \quad (11.32)$$

subject to

$$\dot{P} = a(v) - \delta P, \quad P(0) = P_0, \quad 0 \leq v \leq L. \quad (11.33)$$

From Figure 11.4, it is obvious that $v$ is at most $V$, since the production of DDT beyond that level decreases food production and increases DDT pollution. Hence, (11.34) can be reduced to simply

$$v \geq 0. \quad (11.35)$$
Solution by the Maximum Principle

Form the current-value Lagrangian

$$L(P, v, \lambda, \mu) = u[C(v)] - h(P) + \lambda[a(v) - \delta P] + \mu v$$  \hspace{1cm} (11.36)

using (11.32), (11.33) and (11.35), where

$$\dot{\lambda} = (\rho + \delta)\lambda + h'(P),$$  \hspace{1cm} (11.37)

and

$$\mu \geq 0 \text{ and } \mu v = 0.$$  \hspace{1cm} (11.38)

The optimal solution is given by

$$\frac{L}{v} = u'[C(v)]C'(v) + \lambda a'(v) + \mu = 0.$$  \hspace{1cm} (11.39)
Since the derived Hamiltonian is concave, conditions (11.36)-(11.39) together with
\[ \lim_{t \to \infty} \lambda(t) = \bar{\lambda} = \text{constant} \] (11.40)
are sufficient for optimality; see Theorem 2.1 and Section 2.4. The phase diagram analysis presented below gives \( \lambda(t) \) satisfying (11.40).
From the assumptions on $C'(v)$ or from Figure 11.4, we see that $C'(0) > 0$. This means that $du/dv = u'(C(v))C'(v)\big|_{v=0} > 0$. This along with $h'(0) = 0$ implies that $v > 0$, meaning that it pays to produce some positive amount of DDT in equilibrium. Therefore, the equilibrium value of the Lagrange multiplier is zero, i.e., $\bar{\mu} = 0$. From (11.33), (11.37) and (11.39), we get the equilibrium values $\bar{P}$, $\bar{\lambda}$, and $\bar{v}$ as follows:

$$\bar{P} = \frac{a(\bar{v})}{\delta}, \quad (11.41)$$

$$\bar{\lambda} = -\frac{h'(\bar{P})}{+\delta} = -\frac{u'[C(\bar{v})]C'(\bar{v})}{a'(\bar{v})}. \quad (11.42)$$

From (11.42) and the assumptions on the derivatives of $g$, $C$ and $a$, we know that $\bar{\lambda} < 0$. From this and (11.37), we conclude that $\lambda(t)$ is always negative.
The economic interpretation of $\lambda$ is that $-\lambda$ is the imputed cost of pollution. Let $v = \Phi(\lambda)$ denote the solution of (11.39) with $\mu = 0$. On account of (11.35), define

$$v^* = \max[0, \Phi(\lambda)].$$

We know from the interpretation of $\lambda$ that when $\lambda$ increases, the imputed cost of pollution decreases, which can justify an increase in the DDT production to ensure an increased food output. Thus, it is reasonable to assume that

$$\frac{d\Phi}{d\lambda} > 0.$$ 

It follows that there exists a unique $\lambda^c$ such that

$\Phi(\lambda^c) = 0$, $\Phi(\lambda) < 0$ for $\lambda < \lambda^c$ and $\Phi(\lambda) > 0$ for $\lambda > \lambda^c$. 
Phase Diagram Analysis Continued

- To construct the phase diagram, we must plot the isoclines $\dot{P} = 0$ and $\dot{\lambda} = 0$. These are, respectively,

$$
P = \frac{a(v^*)}{\delta} = \frac{a[\max\{0, \Phi(\lambda)\}]}{\delta},
$$

(11.44)

$$
h'(P) = -(+\delta)\lambda.
$$

(11.45)

- Observe that the assumption $h'(0) = 0$ implies that the graph of (11.45) passes through the origin. Differentiating these equations with respect to $\lambda$ and using (11.43), we obtain

$$
\left. \frac{dP}{d\lambda} \right|_{\dot{P}=0} = \frac{a'(v)}{\delta} \frac{dv}{d\lambda} > 0
$$

(11.46)

as the slope of the $\dot{P} = 0$ isocline, and

$$
\left. \frac{dP}{d\lambda} \right|_{\dot{\lambda}=0} = -\frac{(+\delta)}{h''(P)} < 0.
$$

(11.47)
The intersection point \((\bar{\lambda}, \bar{P})\) of these isoclines denotes the equilibrium levels for the adjoint variable and the pollution stock, respectively. That there exists an optimal path (shown as the solid line in Figure 11.5) converging to the equilibrium \((\bar{\lambda}, \bar{P})\) follows directly from the Global Saddle Point Theorem stated in Appendix D.7.

Given \(\lambda^c\) as the intersection of the \(\dot{P} = 0\) curve and the horizontal axis, the corresponding ordinate \(P^c\) on the optimal trajectory is the related pollution stock level. The significance of \(P^c\) is that if the existing pollution stock \(P\) is larger than \(P^c\), then the optimal control is \(v^* = 0\), meaning no DDT is produced.
Figure 11.5: Phase Diagram for the Pollution Control Model
Given an initial level of pollution $P_0$, the optimal trajectory curve in Figure 11.5 provides the initial value $\lambda_0$ of the adjoint variable. With these initial values, the optimal trajectory is determined by (11.33), (11.37), and (11.43). If $P_0 > P^c$, as shown in Figure 11.5, then $v^* = 0$ until such time that the natural decay of pollution stock has reduced it to $P^c$.

At that time, the adjoint variable has increased to the value $\lambda^c$. The optimal control is $v^* = \phi(\lambda)$ from this time on, and the path converges to $(\bar{\lambda}, \bar{P})$.

At equilibrium, $\bar{v} = \Phi(\bar{\lambda}) > 0$, which implies that it is optimal to produce some DDT forever in the long run. The only time when its production is not optimal is at the beginning when the pollution stock is higher than $P^c$. 
It is important to examine the effects of changes in the parameters on the optimal path. In particular, you are asked in Exercise 11.11. to show that an increase in the natural rate of decay of pollution, $\delta$, will increase $P^c$. That is, when pollution decays at a faster rate, we can increase the threshold level of pollution stock at which to ban the production of the pollutant. For DDT in reality, $\delta$ is small so that its complete ban, which has actually occurred, may not be far from the optimal policy.
In modern contract theory, the term *adverse selection* is used to describe principal-agent models in which an agent has private information before a contract is written. For example, a seller does not know perfectly how much a buyer is willing to pay for a good. A related concept is that of *moral hazard*, when there is present a hidden action not adversely observed by the principal.

In such game situations, clearly the principal would like to know the agent’s private information which he cannot learn simply by asking the agent, because it is in the agent’s interest to distort the truth. Fortunately, according to the theory of *mechanism design*, the principal can design a game whose rules can influence the agent to act in the way the principal would like. Thanks, particularly to the *revelation principle*, the principal needs only consider games in which the agent truthfully reports her private information.
Consider a transaction between a seller (the principal) and a buyer (the agent) of type \( t \in [t_1, t_2] \), \( 0 \leq t_1 \leq t_2 \), represents her willingness-to-pay for seller’s goods. We assume in particular that buyer’s preferences are represented by the utility function

\[
U(q, \phi, t) = ta(q) - \phi,
\]  

where \( q \) is the number of units purchased and \( \phi \) is the total amount paid to the seller. We assume \( a(0) = 0, \ a' > 0, \) and \( a'' < 0. \)

The seller’s unit production cost is \( c > 0 \), so that his profit from selling \( q \) units against a sum of money \( \phi \) is given by

\[
\pi = \phi - cq.
\]
The question of interest here is to obtain a profit-maximizing pair \( \{ \phi, q \} \) that the seller will be able to induce the buyer of type \( \hat{t} \) to choose. Thanks to the revelation principle, the answer is that the seller can offer a menu of contracts \( \{ \phi(t), q(t) \} \) which comes from solving the following maximization problem:

\[
\max_{q(\cdot), \phi(\cdot)} \int_{t_1}^{t_2} [\phi(t) - cq(t)] f(t) dt
\]

subject to

\[
(\text{IR}) \quad \hat{t}a(q(\hat{t})) - \phi(\hat{t}) \geq 0, \quad \hat{t} \in [t_1, t_2], \tag{11.51}
\]

\[
(\text{IC}) \quad \hat{t}a(q(\hat{t})) - \phi(\hat{t}) \geq \hat{t}a(q(t)) - \phi(t), \quad t, \hat{t} \in [t_1, t_2], \quad t \neq \hat{t}. \tag{11.52}
\]
The constraints (11.51), called *individual rationality constraints (IR)*, say that the agent of type $\hat{t}$ will participate in the contract. Clearly, given (11.52), we can replace these constraints by a single constraint

$$t_1 a(q(t_1)) - \phi(t_1) \geq 0. \quad (11.53)$$

The left-hand side of the constraints (11.52), called *incentive compatibility constraints (IC)*, is the utility of agent $\hat{t}$ if she chooses the contract intended for her, whereas the right-hand side represents the utility of agent $\hat{t}$ if she chooses the constraint intended for type $t \neq \hat{t}$. The IC constraints, therefore, imply that type $\hat{t}$ agent is better off choosing the contract intended for her than any other contract in the menu.
Given a menu \( \{ q(\cdot), \phi(\cdot) \} \) that satisfies the seller’s problem (11.50)-(11.52), it must be the case in equilibrium that the buyer \( \hat{t} \) will choose the contract \( \{ q(\hat{t}), \phi(\hat{t}) \} \). In other words, his utility \( \hat{t}a(q(t)) - \phi(t) \) of choosing a contract \( \{ q(t), \phi(t) \} \) will be maximized at \( t = \hat{t} \). Assuming that \( q(\cdot) \) and \( \phi(\cdot) \) are twice differentiable functions, the first-order and second-order conditions are

\[
\hat{t}a'(q(t))\dot{q}(t) - \dot{\phi}(t) \bigg|_{t=\hat{t}} = \hat{t}a'(q(\hat{t}))\dot{q}(\hat{t}) - \dot{\phi}(\hat{t}) = 0, \tag{11.54}
\]

\[
\hat{t}a''(q(t))(\dot{q}(t))^2 + \hat{t}a'(q(t))\ddot{q}(t) - \ddot{\phi}(t) \bigg|_{t=\hat{t}} \leq 0. \tag{11.55}
\]
From (11.54), it follows from replacing \( \hat{t} \) by \( t \) that

\[
ta'(q(t))\dot{q}(t) - \dot{\phi}(t) = 0, \quad t \in [t_1, t_2],
\]

(11.56)
called the *local incentive compatibility condition*, must hold. Differentiating (11.56) gives,

\[
ta''(q(t))(\dot{q}(t))^2 + a'(q(t))\dot{q}(t) + ta'(q(t))\ddot{q}(t) - \ddot{\phi}(t) = 0.
\]

(11.57)

It follows from (11.55), (11.57), and \( a' > 0 \) that

\[
\dot{q}(t) \geq 0.
\]

(11.58)

This is called the *monotonicity condition*. In Exercise 11.12, you are asked to show that (11.56) and (11.58) are sufficient for (11.52) to hold. Since, these conditions are already necessary, we can say that local incentive compatibility (11.56) and monotonicity (11.58) together are equivalent to the IC condition (11.52).
The Optimization Problem

- The seller’s problem can be written as the following optimal control problem:

\[
\max_{u(\cdot)} \int_{t_1}^{t_2} [\phi(t) - cq(t)]f(t)dt
\]

subject to

\[
\dot{q}(t) = u(t), \\
\dot{\phi}(t) = ta'(q(t))u(t), \\
t_1 a(q(t_1)) - \phi(t_1) = 0, \\
u(t) \geq 0.
\]
Here, $q(t)$ and $\phi(t)$ are state variables and $u(t)$ is a control variable satisfying the control constraint $u(t) \geq 0$. The objective function (11.59) is the expected value of the seller’s profit with respect to the density $f(t)$. Equation (11.60) and constraint (11.63) come from the monotonicity condition (11.58). Equation (11.61) with $u(t)$ from (11.60) gives the local incentive compatibility condition (11.56). Finally, (11.62) specifies the IR constraint (11.53) in view of the fact it will be binding for the lowest agent type $t_1$ at the optimum.
We can now use the sense of the maximum principle (3.12) to write the necessary conditions for optimality. Note that (3.12) is written for problem (3.7) that has specified initial states and some constraints on the terminal state vector \( x(T) \) that include the equality constraint

\[ b(x(T), T) = 0. \]

Our problem, on the other hand, has this type of equality constraint, namely (11.62), on the initial states \( q(t_1) \) and \( \phi(t_1) \) and no specified terminal states \( q(t_2) \) and \( \phi(t_2) \).

However, since initial time conditions and terminal time conditions can be treated in a symmetric fashion, we can apply the sense of (3.12), as shown in Remark (3.9), to obtain the necessary optimality conditions to problem (11.59)-(11.63). In Exercise 11.13, you are asked to obtain (11.67) and (11.68) by following Remark (3.9) to account for the presence of the equality constraint (11.62) on the initial state variables rather than on the terminal state as in problem (3.7).
To specify the necessary optimality condition, we first define the Hamiltonian

\[ H(q, \phi, \lambda, \mu, t) = [\phi(t) - cq(t)]f(t) + \lambda(t)u(t) + \mu(t)[ta'(q(t)u(t))] \]

\[ = [\phi(t) - cq(t)]f(t) + [\lambda(t) + \mu(t)ta'(q(t))]u(t). \]  

(11.64)

Then for \( u^* \) with the corresponding state trajectories \( q^* \) and \( \phi^* \) to be optimal, we must have adjoints \( \lambda \) and \( \mu \), and a constant \( \beta \), such that

\[ \dot{q}^* = u^*, \dot{\phi}^* = ta'(q^*)u, \]  

(11.65)

\[ t_1a(q^*(t_1)) - \phi^*(t_1) = 0, \]  

(11.66)

\[ \dot{\lambda} = cf - \mu ta''(q^*)u^*, \lambda(t_1) = \beta t_1a'(q^*(t_1)), \lambda(t_2) = 0, \]  

(11.67)

\[ \dot{\mu} = -f, \mu(t_1) = -\beta, \mu(t_2) = 0, \]  

(11.68)

\[ u^*(t) = \text{bang}[0, \infty; \lambda(t) + \mu(t)ta'(q^*(t))]. \]  

(11.69)
Several remarks are in order at this point. First we see that we have a bang-bang control in (11.69). This means that $u^*(t)$ can be 0, or greater than 0, or an impulse control. Moreover, in the region when $u^*(t) = 0$, which will occur when $\lambda(t) + \mu(t)ta'(q^*(t)) < 0$, we will have a constant $q^*(t)$, and we will have a singular control $u^*(t) > 0$ if we can keep $\lambda(t) + \mu(t)ta'(q^*(t)) = 0$ by an appropriate choice of $u^*(t)$ along the singular path. An impulse control would occur if the initial $q(t_1)$ were above the singular path. Since in our problem, initial states are not exactly specified, we shall not encounter an impulse control here.
The Optimization Problem Continued

The third remark concerns a numerical way of solving the problem. For this, let us rewrite the boundary conditions in (11.67) and (11.68) and the condition (11.66) as below:

\[ t_1 a(q^*(t_1)) - \phi^*(t_1) = 0, \quad \lambda(t_1) = -\mu(t_1) t_1 a'(q^*(t_1)) \quad (11.70) \]

\[ \lambda(t_2) = \mu(t_2) = 0. \quad (11.71) \]

With (11.71) and a guess of \( q(t_2) \) and \( \phi(t_2) \), we can solve the differential equation (11.65), (11.67) and (11.68), with \( u^*(t) \) in (11.69), backward in time. These will give us the values of \( \lambda(t_1), \mu(t_1), q(t_1) \) and \( \phi(t_1) \). We can check if these satisfy the two equations in (11.70). If yes, we have arrived at a solution. If not, we change our guess for \( q(t_2) \) and \( \phi(t_2) \) and start again. As you may have noticed, the procedure is very similar to solving a two-point boundary value problem.
Next we provide an alternative procedure to solve the seller’s problem, a procedure used in the theory of mechanism design. This procedure first ignores the nonnegativity constraint (11.60) and solves the relaxed problem given by (11.59)-(11.62). In view of (11.52), let us define

\[ u^0(\hat{t}) = \hat{t}a(q(\hat{t})) - \phi(\hat{t}) = \max_t [ta(q(t)) - \phi(t)]. \] (11.72)

By the envelope theorem, we have

\[ \frac{du^0(\hat{t})}{d\hat{t}} = \frac{\partial u^0(\hat{t})}{\partial \hat{t}} = a(q(\hat{t})), \] (11.73)

which we can integrate to obtain

\[ u^0(t) = \int_{t_1}^t a(q(x))dx + u^0(t_1) = \int_{t_1}^t a(q(x))dx, \] (11.74)

since \( u^*(t_1) = 0 \) at the optimum.
Also, since \( \phi(t) = ta(q(t)) - u^0(t) \), we can write the seller’s profit as

\[
\int_{t_1}^{t_2} \left[ ta(q(t)) - \int_{t_1}^{t} a(q(x))dx - cq(t) \right] f(t)dt.
\]  

(11.75)

Then, integrating by parts, we have

\[
\int_{t_1}^{t_2} \left\{ ta(q(t)) - cq(t) \right\} f(t) - a(q(t))(1 - F(t)) dt
\]

\[
= \int_{t_1}^{t_2} [ta(q(t)) - cq(t) - a(q(t))/h(t)] f(t)dt,
\]  

(11.76)

where \( h(t) = f(t)/[1 - F(t)] \) is known as the hazard rate.
The Optimization Problem Continued

Since we are interested in maximizing the seller’s profit with respect to the output schedule \( q(\cdot) \), we can maximize the expression under the integral pointwise for each \( t \). The first-order condition for that is

\[
\left[ t - \frac{1 - F(t)}{f(t)} \right] a'(q(t)) = \left[ t - \frac{1}{h(t)} \right] a'(q(t)) = c, \tag{11.77}
\]

which gives us the optimal solution of the relaxed problem as

\[
\hat{q}(t) = a'^{-1} \left[ c \left( t - \frac{1}{h(t)} \right)^{-1} \right]. \tag{11.78}
\]
In obtaining (11.78), we had omitted the nonnegativity constraint (11.63) introduced to ensure that \( q(t) \) is increasing. Thus, it remains to check if \( dq(t)/dt \geq 0 \). It is straightforward to verify that if the hazard rate \( h(t) \) is increasing in \( t \), then \( \hat{q}(t) \) is increasing in \( t \). To show this, we differentiate (11.78) to obtain

\[
\frac{d\hat{q}(t)}{dt} = -\frac{g(t)a'(\hat{q}(t))}{a''(\hat{q}(t))g(t)},
\]

where \( g(t) = [t - 1/h(t)] \). Clearly, if \( h(t) \) is increasing, then \( g(t) \) is increasing, and \( d\hat{q}(t)/dt \geq 0 \).

In this case, \( \hat{q}(t) \) and the corresponding \( \hat{\phi}(t) \) obtained from solving the differential equation given by (11.61) and the boundary condition (11.62) give us the optimal menu \( \{\hat{\phi}(t), \hat{q}(t)\} \).
What if $h(t)$ is not increasing? In that case, there is a procedure called *bunching and ironing* given by the solution of an optimal control problem to be formulated next. This is because $\hat{q}(t)$ in (11.78) is obtained by solving the relaxed problem that ignores the nonnegativity constraint (11.63), and so it may be that $d\hat{q}/dt$ is strictly negative for some $t \in [t, \bar{t}] \subset [t_1, t_2]$ as shown in Figure 11.6.

Then the seller must choose the optimal $q^*(t)$ to maximize the following constrained optimal control problem:

$$
\max_{q(\cdot)} \int_{t_1}^{t_2} \left[ ta(q(t)) - cq(t) - \frac{a(q(t))}{h(t)} \right] f(t)dt
$$

subject to

$$
\dot{q}(t) = u(t), \; u(t) \geq 0.
$$
Figure 11.6: Violation of the Monotonicity Constraint

\[ \dot{q}(t) \]

\[ q^*(\theta) \]

\[ t_1 \quad t \quad \bar{t} \quad t_2 \quad t \]
Now the necessary optimality conditions, with the Hamiltonian defined as

\[
H(q, 0, \lambda, t) = (ta(q) - cq - a(q)/h)f + \lambda u,
\]  

are

\[
\dot{\lambda} = - [(t - 1/h)a'(q) - c] f, \quad \lambda(t_1) = \lambda(t_2) = 0,
\]  

and

\[
u^* = [0, \infty; \lambda].
\]  

We may also note that these conditions are also sufficient since \(H\) in (11.81) is concave in \(q\). Integrating (11.82), we have

\[
\lambda(t) = - \int_{t_1}^{t} \left[ \left( z - \frac{1}{h(z)} \right) a'(q(z)) - c \right] f(z) dz.
\]
Using the transversality conditions in the case when neither the initial nor the terminal state is specified for the state equation (11.80), we obtain

\[ 0 = \lambda(t_1) = \lambda(t_2) = -\int_{t_1}^{t_2} \left[ \left( z - \frac{1}{h(z)} \right) a'(q(z)) - c \right] f(z) \, dz. \]

Then for \( u^*(t) = 0 \) on an interval \( t \in [\theta_1, \theta_2] \subset [t_1, t_2] \), we must have \( \lambda(t) < 0, t \in [\theta_1, \theta_2] \). Moreover, when \( u^*(t) > 0 \), it must be a singular control for which \( \lambda(t) = 0 \).

But \( \lambda(t) = 0 \) is the same as the first-order condition (11.77), which means that if \( q^*(t) \) is strictly increasing, then it must coincide with \( \hat{q}(t) \) in (11.78). It, therefore, only remains to determine the intervals over which \( q^*(t) \) is constant. Consider Figure 11.7.
Figure 11.7: Bunching and Ironing

\[ \dot{q}(t); q^*(t) \]

\[ \lambda = 0 \]

\[ \mu = 0 \]

\[ \lambda < 0 \]
By continuity, we must have \( \lambda(\theta_1) = \lambda(\theta_2) = 0 \), so that

\[
\int_{\theta_1}^{\theta_2} \left[ \left( z - \frac{1}{h(z)} \right) a'(q^*(z)) - c \right] dz = 0.
\]

(11.84)

In addition, we must have

\[
q^*(\theta_1) = q^*(\theta_2)
\]

(11.85)

from the continuity of \( q^*(\cdot) \).

Thus, we have two equations (11.84) and (11.85) and two unknowns, allowing us to obtain the values of \( \theta_1 \) and \( \theta_2 \). An interval \([\theta_1, \theta_2]\) over which \( q^*(t) \) is constant is known as a bunching interval.

Here, we have given a procedure when \( \hat{q}(\cdot) \) has only one interval \([t, \bar{t}]\) over which it is strictly decreasing. If there are more such intervals, this procedure of ironing and bunching can be extended in an obvious manner.
Control theory applications to economics:

- Optimal educational investments.
- Limit pricing and uncertain entry.
- Adjustment costs in the theory of competitive firms.
- International trade.
- Money demand with transaction costs.
- Design of an optimal insurance policy.
- Optimal training and heterogeneous labor.
- Population policy.
- Optimal income tax
- Continuous expanding economies.
- Investment and marketing policies in a duopoly.
- Theory of firm under government regulations.
- Renumeration patterns for medical services.
- Dynamic shareholder behavior under personal taxation.
- Optimal input substitution in response to environmental constraints.
- Optimal crackdowns on a drug market.
Applications to management science and operations research:
- Labor assignments.
- Distribution and transportation applications.
- Scheduling and network planning problems.
- Research and development.
- City congestion problems.
- Warfare models.
- National settlement planning.
- Pricing with dynamic demand and production costs.
- Accelerating diffusion of innovation.
- Optimal acquisition of new technology.
- Optimal pricing and/or advertising for monopolistic diffusion model.
- Manpower planning.
- Optimal quality and advertising under asymmetric information.
- Optimal recycling of tailings for production of building materials.
- Planning for information technology.