Chapter 13
Differential Games

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In previous chapters, we were mainly concerned with the optimal control problems formulated by a single objective function (or a single decision maker).

However, there are situations when there may be more than one decision maker, each having one’s own objective function that each is trying to maximize, subject to a set of differential equations. This extension of optimal control theory is referred to as the *theory of differential games*.
The study of differential games was initiated by Isaacs (1965). From the development of Pontryagin’s maximum principle, we see a connection between differential games and optimal control theory. In fact, differential game problems represent a generalization of optimal control problems in cases where there is more than one controller or player.

However, differential games are conceptually far more complex than optimal control problems in the sense that it is no longer obvious what constitutes a solution.

Indeed, there are a number of different types of solutions such as minimax, Nash, Stackelberg, along with possibilities of cooperation and bargaining; see, e.g., Tolwinski (1982) and Haurie et al. (1983). We will discuss minimax solutions for zero-sum differential games in Sect. 13.1, Nash solutions for nonzero-sum games in Sect. 13.2, and Stackelberg differential games in Sect. 13.3.
Consider the state equation

$$\dot{x} = f(x, u, v, t), \quad x(0) = x_0, \quad (13.1)$$

where we may assume all variables to be scalar for the time being.

Extension to the vector case simply requires appropriate reinterpretations of each of the variables and the equations. In this equation, we let $u$ and $v$ denote the controls applied by players 1 and 2, respectively. We assume that

$$u(t) \in U, \quad v(t) \in V, \quad t \in [0, T],$$

where $U$ and $V$ are convex sets in $E^1$. 
Consider further the objective function

\[ J(u, v) = S[x(T)] + \int_0^T F(x, u, v, t) dt, \]  

(13.2)

where player 1 wants to maximize and player 2 wants to minimize.

Since the gain of player 1 represents a loss to player 2, such games are appropriately termed zero-sum games. Clearly, we are looking for admissible control trajectories \( u^* \) and \( v^* \) such that

\[ J(u^*, v) \geq J(u^*, v^*) \geq J(u, v^*). \]  

(13.3)

The solution \( (u^*, v^*) \) is known as the \textit{minimax} solution.
To obtain the necessary conditions for $u^*$ and $v^*$, we form the Hamiltonian

$$H = F + \lambda f$$  \hspace{1cm} (13.4)

with the adjoint variable $\lambda$ satisfying the equation

$$\dot{\lambda} = -H_x, \quad \lambda(T) = S_x[x(T)].$$  \hspace{1cm} (13.5)
The necessary condition for trajectories $u^*$ and $v^*$ to be a minimax solution is that for $t \in [0, T]$,

$$H(x^*(t), u^*(t), v^*(t), \lambda(t), t) = \min_{v \in V} \max_{u \in U} H(x^*(t), u, v, \lambda(t), t),$$

which can also be stated, with suppression of $(t)$, as

$$H(x^*, u^*, v, \lambda, t) \geq H(x^*, u^*, v^*, \lambda, t) \geq H(x^*, u, v^*, \lambda, t) \quad (13.7)$$

for $u \in U$ and $v \in V$.

Note that $(u^*, v^*)$ is a saddle point of the Hamiltonian function $H$. 
Note also that if $u$ and $v$ are unconstrained, i.e., when $U = V = E^1$, condition (13.6) reduces to the first-order necessary conditions

$$H_u = 0 \text{ and } H_v = 0,$$

and the second-order conditions are

$$H_{uu} \leq 0 \text{ and } H_{vv} \geq 0.$$
We now turn to the treatment of nonzero-sum differential games. Assume that we have $N$ players where $N \geq 2$. Let $u^i \in U^i$, $i = 1, 2, \ldots, N$, represent the control variable for the $i$th player, where $U^i$ is the set of controls from which the $i$th player can choose. Let the state equation be defined as

$$\dot{x} = f(x, u^1, u^2, \ldots, u^N, t).$$

Let $J^i$, defined by

$$J^i = S^i[x(T)] + \int_0^T F^i(x, u^1, u^2, \ldots, u^N, t)dt,$$

denote the objective function which the $i$th player wants to maximize.
In this case, a Nash solution is defined by a set of $N$ admissible trajectories
\[
\{ u^1, u^2, \ldots, u^N \},
\] (13.12)
which have the property that
\[
J^i(u^1, \ldots, u^{i-1}, u^i, u^{i+1}, \ldots, u^N) = \max_{u^i \in U^i} J^i(u^1, \ldots, u^{i-1}, u^i, u^{i+1}, \ldots, u^N)
\] (13.13)
for $i = 1, 2, \ldots, N$. 
Open-Loop Nash Solution

- Define the Hamiltonian functions

\[
H^i(x, u^1, u^2, \ldots, u^N, \lambda^i) = F^i + \lambda^i f
\]

for \( i = 1, 2, \ldots, N \), with \( \lambda^i \) satisfying

\[
\dot{\lambda}^i = -H_x^i, \quad \lambda^i(T) = S^i_x[x(T)].
\]

- The Nash control \( u^i^* \) for the \( i \)-th player is obtained by maximizing the \( i \)-th Hamiltonian \( H^i \) with respect to \( u^i \), i.e., \( u^i^* \) must satisfy

\[
H^i(x^*, u^{1*}, \ldots, u^{(i-1)*}, u^i^*, u^{(i+1)*}, \ldots, u^{N*}, \lambda, t) \geq H^i(x^*, u^{1*}, \ldots, u^{(i-1)*}, u^i, u^{(i+1)*}, \ldots, u^{N*}, \lambda, t), \quad t \in [0, T],
\]

for all \( u^i \in U^i, \ i = 1, 2, \ldots, N \).
A feedback Nash solution is obtained when (13.12) is defined in terms of the current state of the system. To avoid confusion, we let

\[ u^i(x, t) = \phi^i(x, t), \quad i = 1, 2, \ldots, N. \]  

(13.17)

For these controls to represent a feedback Nash strategy, we must recognize the dependence of the other players’ actions on the state variable \( x \). Therefore, we need to replace the adjoint equation (13.15) by

\[ \dot{\lambda}^i = -H^i_x - \sum_{j=1}^{N} H^i_{uj} \phi^j_x = -H^i_x - \sum_{j=1, j \neq i}^{N} H^i_{uj} \phi^j_x. \]  

(13.18)
The summation term in (13.18) is troublesome when it comes to deriving computational algorithms; see Starr and Ho (1969).

It is, however, possible to use a dynamic programming approach for solving extremely simple nonzero-sum games, which require the solution of a partial differential equation. We will use this approach in Section 13.3.
The summation term in (13.18) is absent in three important cases:

(a) in optimal control problems ($N = 1$) since $H_u u_x = 0$,

(b) in two-person zero-sum games because $H^1 = -H^2$ so that $H^1_{u_2} u_2^2 = -H^2_{u_2} u_2^2 = 0$ and $H^2_{u_1} u_1^1 = -H^1_{u_1} u_1^1 = 0$, and

(c) in open-loop nonzero-sum games because $u_j^j = 0$.

The feedback and open-loop Nash solutions are going to be different, in general.
The adjoint variable $\lambda^i$ is the sensitivity of the $i$th player’s profit to a perturbation in the state vector. If the other players are using closed-loop strategies, any perturbation $\delta x$ in the state vector causes them to revise their controls by the amount $\phi^i_x \delta x$.

If the $i$th Hamiltonian $H^i$ were maximized with respect to $u^j$, $j \neq i$, this would not affect the $i$th player’s profit; but since $\partial H^i / \partial u^j \neq 0$ for $i \neq j$, the reactions of the other players to the perturbation influence the $i$th player’s profit, and the $i$th player must account for this effect in considering variations of the trajectory.
Consider extending the fishery model of Section 10.1 by assuming that there are two producers having unrestricted rights to exploit the fish stock in competition with each other.

This gives rise to a nonzero-sum differential game analyzed by Clark (1976).
Equation (10.2) is modified by

\[ \dot{x} = g(x) - q^1 u^1 x - q^2 u^2 x, \quad x(0) = x_0, \]  

(13.19)

where \( u^i(t) \) represents the rate of fishing effort and \( q^i u^i x \) is the rate of catch for the \( i \)th producer, \( i = 1, 2 \). The control constraints are

\[ 0 \leq u^i(t) \leq U^i, \quad i = 1, 2, \]

(13.20)

and the state constraints are

\[ x(t) \geq 0. \]

(13.21)

The objective function for the \( i \)th producer is the total present value of his profits, namely,

\[ J^i = \int_0^\infty (p^i q^i x - c^i) u^i e^{-\rho t} dt, \quad i = 1, 2. \]

(13.22)
To find the feedback Nash solution for this model, we let $\bar{x}^i$ denote the turnpike (or optimal biomass) level given by (10.12) on the assumption that the $i$th producer is the sole-owner of the fishery. Let the bionomic equilibrium $x^i_b$ and the corresponding control $u^i_b$ associated with producer $i$ be defined by (10.4), i.e.,

$$x^i_b = \frac{c^i}{p^iq^i} \quad \text{and} \quad u^i_b = \frac{g(x^i_b)p^i}{c^i}.$$  \hfill (13.23)

As shown in Exercise 10.2, $x^i_b < \bar{x}^i$, and we assume $U^i$ to be sufficiently large so that $u^i_b \leq U^i$. 

We also assume that

\[ x_b^1 < x_b^2, \]  

(13.24)

which means that producer 1 is more efficient than producer 2, i.e., producer 1 can make a positive profit at any level in the interval \((x_b^1, x_b^2]\), while producer 2 loses money in the same interval, except at \(x_b^2\), where he breaks even. For \(x > x_b^2\), both producers make positive profits.
Since $U^1 \geq u^1_b$ by assumption, producer 1 has the capability of driving the fish stock level down to at least $x^1_b$ which, by (13.24), is less than $x^2_b$.

This implies that producer 1 will not allow producer 2 to operate at a sustained level above $x^2_b$; and at a sustained level at or below $x^2_b$, producer 2 cannot make a profit. Hence, his optimal feedback policy is bang-bang:

$$u^2^*(x) = \begin{cases} U^2 & \text{if } x > x^2_b, \\ 0 & \text{if } x \leq x^2_b. \end{cases}$$

(13.25)
As far as producer 1 is concerned, he wants to attain his turnpike level \( \bar{x}^1 \) if \( \bar{x}^1 \leq x^2_b \). If \( \bar{x}^1 > x^2_b \) and \( x_0 \geq \bar{x}^1 \), then from (13.25) producer 2 will fish at his maximum rate until the fish stock is driven to \( x^2_b \).

At this level, it is optimal for producer 1 to fish at a rate which maintains the fish stock at level \( x^2_b \) in order to keep producer 2 from fishing.
Thus, the optimal feedback policy for producer 1 can be stated as

\[
 u^1_*(x) = \begin{cases} 
 U^1, & x > \bar{x}^1 \\
 \bar{u}^1 = \frac{g(\bar{x}^1)}{q^1 \bar{x}^1}, & x = \bar{x}^1 \\
 0, & x < \bar{x}^1 
\end{cases}
\]

, if \( \bar{x}^1 < x^2_b \), (13.26)

\[
 u^1_*(x) = \begin{cases} 
 U^1, & x > x^2_b \\
 \frac{g(x^2_b)}{q^1 x^2_b}, & x = x^2_b \\
 0, & x < x^2_b 
\end{cases}
\]

, if \( \bar{x}^1 \geq x^2_b \). (13.27)
The formal proof that policies (13.25)-(13.27) give a Nash solution requires direct verification using the result of Section 10.1.2. The Nash solution for this case means that for all feasible paths $u^1$ and $u^2$,

$$J^1(u^1\ast, u^2\ast) \geq J^1(u^1, u^2\ast) \quad (13.28)$$

and

$$J^2(u^1\ast, u^2\ast) \geq J^2(u^1\ast, u^2). \quad (13.29)$$
The direct verification involves defining a modified growth function

\[ g^1(x) = \begin{cases} 
  g(x) - q^2 U^2 x & \text{if } x > x_b^2, \\
  g(x) & \text{if } x \leq x_b^2,
\end{cases} \]

and using the Green’s theorem results of Section 10.1.2.

Since \( U^2 \geq u_b^2 \) by assumption, we have \( g^1(x) \leq 0 \) for \( x > x_b^2 \). From (10.12) with \( g \) replaced by \( g^1 \), it can be shown that the new turnpike level for producer 1 is \( \min(\bar{x}^1, x_b^2) \), which defines the optimal policy (13.26)-(13.27) for producer 1. The optimality of (13.25) for producer 2 follows easily.
To interpret the results of the model, suppose that producer 1 originally has sole possession of the fishery, but anticipates a rival entry. Producer 1 will switch from his own optimal sustained yield $\bar{u}_1$ to a more intensive exploitation policy prior to the anticipated entry.

We can now guess the results in situations involving $N$ producers. The fishery will see the progressive elimination of inefficient producers as the stock of fish decreases. Only the most efficient producers will survive. If, ultimately, two or more maximally efficient producers exist, the fishery will converge to a classical bionomic equilibrium, with zero sustained economic rent.
Licensing Entrants

- We have now seen that a feedback Nash solution involving \( N \geq 2 \) competing producers results in the long-run erosion of economic rents.

- This conclusion depends on the assumption that producers face an infinitely elastic supply of all factors of production going into the fishing effort, but typically the methods of licensing entrants to regulated fisheries make some attempt to also control the factors of production such as permitting the licensee to operate only a single vessel of specific size.
In order to develop a model for the licensing of fishermen, we let the control variable $v^i$ denote the capital stock of the $i$th producer and let the concave function $f(v^i)$, with $f(0) = 0$, denote the *fishing mortality function* for $i = 1, 2, \ldots, N$. This requires the replacement of $q^i u^i$ in the previous model by $f(v^i)$.

The extended model becomes nonlinear in control variables. You are asked in Exercise 13.3 to formulate this new model and develop necessary conditions for a feedback Nash solution for this game involving $N$ producers. The reader is referred to Clark (1976) for further details.
In this section, we will study a competitive extension of the Sethi advertising model discussed in Section 12.3. This will give us a stochastic differential game, for which we aim to obtain a feedback Nash equilibrium by using a dynamic programming approach developed in Section 12.1.

Consider a duopoly market in a mature product category where total sales are distributed between two firms, labeled as Firm 1 and Firm 2, which compete for market share through advertising expenditures.
Let $X_t$ denote the market share of Firm 1 at time $t$, so that the market share of Firm 2 is $(1 - X_t)$. Let $U_{1t}$ and $U_{2t}$ denote the advertising effort rates of Firms 1 and 2, respectively, at time $t$.

Using the subscript $i \in \{1, 2\}$ to reference the two firms, let $r_i > 0$ denote the advertising effectiveness parameter, $\pi_i > 0$ denote the sales margin, $\rho_i > 0$ denote the discount rate, and $c_i > 0$ denote the cost parameter so that the cost of advertising effort $u$ by Firm $i$ is $c_i u^2$.

Further, let $\delta > 0$ be the churn parameter, $Z_t$ be the standard one-dimensional Wiener process, and $\sigma(x)$ be the diffusion coefficient function as defined in Section 12.3.
Then, in view of the competition between the firms, Prasad and Sethi (2004) extend the Sethi model dynamics in (12.42) as the Itô stochastic differential equation

\[
dX_t = \left[ r_1 U_{1t} \sqrt{1 - X_t} - \delta X_t - r_2 U_{2t} \sqrt{X_t} + \delta (1 - X_t) \right] dt + \sigma(X_t)dZ_t, \quad X(0) = x_0 \in [0, 1].
\]  (13.30)

We formulate the optimal control problem faced by the two firms which seek to maximize their respective expected discounted profit streams

\[
\max_{U_1 \geq 0} \left\{ V^1(x_0) = E \int_0^\infty e^{-\rho_1 t} \left[ \pi_1 X_t - c_1 U_{1t}^2 \right] dt \right\}, \quad (13.31)
\]

\[
\max_{U_2 \geq 0} \left\{ V^2(x_0) = E \int_0^\infty e^{-\rho_2 t} \left[ \pi_2 (1 - X_t) - c_2 U_{2t}^2 \right] dt \right\}, \quad (13.32)
\]

subject to the market share dynamics (13.30).
We form the Hamilton-Jacobi-Bellman (HJB) equations for the value functions $V^1(x)$ and $V^2(x)$:

$$\rho_1 V^1 = \max_{U_1 \geq 0} \{ H^1(x, U_1, U_2, V^1_x) + (\sigma(x))^2 V^1_{xx}/2 \}$$

$$= \max_{U_1 \geq 0} \{ \pi_1 x - c_1 U_1^2 + V^1_x [r_1 U_1 \sqrt{1-x} - r_2 U_2 \sqrt{x} - \delta (2x-1)]$$

$$+ (\sigma(x))^2 V^1_{xx}/2 \}, \quad (13.33)$$

$$\rho_2 V^2 = \max_{U_2 \geq 0} \{ H^2(x, U_1, U_2, V^2_x) + (\sigma(x))^2 V^2_{xx}/2 \}$$

$$= \max_{U_2 \geq 0} \{ \pi_2 (1-x) - c_2 U_2^2 + V^2_x [r_1 U_1 \sqrt{1-x}$$

$$- r_2 U_2 \sqrt{x} - \delta (2x-1)] + (\sigma(x))^2 V^2_{xx}/2 \}, \quad (13.34)$$

where the Hamiltonians are as defined in (13.14).
We use the first-order conditions for Hamiltonian maximization to obtain the optimal feedback advertising decisions

\[ U_1^*(x) = V_x^1(x)r_1\sqrt{1 - x/2c_1} \quad \text{and} \quad U_2^*(x) = -V_x^2(x)r_2\sqrt{x/2c_2}. \]  

(13.35)

Since it is reasonable to expect that \( V_x^1 \geq 0 \) and \( V_x^2 \leq 0 \), these controls will turn out to be nonnegative as we will see later.
Substituting (13.35) in (13.33) and (13.34), we obtain the Hamilton-Jacobi equations

\[ \rho_1 V^1 = \pi_1 x + (V_x^1)^2 r_1^2 (1 - x)/4c_1 + V_x^1 V_x^2 r_2^2 x/2c_2 \\
- V_x^1 \delta(2x - 1) + (\sigma(x))^2 V_{xx}^1 /2, \] (13.36)

\[ \rho_2 V^2 = \pi_2 (1 - x) + (V_x^2)^2 r_2^2 x/4c_2 + V_x^1 V_x^2 r_1^2 (1 - x)/2c_1 \\
- V_x^2 \delta(2x - 1) + (\sigma(x))^2 V_{xx}^2 /2. \] (13.37)

As in Section 12.3, we look for the following forms for the value functions

\[ V^1 = \alpha_1 + \beta_1 x \text{ and } V^2 = \alpha_2 + \beta_2 (1 - x). \] (13.38)
These are inserted into (13.36) and (13.37) to determine the unknown coefficients $\alpha_1, \beta_1, \alpha_2,$ and $\beta_2$. Equating the coefficients of $x$ and the constants on both sides of (13.36) and the coefficients of $(1-x)$ and the constants on both sides of (13.37), the following four equations emerge, which can be solved for the unknowns $\alpha_1, \beta_1, \alpha_2,$ and $\beta_2$:

\[
\begin{align*}
\rho_1 \alpha_1 &= \beta_1^2 r_1^2 / 4c_1 + \beta_1 \delta, \\
\rho_1 \beta_1 &= \pi_1 - \beta_1^2 r_1^2 / 4c_1 - \beta_1 \beta_2 r_2^2 / 2c_2 - 2\beta_1 \delta, \\
\rho_2 \alpha_2 &= \beta_2^2 r_2^2 / 4c_2 + \beta_2 \delta, \\
\rho_2 \beta_2 &= \pi_2 - \beta_2^2 r_2^2 / 4c_2 - \beta_1 \beta_2 r_1^2 / 2c_1 - 2\beta_2 \delta.
\end{align*}
\]
Let us first consider the special case of symmetric firms, i.e., when \( \pi = \pi_1 = \pi_2, \ c = c_1 = c_2, \ r = r_1 = r_2, \) and \( \rho = \rho_1 = \rho_2, \) and therefore \( \alpha = \alpha_1 = \alpha_2, \ \beta = \beta_1 = \beta_2. \) The four equations in (13.39-13.42) reduce to the following two:

\[
\rho \alpha = \beta^2 r^2 / 4c + \beta \delta \quad \text{and} \quad \rho \beta = \pi - 3\beta^2 r^2 / 4c - 2\beta \delta. \quad (13.43)
\]

There are two solutions for \( \beta. \) One is negative, which clearly makes no sense. Thus, the remaining positive solution is the correct one.
This also allows us to obtain the corresponding $\alpha$. The solution is

$$\alpha = \frac{[(\rho - \delta)(D - \sqrt{D^2 + 12R\pi}) + 6R\pi]/18R\rho}{18R}, \quad (13.44)$$

$$\beta = \frac{D}{6R}, \quad (13.45)$$

where $R = r^2/4c$ and $W = \rho + 2\delta$. With this the value functions in (13.38) are defined, and the controls in (13.35) for the case of symmetric firms can be written as

$$u_1^*(x) = \frac{\beta_1 r_1 \sqrt{1-x}}{2c_1} = \frac{\beta r \sqrt{1-x}}{2c} \quad \text{and} \quad u_2^*(x) = \frac{\beta_2 r_2 \sqrt{x}}{2c_2} = \frac{\beta r \sqrt{x}}{2c},$$

which are clearly nonnegative as required.
We return now to the general case of asymmetric firms. For this, we re-express equations (13.39-13.42) in terms of a single variable $\beta_1$, which is determined by solving the quartic equation

$$3R_1^2\beta_1^4 + 2R_1(W_1 + W_2)\beta_1^3 + (4R_2\pi_2 - 2R_1\pi_1 - W_1^2 + 2W_1W_2)\beta_1^2 + 2\pi_1(W_1 - W_2)\beta_1 - \pi_1^2 = 0. \tag{13.46}$$

We solve this explicitly for four roots. We find that only one root is positive, and select it as $\beta_1$. With that, other coefficients can be obtained by solving for $\alpha_1$ and $\beta_2$, and then for $\alpha_2$ as follows:

$$\alpha_1 = \beta_1(\beta_1 R_1 + \delta)/\rho_1, \tag{13.47}$$
$$\beta_2 = (\pi_1 - \beta_1^2 R_1 - \beta_1 W_1)/2\beta_1 R_2, \tag{13.48}$$
$$\alpha_2 = \beta_2(\beta_2 R_2 + \delta)/\rho_2, \tag{13.49}$$

where $R_1 = r_1^2/4c_1$, $R_2 = r_2^2/4c_2$, $W_1 = \rho_1 + 2\delta$, and $W_2 = \rho_2 + 2\delta$. 
It is worthwhile to mention that firm $i$’s advertising effectiveness parameter $r_i$ and advertising cost parameter $c_i$ manifest themselves through $R_i = r_i^2 / 4c_i$.

This would suggest that $R_i$ is a measure of firm $i$’s advertising power. This can be seen more clearly in Exercise 13.6 involving two firms that are identical in all other aspects except that $R_2 > R_1$. Specifically in that exercise, you are asked to use Mathematica or another suitable software program to solve (13.46) to obtain $\beta_1$ and then obtain the coefficients $\alpha_1, \alpha_2,$ and $\beta_2$ by using (13.47)-(13.49), when $\rho_1 = \rho_2 = 0.05$, $\pi_1 = \pi_2 = 1$, $\delta = 0.01$, $R_1 = 1$, $R_2 = 4$, $x_0 = 0.5$, and $\sigma(x) = \sqrt{0.5x(1-x)}$. Figure 13.1 represents a sample path of the market share of the two firms with this data.
Figure 13.1: A Sample Path of Optimal Market Share Trajectories
It is noteworthy to see that both firms are identical except in their advertising powers $R_1$ and $R_2$. With $R_2 > R_1$, firm 2 is more powerful and we see that this results in its capture of an increasing share of the market average over time beginning with exactly one half of the market at time 0.
Feedback Stackelberg Stochastic Differential Game of Cooperative Advertising

- The preceding sections in this chapter dealt with differential games in which all players make their decisions simultaneously. We now discuss a differential game in which two players make their decisions in a hierarchical manner.

- The player having the right to move first is called the leader and the other player is called the follower. If there are two or more leaders, they play Nash, and the same goes for the followers.

- In terms of solutions of Stackelberg differential games, we have open-loop and feedback solutions.

- An open-loop Stackelberg equilibrium specifies, at the initial time (say, $t = 0$), the decisions over the entire horizon.

- Typically, open-loop solutions are not time consistent in the sense that at any time $t > 0$, the remaining decision may no longer be optimal; see Exercise 13.2.
Feedback Stackelberg Stochastic Differential Game of Cooperative Advertising Continued

- A *feedback* or *Markovian Stackelberg equilibrium*, on the other hand, consists of decisions expressed as functions of the current state and time. Such a solution is time consistent.

- In this section, we will not develop the general theory, for which we refer the reader to Basar and Olsder (1999), Dockner et al. (2000), Bensoussan, Chen, and Sethi (2014, 2015), and Bensoussan et al. (2019). Instead, we will formulate a Stackelberg differential game of cooperative advertising between a manufacturer as the leader and a retailer as the follower, and obtain a feedback Stackelberg solution. This formulation is due to He, Prasad and Sethi (2009).
The manufacturer sells a product to end users through the retailer. The product is in a mature category where sales, expressed as a fraction of the potential market, is influenced through advertising expenditures.

The manufacturer as the leader decides on an advertising support scheme via a subsidy rate, i.e., he will contribute a certain percentage of the advertising expenditure by the retailer. Specifically, the manufacturer decides on a subsidy rate $W_t$, $0 \leq W_t \leq 1$, and the retailer as the follower decides on the advertising effort level $U_t \geq 0$, $t \geq 0$. 
As in Section 12.3, the cost of advertising is quadratic in the advertising effort $U_t$. Then, with the advertising effort $U_t$ and the subsidy rate $W_t$, the manufacturer's and the retailer's advertising expenditures are $W_t U_t^2$ and $(1 - W_t) U_t^2$, respectively. The market share dynamics is given by the Sethi model

$$dX_t = (rU_t \sqrt{1 - X_t} - \delta X_t) dt + \sigma(X_t) dZ_t, \quad X_0 = x_0. \quad (13.50)$$

The corresponding expected profits of the retailer and the manufacturer are, respectively, as follows:

$$J_R = E \left[ \int_0^\infty e^{-\rho t} (\pi X_t - (1 - W_t) U_t^2) dt \right], \quad (13.51)$$

$$J_M = E \left[ \int_0^\infty e^{-\rho t} \left( \pi_M X_t - W_t U_t^2 \right) dt \right]. \quad (13.52)$$
A solution of this Stackelberg differential game depends on the available information structure. We shall assume that at each time $t$, both players know the current system state and the follower knows the action of the leader.

The concept of equilibrium that applies in this case is that of feedback Stackelberg equilibrium. For this and other information structures and equilibrium concepts, see Bensoussan, Chen, and Sethi (2015).
Next we define the rules, governing the sequence of actions, by which this game will be played over time. To be specific, the sequence of plays at any time $t \geq 0$ is as follows.

- First, the manufacturer observes the market share $X_t$ at time $t$ and selects the subsidy rate $W_t$. Then, the retailer observes this action $W_t$ and, knowing also the market share $X_t$ at time $t$, sets the advertising effort rate $U_t$ as his response to $W_t$.

- The system evolves over time as this game is played in continuous time beginning at time $t = 0$. One could visualize this game as being played at times $0, \delta t, 2\delta t, \ldots$, and then let $\delta t \to 0$. 
Next, we will address the question of how players choose their actions at any given $t$.

Specifically, we are interested in deriving an equilibrium menu $W(x)$ for the leader representing his decision when the state is $x$ at time $t$, and a menu $U(x, W)$ for the follower representing his decision when he observes the leader’s decision to be $W$ in addition to the state $x$ at time $t$.

For this, let us first define a feedback Stackelberg equilibrium, and then develop a procedure to obtain it.
We begin with specifying the admissible strategy spaces for the manufacturer and the retailer, respectively:

\[ \mathcal{W} = \{ W | W : [0, 1] \rightarrow [0, 1] \text{ and } W(x) \text{ is Lipschitz continuous in } x \}, \]

\[ \mathcal{U} = \{ U | U : [0, 1] \times [0, 1] \rightarrow [0, \infty) \text{ and } U(x, W) \text{ is Lipschitz continuous in } (x, W) \} \].

For a pair of strategies \((W, U) \in \mathcal{W} \times \mathcal{U}\), let \(Y_s, s \geq t\), denote the solution of the state equation

\[ dY_s = (rU(Y_s, W_s)\sqrt{1 - Y_s - \delta Y_s})ds + \sigma(Y_s)dZ_s, \quad Y_t = x. \quad (13.53) \]

We should note that \(Y_s\) here stands for \(Y_s(t, x; W, U)\), as the solution depends on the specified arguments.
Then $J^{t,x}_M(W(\cdot), U(\cdot, W(\cdot)))$ and $J^{t,x}_R(W(\cdot), U(\cdot, W(\cdot)))$ representing the current-value profits of the manufacturer and retailer at time $t$ are, respectively,

$$
J^{t,x}_M(W(\cdot), U(\cdot, W(\cdot)))
= E \int_t^\infty e^{-\rho(s-t)} [\pi_M Y_s - W(Y_s)\{U(Y_s, W(Y_s))\}^2], \quad (13.54)
$$

$$
J^{t,x}_R(W(\cdot), U(\cdot, W(\cdot)))
= E \int_t^\infty e^{-\rho(s-t)} [\pi Y_s - (1 - W(Y_s))\{U(Y_s, W(Y_s))\}^2], \quad (13.55)
$$

where we should stress that $W(\cdot), U(\cdot, W(\cdot))$ evaluated at any state $\zeta$ are $W(\zeta), U(\zeta, W(\zeta))$. We can now define our equilibrium concept.
A pair of strategies \((W^*, U^*) \in \mathcal{W} \times \mathcal{U}\) is called a feedback Stackelberg equilibrium if

\[
J^t,x_M (W^*(\cdot), U^*(\cdot, W^*(\cdot))) \\
\geq J^t,x_M (W(\cdot), U^*(\cdot, W(\cdot))), \ W \in \mathcal{W}, x \in [0, 1], \ t \geq 0, \ (13.56)
\]

and

\[
J^t,x_R (W^*(\cdot), U^*(\cdot, W^*(\cdot))) \\
\geq J^t,x_R (W^*(\cdot), U(\cdot, W^*(\cdot))), \ U \in \mathcal{U}, x \in [0, 1], \ t \geq 0. \ (13.57)
\]
It has been shown in Bensoussan et al. (2019) that this equilibrium is obtained by solving a pair of Hamilton-Jacobi-Bellman equations where a static Stackelberg game is played at the Hamiltonian level at each \( t \), and where

\[
H^M(x, W, U, \lambda^M) = \pi_M x - WU^2 + \lambda^M (rU \sqrt{1 - x - \delta x}) \tag{13.58}
\]

\[
H^R(x, W, U, \lambda^R) = \pi x - (1 - W)U^2 + \lambda^R (rU \sqrt{1 - x - \delta x}) \tag{13.59}
\]

are the Hamiltonians for the manufacturer and the retailer, respectively.
Feedback Stackelberg Stochastic Differential Game of Cooperative Advertising Continued

To solve this Hamiltonian level game, we first maximize $H^R$ with respect to $U$ in terms of $x$ and $W$. The first-order condition gives

$$U^*(x, W) = \frac{\lambda^R r \sqrt{1 - x}}{2(1 - W)},$$  \hspace{1cm} (13.60)

as the optimal response of the follower for any decision $W$ by the leader. We then substitute this for $U$ in $H^M$ to obtain

$$H^M(x, W, U^*(x, W), \lambda^M) = \pi_M x - \frac{W(\lambda^R r)^2(1 - x)}{4(1 - W)^2} + \lambda^M \left( \frac{(\lambda^R r)^2(1 - x)}{2(1 - W)} - \delta x \right).$$  \hspace{1cm} (13.61)
The first-order condition of maximizing $H^M$ with respect to $W$ gives us

$$W(x) = \frac{2\lambda^M - \lambda^R}{2\lambda^M + \lambda^R}.$$  \hspace{1cm} (13.62)

Clearly $W(x) \geq 1$ makes no intuitive sense because it would induce the retailer to spend an infinite amount on advertising, and that would not be optimal for the leader. Moreover, $\lambda^M$ and $\lambda^R$, the marginal valuations of the market share of the leader and the follower, respectively, are expected to be positive, and therefore it follows from (13.62) that $W(x) < 1$. Thus, we set,

$$W^*(x) = \max \left\{ 0, \frac{2\lambda^M - \lambda^R}{2\lambda^M + \lambda^R} \right\}.$$  \hspace{1cm} (13.63)
We can now write the HJB equations as

\[
\rho V_R^R = H_R^R(x, W^*(x), U^*(x, W^*(x)), V_x^R) + (\sigma(x))^2 V_{xx}^R / 2 \\
= \pi x + \frac{(V_x^R r)^2(1 - x)}{4(1 - W^*(x))} - V_x^R \delta x + \frac{(\sigma(x))^2 V_{xx}^R}{2},
\]

(13.64)

\[
\rho V_M^M = H_M^M(x, W^*(x), U^*(x, W^*(x)), V_x^M) + (\sigma(x))^2 V_{xx}^M / 2 \\
= \pi_M x - \frac{(V_x^M r)^2(1 - x) W^*(x)}{4(1 - W^*(x))^2} + \frac{V_x^M V_x^R r^2(1 - x)}{2(1 - W^*(x))} \\
- V_x^M \delta x + (\sigma(x))^2 V_{xx}^M / 2.
\]

(13.65)

The solution of these equations will yield the value functions \( V_M^M(x) \) and \( V_R^R(x) \).
With these in hand, we can give the equilibrium menu of actions to the manufacturer and the retailer to guide their decisions at each $t$. These menus are

$$W^*(x) = \max \left\{ 0, \frac{2V^M_x - V^R_x}{2V^M_x + V^R_x} \right\} \quad \text{and} \quad U^*(x, W) = \frac{V^R_x r \sqrt{1 - x}}{2(1 - W)}.$$  \hspace{1cm} (13.66)

To solve for the value function, we next investigate the two cases where the subsidy rate is (a) zero and (b) positive, and determine the condition required for no subsidy to be optimal.
Case (a): No Co-op Advertising ($W^* = 0$). Inserting $W^*(x) = 0$ into (13.66) gives

$$U^*(x, 0) = \frac{rV_x R \sqrt{1 - x}}{2}.$$  \hfill (13.67)

Inserting $W^*(x) = 0$ into (13.65) and (13.64), we have

$$\rho V^M = \pi M x + \frac{V^M x^2 V^R r^2 (1 - x)}{2} - V^M x \delta x + \frac{(\sigma(x))^2 V^M_{xx}}{2}, \hfill (13.68)$$

$$\rho V^R = \pi x + \frac{(V^R x)^2 r^2 (1 - x)}{4} - V^R x \delta x + \frac{(\sigma(x))^2 V^R_{xx}}{2}. \hfill (13.69)$$
Let $V^M(x) = \alpha_M + \beta_M x$ and $V^R(x) = \alpha + \beta x$. Then, $V^M_x = \beta_M$ and $V^R_x = \beta$. Substituting these into (13.68) and (13.69) and equating like powers of $x$, we can express all of the unknowns in terms of $\beta$, which itself can be explicitly solved. That is, we obtain

$$\beta = \frac{2\pi}{\sqrt{(\rho + \delta)^2 + r^2\pi + (\rho + \delta)}}, \quad \beta_M = \frac{2\pi_M}{2(\rho + \delta) + \beta r^2},$$

$$\alpha = \frac{\beta^2 r^2}{4\rho}, \quad \alpha_M = \frac{\beta \beta_M r^2}{2\rho}.$$  

(13.70)
Using $V_x^R = \beta$ and (13.70) in (13.67), we can write $U^*(x) = \sqrt{\rho \alpha (1 - x)}$. Finally, we can derive the required condition from the right-hand side of $W^*(x)$ in (13.66), which is $2V_x^M \leq V_x^R$, for no co-op advertising ($W^* = 0$) in the equilibrium. This is given by $2\beta^M \leq \beta$, or

$$
\frac{4\pi M}{2(\rho + \delta)} + \frac{2\pi r^2}{\sqrt{(\rho + \delta)^2 + r^2\pi + (\rho + \delta)}} \leq \frac{2\pi}{\sqrt{(\rho + \delta)^2 + r^2\pi + (\rho + \delta)}}.
$$

(13.71)

After a few steps of algebra, this yields the required condition

$$
\theta := \frac{\pi M}{\sqrt{(\rho + \delta)^2 + r^2\pi}} - \frac{\pi}{\sqrt{(\rho + \delta)^2 + r^2\pi + (\rho + \delta)}} \leq 0.
$$

(13.72)

Next, we obtain the solution when $\theta > 0$. 
**Case (b):** Co-op Advertising ($W^* > 0$). Then, $W^*(x)$ in (13.66) reduces to

$$W^*(x) = \frac{2V_x^M - V_x^R}{2V_x^M + V_x^R}. \quad (13.73)$$

Inserting this for $W^*(x)$ into (13.65) and (13.64), we have

$$\rho V^M = \pi M x - \frac{r^2(1 - x)[4(V_x^M)^2 - (V_x^R)^2]}{16} + \frac{V_x^M r^2(1 - x)[2V_x^M + V_x^R]}{4} - V_x^M \delta x + \frac{(\sigma(x))^2 V_x^M}{2}, \quad (13.74)$$

$$\rho V^R = \pi x + \left[\frac{(V_x^R)^2 r^2(1 - x)}{4}\right] \left[\frac{2V_x^M + V_x^R}{2V_x^R}\right] - V_x^R \delta x + \frac{(\sigma(x))^2 V_x^R}{2}. \quad (13.75)$$
Once again, $V^M(x) = \alpha_M + \beta_M x$, $V^R = \alpha + \beta x$, $V^M_x = \beta_M$, and $V^R_x = \beta$. Substituting these into (13.74) and (13.75) and equating like powers of $x$, we have

$$\alpha = \frac{\beta(\beta + 2\beta_M)r^2}{8\rho}, \quad (13.76)$$

$$(\rho + \delta)\beta = \pi - \frac{\beta(\beta + 2\beta_M)r^2}{8}, \quad (13.77)$$

$$\alpha_M = \frac{(\beta + 2\beta_M)^2r^2}{16\rho}, \quad (13.78)$$

$$(\rho + \delta)\beta_M = \pi_M - \frac{(\beta + 2\beta_M)^2r^2}{16}. \quad (13.79)$$
Using (13.66), (13.73), and (13.78), we can write $U^*(x, W^*(x))$, with a slight abuse of notation, as

$$U^*(x) = \frac{r(V_x^R + 2V_x^M)\sqrt{1 - x}}{4} = \sqrt{\rho \alpha_M (1 - x)}.$$  \hspace{1cm} (13.80)

The four equations (13.76)–(13.79) determine the solutions for the four unknowns, $\alpha, \beta, \alpha_M,$ and $\beta_M$. From (13.77) and (13.79), we can obtain

$$\beta^3 + a_1 \beta^2 + a_2 \beta + a_3 = 0,$$ \hspace{1cm} (13.81)

where

$$a_1 = \frac{2\pi M}{\rho + \delta} > 0, \quad a_2 = \frac{8\pi}{r^2} > 0, \quad \text{and} \quad a_3 = \frac{-8\pi^2}{(\rho + \delta)r^2} < 0.$$
From Descarte’s Rule of Signs, there exists a unique, positive real root. The two remaining roots may be both imaginary or both real and negative.

Since this is a cubic equation, a complete solution can be obtained. Using *Mathematica* or following Spiegel, Lipschutz, and Liu (2008), we can write down the three roots as
Feedback Stackelberg Stochastic Differential Game of Cooperative Advertising Continued

\[ \beta(1) = S + T - \frac{1}{3}a_1, \]

\[ \beta(2) = -\frac{1}{2}(S + T) - \frac{1}{3}a_1 + \frac{\sqrt{3}}{2}i(S - T), \]

\[ \beta(3) = -\frac{1}{2}(S + T) - \frac{1}{3}a_1 - \frac{\sqrt{3}}{2}i(S - T), \]

with

\[ S = \sqrt[3]{R + \sqrt{Q^3 + R^2}}, \quad T = \sqrt[3]{R - \sqrt{Q^3 + R^2}}, \quad i = \sqrt{-1}, \]

where

\[ Q = \frac{3a_2 - a_1^2}{9}, \quad R = \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54}. \]
Next, we identify the positive root in each of the following three cases:

**Case 1:** \((Q > 0)\). We have \(S > 0 > T\) and \(Q^3 + R^2 > 0\). There is one positive root and two imaginary roots. The positive root is \(\beta = S + T - (1/3)a_1\).

**Case 2:** \((Q < 0\) and \(Q^3 + R^2 > 0\).\) There are three real roots with one positive root, which is \(\beta = S + T - (1/3)a_1\).

**Case 3:** \((Q < 0\) and \(Q^3 + R^2 < 0\).\) \(S\) and \(T\) are both imaginary. We have three real roots with one positive root. While subcases can be given to identify the positive root, for our purposes, it is enough to identify it numerically.

Finally, we can conclude that \(2\beta_M - \beta > 0\) so that \(W^* > 0\), since if this were not the case, then \(W^*\) would be zero, and we would once again be in Case (a).
Table 13.1: Optimal Feedback Stackelberg Solution

<table>
<thead>
<tr>
<th></th>
<th>(a) if $\theta \leq 0$</th>
<th>(b) if $\theta &gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No Co-op Equilibrium</td>
<td>Co-op Equilibrium</td>
</tr>
<tr>
<td>Retailer’s profit $V^R$</td>
<td>$V^R(x) = \alpha + \beta x$</td>
<td>$V^R(x) = \alpha + \beta x$</td>
</tr>
<tr>
<td>Manufacturer’s profit $V^M$</td>
<td>$V^M(x) = \alpha_M + \beta_M x$</td>
<td>$V^M(x) = \alpha_M + \beta_M x$</td>
</tr>
<tr>
<td>Coefficients of</td>
<td>$\beta = \frac{2\pi}{\sqrt{(\rho+\delta)^2 + r\pi + (\rho+\delta)}}$</td>
<td>$\beta = \frac{\pi}{\rho+\delta} - \frac{\beta(\beta + 2\beta_M)^2 r^2}{8(\rho+\delta)}$</td>
</tr>
<tr>
<td>profit functions,</td>
<td>$\beta_M = \frac{2\pi M}{2(\rho+\delta) + \beta r^2}$</td>
<td>$\beta_M = \frac{\pi M}{\rho+\delta} - \frac{\beta(\beta + 2\beta_M)^2 r^2}{16(\rho+\delta)}$</td>
</tr>
<tr>
<td>$\alpha, \beta, \alpha_M, \beta_M$</td>
<td>$\alpha = \frac{\beta^2 r^2}{4\rho}$</td>
<td>$\alpha_M = \frac{\beta(\beta + 2\beta_M)^2 r^2}{8\rho}$</td>
</tr>
<tr>
<td>obtained from:</td>
<td>$\alpha_M = \frac{\beta\beta_M r^2}{2\rho}$</td>
<td>$\alpha_M = \frac{(\beta + 2\beta_M)^2 r^2}{16\rho}$</td>
</tr>
<tr>
<td>Subsidy rate $W^*(x) =$</td>
<td>0</td>
<td>$\frac{2\beta M - \beta}{2\beta M + \beta} = 1 - \frac{\alpha}{\alpha_M}$</td>
</tr>
<tr>
<td>Advertising effort $U^*(x) =$</td>
<td>$\frac{r\beta \sqrt{1-x}}{2} = \sqrt{\rho\alpha (1-x)}$</td>
<td>$\frac{r(\beta + 2\beta_M) \sqrt{1-x}}{4} = \sqrt{\rho\alpha_M (1-x)}$</td>
</tr>
</tbody>
</table>
Figure 13.2: Optimal subsidy rate vs. Retailer’s margin and Figure 13.3: Optimal subsidy rate vs. Manufacturer’s margin

- We can now summarize the optimal feedback Stackelberg equilibrium in Table 13.1. In Exercises 13.7-13.10, you are asked to further explore the model of this section when the parameters \( \pi = 0.25, \pi_M = 0.5, r = 2, \rho = 0.05, \delta = 1, \) and \( \sigma(x) = 0.25\sqrt{x(1-x)}. \)
For this case, He et al. (2009) obtain the comparative statics as shown in Fig. 13.2 and Fig. 13.3 respectively.
Figure 13.3: Optimal subsidy rate vs. Manufacturer’s margin

![Figure 13.3: Optimal Subsidy Rate vs. Manufacturer Margin](image)

Figure 13.3: Optimal Subsidy Rate vs. Manufacturer’s Margin
Feedback Stackelberg Stochastic Differential Game of Cooperative Advertising

- There have been many applications of differential games in marketing in general and optimal advertising in particular.