Main Purpose: To introduce the maximum principle as a necessary condition that must be satisfied by any optimal control.

Necessary condition for optimization of dynamic systems.

General derivation by Pontryagin et al. in 1956-60.

A simple (but not completely rigorous) proof using dynamic programming.

Examples.

Statement of sufficiency conditions.

Computational method.
Optimal control theory deals with the problem of optimizing dynamic systems.

It requires a clear mathematical description of the system to be optimized, the constraints imposed on the system, and the objective function to be maximized (or minimized).
State Equation: Given the initial state $x_0$ of the system and control history $u(t), \ t \in [0, T]$, of the process, the evolution of the system may be described by the first-order differential equation, known also as the *state equation*.

$$\dot{x}(t) = f(x(t), u(t), t), \ x(0) = x_0, \quad (2.1)$$

where,

- $x(t) \in \mathbb{E}^n$ is the vector of *state variables*,
- $u(t) \in \mathbb{E}^m$ is the vector of *control variables*,
- and the function $f : \mathbb{E}^n \times \mathbb{E}^m \times \mathbb{E}^1 \rightarrow \mathbb{E}^n$.

The function $f$ is assumed to be continuously differentiable. We also assume $x$ to be a column vector and $f$ to be a column vector of functions.

The path $x(t), \ t \in [0, T]$, is called a *state trajectory* and $u(t), \ t \in [0, T]$, is called a *control trajectory* or simply, a *control*. 
Let us define an *admissible control* to be a control trajectory $u(t)$, $t \in [0, T]$, which is piecewise continuous and satisfies, in addition,

$$u(t) \in \Omega(t) \subset E^m, \quad t \in [0, T]. \quad (2.2)$$

Usually, the set $\Omega(t)$ is determined by physical or economic constraints on the values of the control variables at time $t$.

An objective function is a quantitative measure of the performance of the system over time. Objective function is mathematically defined as follows,

$$J = \int_0^T F(x(t), u(t), t)dt + S(x(T), T) \quad (2.3)$$

where the functions $F : E^n \times E^m \times E^1 \rightarrow E^1$ and $S : E^n \times E^1 \rightarrow E^1$ are assumed to be continuously differentiable.
The problem is to find an admissible control \( u^* \), which maximizes the objective function subject to the state equation and the control constraints as defined in the previous slides. We now restate the optimal control problem as:

\[
\left\{ \begin{array}{l}
\max_{u(t) \in \Omega(t)} \left\{ J = \int_0^T F(x, u, t) dt + S(x(T), T) \right\} \\
\text{subject to} \\
\dot{x} = f(x, u, t), \quad x(0) = x_0.
\end{array} \right.
\]

The control \( u^* \) is called an optimal control and \( x^* \), determined by means of the state equation with \( u = u^* \), is called the optimal trajectory or an optimal path.

The optimal value \( J(u^*) \) of the objective function will be denoted as \( J^* \).
Three Different Forms of Optimal Control Problem

- **Case 1:** The optimal control problem (2.4) is said to be in *Bolza form* because of the form of the objective function in (2.3).

- **Case 2:** When $S \equiv 0$, it is said to be in *Lagrange form*.

- **Case 3:** When $F \equiv 0$, it is said to be in *Mayer form*.

- **Case 3a:** And if $F \equiv 0$ and $S$ is linear, it is in *linear Mayer form*, i.e.,

\[
\begin{cases}
\max_{u(t) \in \Omega(t)} \{ J = cx(T) \} \\
\text{subject to} \\
\dot{x} = f(x, u, t), \quad x(0) = x_0,
\end{cases}
\]

where $c = (c_1, c_2, \cdots, c_n)$ is an $n$-dimensional row vector of constants.
The Bolza form can be reduced to the linear Mayer form by defining a new state vector \( y = (y_1, y_2, \ldots, y_{n+1}) \), having \( n + 1 \) components defined as follows:

\[
y_i = x_i \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad y_{n+1} \quad \text{defined by the solution of the equation}
\]

\[
\dot{y}_{n+1} = F(x, u, t) + \frac{\partial S(x, t)}{\partial x} f(x, u, t) + \frac{\partial S(x, t)}{\partial t},
\]

with \( y_{n+1}(0) = S(x_0, 0) \).

By writing \( f(x, u, t) \) as \( f(y, u, t) \), and by denoting the right-hand side of (2.6) as \( f_{n+1}(y, u, t) \), we can write the new state equation in the vector form as,

\[
\dot{y} = \begin{pmatrix} \dot{x} \\ \dot{y}_{n+1} \end{pmatrix} = \begin{pmatrix} f(y, u, t) \\ f_{n+1}(y, u, t) \end{pmatrix}, \quad y(0) = \begin{pmatrix} x_0 \\ S(x_0, 0) \end{pmatrix}. \quad (2.7)
\]
Now we integrate (2.6) from 0 to $T$, we see that

$$y_{n+1}(T) - y_{n+1}(0) = \int_0^T F(x,u,t) dt + S(x(T), T) - S(x_0, 0).$$

In view of setting the initial condition as $y_{n+1}(0) = S(x_0, 0)$, the problem in (2.4) can be expressed as that of maximizing

$$J = \int_0^T F(x,u,t) dt + S(x(T), T) = y_{n+1}(T) = cy(T) \quad (2.8)$$

where $c = (0, \cdots, 0, 1)$ is an $(n+1)$-dimensional row vector with the first $n$ terms all 0.
Richard Bellman (1957) in his book on dynamic programming states the principle of optimality as follows:

"An optimal policy has the property that, whatever the initial state and initial decision are, the remaining decision must constitute an optimal policy with regard to the outcome resulting from the initial decision."

We will use the principle of optimality to derive conditions on the value function \( V(x, t) \).
**Assertion:** If ABE (shown in blue) is an optimal path from A to E, then BE (in blue) is an optimal path from B to E.

**Proof:** Suppose it is not. Then there is another path (existence is assumed here) BCE (in red), which is optimal from B to E, i.e.,

\[ J_{BCE} > J_{BE} \]

But then,

\[ J_{ABE} = J_{AB} + J_{BE} < J_{AB} + J_{BCE} = J_{ABCE} \]

This contradicts the hypothesis that ABE is an optimal path from A to E.
Example 2.1 and it’s Solution

- Convert the following single-state problem in Bolza form to its linear Mayer form:

$$\max \left\{ J = \int_0^T \left( x - \frac{u^2}{2} \right) dt + \frac{1}{4} [x(T)]^2 \right\}$$

subject to

$$\dot{x} = u, \ x(0) = x_0.$$

- **Solution.** We use (2.6) to introduce the additional state variable $y_2$ as follows:

$$\dot{y}_2 = x - \frac{u^2}{2} + \frac{1}{2} xu, \ y_2(0) = \frac{1}{4} x_0^2.$$
Solution of Example 2.1

Then,

\[
y_2(T) = y_2(0) + \int_0^T \left( x - \frac{u^2}{2} + \frac{1}{2} xu \right) \, dt
\]

\[
= \int_0^T \left( x - \frac{u^2}{2} \right) \, dt + \int_0^T \left( \frac{1}{2} x \dot{x} \right) \, dt + y_2(0)
\]

\[
= \int_0^T \left( x - \frac{u^2}{2} \right) \, dt + \int_0^T d \left( \frac{1}{4} x^2 \right) + y_2(0)
\]

\[
= \int_0^T \left( x - \frac{u^2}{2} \right) \, dt + \frac{1}{4} [x(T)]^2 - \frac{1}{4} x_0^2 + y_2(0)
\]

\[
= \int_0^T \left( x - \frac{u^2}{2} \right) \, dt + \frac{1}{4} x(T)^2
\]

\[= J.\]
Thus, the linear Mayer form version with the two-dimensional state $y = (x, y_2)$ can be stated as

$$\max \{ J = y_2(T) \}$$

subject to

$$\dot{x} = u, \quad x(0) = x_0,$$

$$\dot{y}_2 = x - \frac{u^2}{2} + \frac{1}{2} xu, \quad y_2(0) = \frac{1}{4} x_0^2.$$
A Dynamic Programming Example

Stagecoach Problem:

Stages 1 2 3 4

Costs \( C_{i,j} \):

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<tr>
<th></th>
<th>5</th>
<th>6</th>
<th>7</th>
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<tbody>
<tr>
<td>2</td>
<td>7</td>
<td>4</td>
<td>6</td>
</tr>
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<td>3</td>
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<td>4</td>
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<td>5</td>
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<td>7</td>
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</table>
Stagecoach Problem:

Note that 1-2-6-9-10 is a greedy (or myopic) path that minimizes cost at each stage. However, this may not be the minimal cost solution. For example, 1-4-6 yields a cheaper cost than 1-2-6.
Solution of Dynamic Programming Example

Let $1 - u_1 - u_2 - u_3 - 10$ be the optimal path. Let $f_n(s, u_n)$ be the minimal cost at stage $n$ given that the current state is $s$ and the decision taken is $u_n$. Let $f^*_n(s)$ be the minimal cost at stage $n$ if the current state is $s$. Then

$$f^*_n(s) = \min_{u_n} f_n(s, u_n) = \min_{u_n} \{c_{s, u_n} + f^*_{n+1}(u_n)\}$$

This is the Recursive Equation of DP. It can be solved by a backward procedure, which starts at the terminal stage and stops at the initial stage.
Solution of Dynamic Programming Example Cont.

- Stage 4:

<table>
<thead>
<tr>
<th>s</th>
<th>$f^*_4(s)$</th>
<th>$u^*_4$</th>
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<tbody>
<tr>
<td>8</td>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>10</td>
</tr>
</tbody>
</table>

- Stage 3:

<table>
<thead>
<tr>
<th>s</th>
<th>$f_3(s, u_3) = C_{s,u_3} + f^*_4(u_3)$</th>
<th>$f^*_3(s)$</th>
<th>$u^*_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td></td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$1 + 3 = 4$</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>6</td>
<td>$6 + 3 = 9$</td>
<td>7</td>
<td>9</td>
</tr>
<tr>
<td>7</td>
<td>$3 + 3 = 6$</td>
<td>6</td>
<td>8</td>
</tr>
</tbody>
</table>
Stage 2:

\[ f_2(s, u_2) = C_{s,u_2} + f_3^*(u_2) \]

<table>
<thead>
<tr>
<th>( s )</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>( f_2^*(s) )</th>
<th>( u_2^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>7 + 4 = 11</td>
<td>4 + 7 = 11</td>
<td>6 + 6 = 12</td>
<td>11</td>
<td>5, 6</td>
</tr>
<tr>
<td>3</td>
<td>3 + 4 = 7</td>
<td>2 + 7 = 9</td>
<td>4 + 6 = 10</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>4 + 4 = 8</td>
<td>1 + 7 = 8</td>
<td>5 + 6 = 11</td>
<td>8</td>
<td>5, 6</td>
</tr>
</tbody>
</table>
Solution of Dynamic Programming Example Cont.

Stage 1:

<table>
<thead>
<tr>
<th>s</th>
<th>$f_1(s, u_1) = C_{s,u_1} + f_2^*(u_1)$</th>
<th>$f_1^*(s)$</th>
<th>$u_1^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2 + 11 = 13</td>
<td>11</td>
<td>3, 4</td>
</tr>
<tr>
<td>3</td>
<td>4 + 7 = 11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>3 + 8 = 11</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The optimal paths:

\[
1 \rightarrow 3 \rightarrow 5 \rightarrow 8 \rightarrow 10 \\
1 \rightarrow 4 \rightarrow 5 \rightarrow 8 \rightarrow 10 \\
1 \rightarrow 4 \rightarrow 6 \rightarrow 9 \rightarrow 10 \]

Total Cost = 11.
Suppose $V(x, t) : E^n \times E^1 \rightarrow E^1$ is a function whose value is the maximum value of the objective function of the control problem for the system, given that we start at time $t$ in state $x$. That is,

$$V(x, t) = \max_{u(s) \in \Omega(s)} \left[ \int_t^T F(x(s), u(s), s) ds + S(x(T), T) \right], \quad (2.9)$$

where for $s \geq t$,

$$\frac{dx(s)}{ds} = f(x(s), u(s), s), \quad x(t) = x.$$
The Hamilton-Jacobi-Bellman Equation Cont.

Figure 2.1: An Optimal Path in the State-Time Space

\[ V(x + \delta x, t + \delta t) \]

\[ V(x, t) \]

Optimal Path \( x^*(t) \)
By the principle of optimality, the change in the objective function is made up of two parts:

1. the incremental change in $J$ from $t$ to $t + \delta t$, which is given by the integral of $F(x, u, t)$ from $t$ to $t + \delta t$;

2. the value function $V(x + \delta x, t + \delta t)$ at time $t + \delta t$.

The control actions $u(\tau)$ should be chosen to lie in $\Omega(\tau)$, $\tau \in [t, t + \delta t]$ and to maximize the sum of these two terms.

In equation form this is,

$$V(x, t) = \max_{u(\tau) \in \Omega(\tau)} \left\{ \int_{t}^{t+\delta t} F[x(\tau), u(\tau), \tau] d\tau + V[x(t + \delta t), t + \delta t] \right\},$$

(2.10)

where $\delta t$ represents a small increment in $t$. 
Since $F$ is a continuous function, the integral in (2.10) is approximately $F(x, u, t)\delta t$ so we can rewrite (2.10) as

$$V(x, t) = \max_{u \in \Omega(t)} \{ F(x, u, t)\delta t + V[x(t + \delta t), t + \delta t] \} + o(\delta t), \quad (2.11)$$

where $o(\delta t)$ denotes a collection of higher-order terms in $\delta t$, see section 1.4.4.

Let us use the Taylor series expansion of $V$ with respect to $\delta t$, as we assume that the value function $V$ is a continuously differentiable, and obtain

$$V[x(t + \delta t), t + \delta t] = V(x, t) + [V_x(x, t) \dot{x} + V_t(x, t)]\delta t + o(\delta t), \quad (2.12)$$
Substituting for $\dot{x}$ from (2.1) in the above equation and then using it in (2.11), we obtain

$$V(x, t) = \max_{u \in \Omega(t)} \left\{ F(x, u, t) \delta t + V(x, t) + V_x(x, t) f(x, u, t) \delta t + V_t(x, t) \delta t \right\} + o(\delta t).$$ \hspace{1cm} (2.13)

Canceling $V(x, t)$ on both sides and then dividing by $\delta t$ we get

$$0 = \max_{u \in \Omega(t)} \left\{ F(x, u, t) + V_x(x, t) f(x, u, t) + V_t(x, t) \right\} + \frac{o(\delta t)}{\delta t}.$$ \hspace{1cm} (2.14)

Now we let $\delta t \to 0$ and obtain the following equation

$$0 = \max_{u \in \Omega(t)} \left\{ F(x, u, t) + V_x(x, t) f(x, u, t) + V_t(x, t) \right\},$$ \hspace{1cm} (2.15)

for which the boundary condition is

$$V(x, T) = S(x, T).$$ \hspace{1cm} (2.16)
The components of the vector \( V_x(x, t) \) can be interpreted as the marginal contributions of the state variables \( x \) to the value function or the maximized objective function. We denote the marginal return vector (along the optimal path \( x^*(t) \)) by the *adjoint* (row) vector \( \lambda(t) \in E^n \), i.e.,

\[
\lambda(t) = V_x(x^*(t), t) := V_x(x, t) \mid_{x=x^*(t)} .
\]  

(2.17)

We introduce a function \( H : E^n \times E^m \times E^n \times E^1 \to E^1 \) called the *Hamiltonian*

\[
H(x, u, \lambda, t) = F(x, u, t) + \lambda f(x, u, t) .
\]  

(2.18)

We can then rewrite equation (2.15) as the equation

\[
\max_{u \in \Omega(t)} [H(x, u, V_x, t) + V_t] = 0 ,
\]  

(2.19)

called the *Hamilton-Jacobi-Bellman equation* or, simply, the HJB equation.
From (2.19) and (2.17), we can get the Hamiltonian maximizing condition of the maximum principle,

\[
H[x^*(t), u^*(t), \lambda(t), t] + V_t(x^*(t), t) \geq H[x^*(t), u, \lambda(t), t] + V_t(x^*(t), t). \tag{2.20}
\]

Canceling the term \( V_t \) on both sides, we obtain the Hamiltonian maximizing condition

\[
H[x^*(t), u^*(t), \lambda(t), t] \geq H[x^*(t), u, \lambda(t), t] \tag{2.21}
\]

for all \( u \in \Omega(t) \).

**Remark 2.1:** We use \( u^* \) and \( x^* \) for optimal control and state to distinguish them from an admissible control \( u \) and the corresponding state \( x \), respectively. However, since the adjoint variable \( \lambda \) is defined only along the optimal path, there is no need for such a distinction, and therefore we do not use the superscript \( * \) on \( \lambda \).
Derivation of the Adjoint Equation

Let

\[ x(t) = x^*(t) + \delta x(t), \quad (2.22) \]

where \( \| \delta x(t) \| < \varepsilon \) for a small positive \( \varepsilon \).

Fix time instant \( t \) and use the HJB equation in (2.19) as

\[
0 = H[x^*(t), u^*(t), V_x(x^*(t), t), t] + V_t(x^*(t), t) \\
\geq H[x(t), u^*(t), V_x(x(t), t), t] + V_t(x(t), t). \quad (2.23)
\]

LHS = 0 from (19) since \( u^*(t) \) maximizes \( H + V_t \). RHS will be zero if \( u^*(t) \) also maximizes \( H + V_t \), with \( x(t) \) as the state. In general \( x(t) \neq x^*(t) \), and thus RHS \( \leq 0 \).

But then \( \text{RHS} \big|_{x(t)=x^*(t)} = 0 \Rightarrow \text{RHS is maximized at } x(t) = x^*(t) \).
Since $x(t)$ is not explicitly constrained, $x^*(t)$ is an unconstrained local maximum of the RHS of (2.23), and so we have
\[
\frac{\partial \text{RHS}_{(2.23)}}{\partial x} \bigg|_{x(t)=x^*(t)} = 0 \text{ or }
\]
\[
H_x[x^*(t), u^*(t), V_x(x^*(t), t), t] + V_{tx}(x^*(t), t) = 0.
\] (2.24)

With $H = F + V_x f$ from (2.17) and (2.18), we obtain
\[
H_x = F_x + V_x f_x + f^T V_{xx} = F_x + V_x f_x + (V_{xx} f)^T.
\]

Substituting this in (2.24) and knowing that $V_{xx} = (V_{xx})^T$, we have
\[
F_x + V_x f_x + f^T V_{xx} + V_{tx} = F_x + V_x f_x + (V_{xx} f)^T + V_{tx} = 0.
\] (2.25)

**Note:** (2.25) assumes $V$ to be twice continuously differentiable. See (1.16) or Exercise 1.10 for details.
To obtain the so-called adjoint equation from (2.25), which is the necessary condition, we begin by taking the time derivative of $V_x(x,t)$. Thus,

$$\frac{dV_x}{dt} = \left( \frac{dV_{x_1}}{dt}, \frac{dV_{x_2}}{dt}, \ldots, \frac{dV_{x_n}}{dt} \right)$$

$$= \left( V_{x_1} \dot{x} + V_{x_1} t, V_{x_2} \dot{x} + V_{x_2} t, \ldots, V_{x_n} \dot{x} + V_{x_n} t \right)$$

$$= \left( \sum_{i=1}^{n} V_{x_1} \dot{x}_i, \sum_{i=1}^{n} V_{x_2} \dot{x}_i, \ldots, \sum_{i=1}^{n} V_{x_n} \dot{x}_i \right) + (V_x)_t$$

$$= (V_{xx} \dot{x})^T + V_{xt}$$

$$= (V_{xx} f)^T + V_{tx}.$$  

(2.26)
In (2.26) note that

\[ V_{xi}x = (V_{xi}x_1, V_{xi}x_2, \ldots, V_{xi}x_n) \]

and

\[ V_{xx}\dot{x} = \begin{pmatrix} V_{x1}x_1 & V_{x1}x_2 & \cdots & V_{x1}x_n \\ V_{x2}x_1 & V_{x2}x_2 & \cdots & V_{x2}x_n \\ \vdots & \vdots & \ddots & \vdots \\ V_{xn}x_1 & V_{xn}x_2 & \cdots & V_{xn}x_n \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix}. \quad (2.27) \]

Using (2.25) and (2.26), we have

\[
\frac{dV_x}{dt} = -F_x - V_x f_x. \quad (2.28)
\]
Using (2.17) we can rewrite (2.28) as

\[ \dot{\lambda} = -F_x - \lambda f_x. \]

From \( H \) as defined in (2.18), we have

\[ \dot{\lambda} = -H_x. \] (2.29)

From the definition of \( \lambda \) in (2.17) and the boundary condition (2.16), we have the terminal boundary condition, which is also called the transversality condition

\[ \lambda(T) = \frac{\partial S(x, T)}{\partial x} \bigg|_{x=\mathbf{x}^*(T)} = S_x(\mathbf{x}^*(T), T). \] (2.30)

The adjoint equation (2.29) together with its boundary condition (2.30) determine the adjoint variables.
The maximum principle

The necessary conditions for \( u^*(t), \ t \in [0, T], \) to be an optimal control are:

\[
\begin{align*}
\dot{x}^* &= f(x^*, u^*, t), \quad x^*(0) = x_0, \\
\dot{\lambda} &= -H_x[x^*, u^*, \lambda, t], \quad \lambda(T) = S_x(x^*(T), T), \\
H[x^*, u^*, \lambda, t] &\geq H[x^*, u, \lambda, t], \quad \forall u \in \Omega(t), \ t \in [0, T].
\end{align*}
\] (2.31)
Two-Point Boundary value Problem (TPBVP)

- Two-Point Boundary value Problem (TPBVP) is a system of equations where initial values of some variables and final values of other variables are specified.

- Note also that if we can solve the Hamiltonian maximizing condition for an optimal control function in closed form $u^*(x, \lambda, t)$ so that

$$u^*(t) = u^*[x^*(t), \lambda(t), t],$$

then we can substitute this into the state and adjoint equations to get the TPBVP just in terms of a set of differential equations, i.e.,

$$\begin{cases}
\dot{x}^* = f(x^*, u^*(x^*, \lambda, t), t), & x^*(0) = x_0, \\
\dot{\lambda} = -H_x(x^*, u^*(x^*, \lambda, t), \lambda, t), & \lambda(T) = S_x(x^*(T), T).
\end{cases} \quad (2.32)$$
Let us recall the objective function (2.3):

\[ J = \int_0^T F(x, u, t) dt + S(x(T), T), \]

where

$F$ is considered to be the instantaneous profit rate measured in dollars per unit of time, and

$S(x, T)$ is the salvage value, in dollars, of the system at time $T$ when the terminal state is $x$. 
Multiplying (2.18) formally by $dt$ and using the state equation (2.1) gives
\[
H dt = F dt + \lambda f dt = F dt + \lambda \dot{x} dt = F dt + \lambda dx.
\]

$F(x, u, t) dt$: direct contribution to $J$ from time $t$ to $t + dt$,

$\lambda dx$: indirect contribution to $J$ due to added capital stock $dx$,

$H dt$: total contribution to $J$ from time $t$ to $t + dt$, when $x(t) = x$ and $u(t) = u$ in the interval $[t, t + dt]$. 
By (2.29) and (2.30), we have

\[ \dot{\lambda} = -\frac{\partial H}{\partial x} = -\frac{\partial F}{\partial x} - \lambda \frac{\partial f}{\partial x}, \quad \lambda(T) = S_x(x(T), T). \]

Rewrite the first equation as

\[ -d\lambda = H_x dt = F_x dt + \lambda f_x dt. \]

- \(d\lambda\): marginal cost of holding capital \(x\) from \(t\) to \(t + dt\),
- \(H_x dt\): marginal revenue of investing the capital,
- \(F_x dt\): direct marginal contribution,
- \(\lambda f_x dt\): indirect marginal contribution.

Thus, the adjoint equation implies \(MC = MR\).
Further insight can be obtained by integrating the above adjoint equation from $t$ to $T$ as follows:

$$
\lambda(t) = \lambda(T) + \int_t^T H_x(x(\tau), u(\tau), \lambda(\tau), \tau) d\tau
= S_x(x(T), T) + \int_t^T H_x d\tau.
$$

The price $\lambda(t)$ of a unit capital at time $t$ is the sum of its terminal price $\lambda(T)$ plus the integral of the marginal surrogate profit rate $H_x$ from $t$ to $T$.

Note that the price $\lambda(T)$ of a unit of capital at time $T$ is its marginal salvage value $S_x(x(T), T)$. In the special case when $S \equiv 0$, we have $\lambda(T) = 0$, as clearly no value can be derived or lost from an infinitesimal increase in $x(T)$. 
Consider the problem:

\[
\max \left\{ J = \int_0^1 -x dt \right\} \tag{2.33}
\]

subject to the state equation

\[
\dot{x} = u, \ x(0) = 1 \tag{2.34}
\]

and the control constraint

\[ u \in \Omega = [-1, 1]. \tag{2.35} \]

Note that this is problem (2.4) with \( T = 1, \ F = -x, \ S = 0, \) and \( f = u. \) Because \( F = -x, \) we can interpret the problem as one of minimizing the (signed) area under the curve \( x(t) \) for \( 0 \leq t \leq 1. \)
Solution of Example 2.2

- We form the Hamiltonian

\[ H = -x + \lambda u. \]  (2.36)

- Because the Hamiltonian is linear in \( u \), the form of the optimal control, i.e., the one that would maximize the Hamiltonian, is

\[ u^*(t) = \begin{cases} 
1 & \text{if } \lambda(t) > 0, \\
\text{arbitrary} & \text{if } \lambda(t) = 0, \\
-1 & \text{if } \lambda(t) < 0.
\end{cases} \]  (2.37)

- It is called Bang-Bang Control. In the notation of Section 1.4,

\[ u^*(t) = \text{bang}[-1, 1; \lambda(t)]. \]  (2.38)
Solution of Example 2.2 Cont.

- To find $\lambda$, we write the adjoint equation

$$\dot{\lambda} = -H_x = 1, \quad \lambda(1) = S_x(x(T), T) = 0. \quad (2.39)$$

- Because this equation does not involve $x$ and $u$, we can easily solve it as

$$\lambda(t) = t - 1. \quad (2.40)$$

- It follows that $\lambda(t) = t - 1 < 0$ for $t \in [0, 1)$ and so $u^*(t) = -1, \quad t \in [0, 1)$. Since $\lambda(1) = 0$, for simplicity we can also set $u^*(t) = -1$ at the single point $t = 1$. We can then specify the optimal control to be

$$u^*(t) = -1 \quad \text{for all } t \in [0, 1].$$
Substituting this into the state equation (2.34) we have

\[
\dot{x} = -1, \quad x(0) = 1, \tag{2.41}
\]

whose solution is

\[
x^*(t) = 1 - t \quad \text{for} \quad t \in [0, 1]. \tag{2.42}
\]

The graphs of the optimal state and adjoint trajectories appear in Figure 2.2. Note that the optimal value of the objective function is

\[ J^* = -1/2. \]
Solution of Example 2.2 Cont.

Figure 2.2: Optimal State and Adjoint Trajectories for Example 2.2
Illustration of $\lambda(t)$ as a marginal value of the state

- Let us illustrate that the adjoint variable $\lambda(t)$ gives the marginal value per unit increment in the state variable $x(t)$ at time $t$.
- Note from (2.40) that $\lambda(0) = -1$. Thus, if we increase the initial value $x(0)$ from 1, by a small amount $\varepsilon$, to a new value $1 + \varepsilon$, where $\varepsilon$ may be positive or negative, then we expect the optimal value of the objective function to change from

$$J^* = -1/2$$

... to

$$J^*_{(1+\varepsilon)} = -1/2 + \lambda(0)\varepsilon + o(\varepsilon) = -1/2 - \varepsilon + o(\varepsilon),$$

where we use the subscript $(1+\varepsilon)$ to distinguish the new value from $J^*$ as well as to emphasize its dependence on the new initial condition $x(0) = 1 + \varepsilon$. 
Illustration of $\lambda(t)$ as a marginal value of the state Cont.

- Observe that $u^*(t) = -1$, $t \in [0, 1]$, remains optimal. Then from (2.41) with $x(0) = 1 + \varepsilon$, we can obtain the new optimal state trajectory, shown by the dotted line in Figure 2.2, as

$$x^*_{(1+\varepsilon)}(t) = 1 + \varepsilon - t, \quad t \in [0, 1],$$

where $x^*_y(t)$ indicates the dependence of the optimal trajectory on the initial value $x(0) = y$.

- Substituting this new state trajectory in (2.33) and integrating gives the new objective function value as $-1/2 - \varepsilon$. Since 0 is of the order $o(\varepsilon)$, the claim has been illustrated.

- In general it may be necessary to perform separate calculations for positive and negative $\varepsilon$. 
Simple Examples: Example 2.3

Let us solve the same problem as in Example 2.2 over the interval $[0, 2]$ so that the objective is:

$$\max \left\{ J = \int_0^2 -x \, dt \right\}.$$  \hfill (2.43)

subject to the state equation

$$\dot{x} = u, \quad x(0) = 1$$

and the control constraint

$$u \in \Omega = [-1, 1].$$

The dynamics and constraints are (2.34) and (2.35), respectively, as before. Here we want to minimize the signed area between the horizontal axis and the trajectory of $x(t)$ for $0 \leq t \leq 2$. 
As before, the Hamiltonian is defined by (2.36) and the optimal control is as in (2.38). The adjoint equation

\[
\dot{\lambda} = 1, \quad \lambda(2) = 0 \tag{2.44}
\]

is the same as (2.39) except that now \( T = 2 \) instead of \( T = 1 \).

The solution of (2.44) is easily found to be

\[
\lambda(t) = t - 2, \quad t \in [0, 2]. \tag{2.45}
\]

The state equation (2.41) and its solution (2.42) for \( t \in [0, 2] \) are exactly the same. The graph of \( \lambda(t) \) is shown in Figure 2.3. The optimal value of the objective function is \( J^* = 0 \).
Figure 2.3: Optimal State and Adjoint Trajectories for Example 2.3
Simple Examples: Example 2.4

The next example is:

\[
\max \left\{ J = \int_0^1 -\frac{1}{2} x^2 dt \right\} \tag{2.46}
\]

subject to the same constraints as in Example 2.2, namely,

\[
\dot{x} = u, \ x(0) = 1, \ u \in \Omega = [-1, 1]. \tag{2.47}
\]

Here \( F = -(1/2)x^2 \) so that the interpretation of the objective function (2.46) is that we are trying to find the trajectory \( x(t) \) in order that the area under the curve \((1/2)x^2\) is minimized.
Solution of Example 2.4

- The Hamiltonian is
  \[ H = -\frac{1}{2}x^2 + \lambda u. \]  
  (2.48)

- The control function \( u^*(x, \lambda) \) that maximizes the Hamiltonian in this case depends only on \( \lambda \), and it has the form
  \[ u^*(x, \lambda) = \text{bang}[-1, 1; \lambda]. \]  
  (2.49)

- The adjoint equation is
  \[ \dot{\lambda} = -H_x = x, \quad \lambda(1) = 0. \]  
  (2.50)

- Here the adjoint equation involves \( x \), so we cannot solve it directly. Because the state equation (2.47) involves \( u \), which depends on \( \lambda \), we also cannot integrate it independently without knowing \( \lambda \).
Solution of Example 2.4 Cont.

- A way out of this dilemma is to use some intuition. Since we want to minimize the area under \((1/2)x^2\) and since \(x(0) = 1\), it is clear that we want \(x\) to decrease as quickly as possible. Let us therefore temporarily assume that \(\lambda\) is nonpositive in the interval \([0, 1]\) so that from (2.49) we have \(u = -1\) throughout the interval. With this assumption, we can solve (2.47) as

\[
x(t) = 1 - t.
\]

- Substituting this into (2.50) gives

\[
\dot{\lambda} = 1 - t.
\]
Integrating both sides of this equation from $t$ to 1 gives

\[ \int_t^1 \dot{\lambda}(\tau) d\tau = \int_t^1 (1 - \tau) d\tau, \]

or

\[ \lambda(1) - \lambda(t) = (\tau - \frac{1}{2} \tau^2) \bigg|_t^1, \]

which, using $\lambda(1) = 0$, yields

\[ \lambda(t) = -\frac{1}{2} t^2 + t - \frac{1}{2}. \quad (2.52) \]

The reader may now verify that $\lambda(t)$ is nonpositive in the interval $[0, 1]$, verifying our original assumption. Hence, (2.51) and (2.52) satisfy the necessary conditions. Figure 2.4 shows the graphs of the optimal state and adjoint trajectories.
Solution of Example 2.4 Cont.

Figure 2.4: Optimal Trajectories for Examples 2.4 and 2.5

\[ x^*(t) = 1 - t \]

\[ \lambda(t) = -\frac{t^2}{2} + t - 1/2 \]
Let us rework Example 2.4 with $T = 2$, i.e., with the objective function:

$$\max \left\{ J = \int_0^2 -\frac{1}{2}x^2 dt \right\}$$

subject to the constraints

$$\dot{x} = u, \ x(0) = 1, \ u \in \Omega = [-1, 1].$$

It would be clearly optimal if we could keep $x^*(t) = 0, t \in [1, 2]$. This is possible by setting $u^*(t) = 0, t \in [1, 2]$. Note that this is a singular control, since it gives $\lambda(t) = 0, t \in [1, 2]$. Please see Figure 2.4.
The Hamiltonian is still as in (2.48),

\[ H = -\frac{1}{2} x^2 + \lambda u. \]

The form of the optimal policy remains as in (2.49),

\[ u^*(x, \lambda) = \text{bang}[-1, 1; \lambda]. \]

The adjoint equation is

\[ \dot{\lambda} = x, \quad \lambda(2) = 0. \]
With the trajectory \( x^*(t), \ 0 \leq t \leq 2, \) thus obtained, we can use (2.53) to compute the optimal value of the objective function as

\[
J^* = \int_0^1 -(1/2)(1 - t)^2 dt + \int_1^2 -(1/2)(0) dt = -1/6.
\]

Now suppose that the initial \( x(0) \) is perturbed by a small amount \( \varepsilon \) to \( x(0) = 1 + \varepsilon \), where \( \varepsilon \) may be positive or negative. According to the marginal value interpretation of \( \lambda(0) \), whose value is \(-1/2\) in this example, we can estimate the change in the objective function to be

\[
\lambda(0)\varepsilon + o(\varepsilon) = -\varepsilon/2 + o(\varepsilon).
\]
We calculate directly the impact of the perturbation in the initial value. For this we must obtain new control and state trajectories

\[ u^{(1+\epsilon)}(t) = \begin{cases} 
-1, & t \in [0, 1 + \epsilon], \\
0, & t \in (1 + \epsilon, 2], 
\end{cases} \]

and

\[ x^{(1+\epsilon)}(t) = \begin{cases} 
1 + \epsilon - t, & t \in [0, 1 + \epsilon], \\
0, & t \in (1 + \epsilon, 2], 
\end{cases} \]

where we have used the subscript \((1 + \epsilon)\) to distinguish these from the original trajectories as well as to indicate their dependence on the initial value \(x(0) = 1 + \epsilon\).
We obtain the corresponding optimal value of the objective function:

\[
J^*_0(1 + \varepsilon) = \int_0^{1+\varepsilon} -(1/2)(1 + \varepsilon - t)^2 dt = -1/6 - \varepsilon/2 - \varepsilon^2/2 - \varepsilon^3/6
\]

\[
= -1/6 + \lambda(0)\varepsilon + o(\varepsilon),
\]

where \( o(\varepsilon) = -\varepsilon^2/2 - \varepsilon^3/6. \)

In order to completely specify \( V(x, t) \) for all \( x \in E^1 \) and all \( t \in [0, 2] \), we need to deal with a number of cases.

We will carry out the details only in the case of any \( t \in [0, 2] \) and \( 0 \leq x \leq 2 - t \), and leave the listing of the other cases and the required calculations as Exercise 2.13.
The optimal control is
\[
  u^*_t(x,t)(s) = \begin{cases} 
-1, & s \in [t, t + x], \\
0, & s \in (t + x, 2],
\end{cases}
\]
and the corresponding state trajectory is
\[
x^*_t(x,t)(s) = \begin{cases} 
x - (s - t), & s \in [t, t + x], \\
0, & s \in (t + x, 2],
\end{cases}
\]
where we use the subscript to show the dependence of the control and state trajectories of a problem beginning at time \(t\) with the state \(x(t) = x\).

Thus,
\[
V(x, t) = \int_t^{t+x} -\frac{1}{2}[x^*_t(x,t)(s)]^2 ds = -\frac{1}{2} \int_t^{t+x} (x - s + t)^2 ds.
\]
Differentiating the right-hand side with respect to $x$, we obtain

$$V_x(x, t) = -\frac{1}{2} \int_t^{x+t} 2(x - s + t) ds.$$ 

Furthermore, since

$$x^*(t) = \begin{cases} 
1 - t, & t \in [0, 1], \\
0, & t \in (1, 2],
\end{cases}$$

we obtain

$$V_x(x^*(t), t) = \begin{cases} 
-\frac{1}{2} \int_t^1 2(1 - s) ds = -\frac{1}{2} t^2 + t - \frac{1}{2}, & t \in [0, 1], \\
0, & t \in (1, 2],
\end{cases}$$

which equals $\lambda(t)$ obtained as the adjoint variable in Example 2.5.
The problem is:

$$\max \left\{ J = \int_0^2 (2x - 3u - u^2)dt \right\}$$

subject to

$$\dot{x} = x + u, \ x(0) = 5$$

and the control constraint

$$u \in \Omega = [0, 2].$$
The Hamiltonian is

\[ H = (2x - 3u - u^2) + \lambda(x + u) \]
\[ = (2 + \lambda)x - (u^2 + 3u - \lambda u). \] (2.57)

The optimal control is

\[ u^*(t) = \frac{\lambda(t) - 3}{2}. \] (2.58)

The adjoint equation is

\[ \dot{\lambda} = -\frac{\partial H}{\partial x} = -2 - \lambda, \quad \lambda(2) = 0. \] (2.59)
Use the integrating factor $e^t$ as shown in Appendix A.6, to obtain

$$\lambda(t) = 2(e^{2-t} - 1).$$

If we substitute this into (2.58) and impose the control constraint (2.56), we see that the optimal control is

$$u^*(t) = \begin{cases} 
2 & \text{if } e^{2-t} - 2.5 > 2, \\
2e^{2-t} - 2.5 & \text{if } 0 \leq e^{2-t} - 2.5 \leq 2, \\
0 & \text{if } e^{2-t} - 2.5 < 0.
\end{cases}$$

It can also be written as

$$u^*(t) = \text{sat}[0, 2; e^{2-t} - 2.5].$$
Solution of Example 2.6 Cont.

Figure 2.5: Optimal Control for Example 2.6
Sufficiency Conditions

- **Derived Hamiltonian** function $H^0 : E^n \times E^m \times E^1 \rightarrow E^1$ is defined as follows:

\[
H^0(x, \lambda, t) = \max_{u \in \Omega(t)} H(x, u, \lambda, t)
\]  
(2.61)

OR

\[
H^0(x, \lambda, t) = H(x, u^*, \lambda, t).
\]  
(2.62)

- The derivative $H^0_x(x, \lambda, t)$ by use of the Envelope Theorem is:

\[
H^0_x(x, \lambda, t) = H_x(x, u^*, \lambda, t) := H_x(x, u, \lambda, t)|_{u=u^*}.
\]  
(2.63)

- When $u^*(x, \lambda, t)$ is differentiable in $x$, let us differentiate (2.62) with respect to $x$:

\[
H^0_x(x, \lambda, t) = H_x(x, u^*, \lambda, t) + H_u(x, u^*, \lambda, t) \frac{\partial u^*}{\partial x}.
\]  
(2.64)
But

\[ H_u(x, u^*, \lambda, t) \frac{\partial u^*}{\partial x} = 0. \] (2.65)

There are two cases to consider here:

**Case I:** If \( u^* \) is in the interior of \( \Omega(t) \), then it satisfies the first-order condition \( H_u(x, u^*, \lambda, t) = 0 \), thereby implying (2.65).

**Case II:** If \( u^* \) is on the boundary of \( \Omega(t) \). Then, for each \( i, j \), either \( H_{u_i} = 0 \) or \( \partial u^*_i / \partial x_j = 0 \) or both.
Theorem 2.1

**Theorem 2.1: (Sufficiency Conditions).** Let \( u^*(t) \), and the corresponding \( x^*(t) \) and \( \lambda(t) \) satisfy the maximum principle necessary condition (2.31) for all \( t \in [0, T] \). Then, \( u^* \) is an optimal control if \( H^0(x, \lambda(t), t) \) is concave in \( x \) for each \( t \) and \( S(x, T) \) is concave in \( x \).

**Proof:** By definition

\[
H[x(t), u(t), \lambda(t), t] \leq H^0[x(t), \lambda(t), t]. \tag{2.66}
\]

Since \( H^0 \) is differentiable and concave,

\[
H^0[x(t), \lambda(t), t] \leq H^0[x^*(t), \lambda(t), t] + H^0_x[x^*(t), \lambda(t), t] [x(t) - x^*(t)]. \tag{2.67}
\]
Proof for Theorem 2.1

- Using Envelope Theorem we can write

\[
H[x(t), u(t), \lambda(t), t] \leq H[x^*(t), u^*(t), \lambda(t), t] \\
+ H_x[x^*(t), u^*(t), \lambda(t), t][x(t) - x^*(t)]. \tag{2.68}
\]

- By definition of \( H \) and the adjoint equation,

\[
F[x(t), u(t), t] + \lambda(t)f[x(t), u(t), t] \leq F[x^*(t), u^*(t), t] \\
+ \lambda(t)f[x^*(t), u^*(t), t] - \dot{\lambda}(t)[x(t) - x^*(t)]. \tag{2.69}
\]

- Using the state equation,

\[
F[x^*(t), u^*(t), t] - F[x(t), u(t), t] \\
\geq \dot{\lambda}(t)[x(t) - x^*(t)] + \lambda(t)[\dot{x}(t) - \dot{x}^*(t)]. \tag{2.70}
\]
Since $S(x, T)$ is a differential and concave function, we have

\[ S(x(T), T) \leq S(x^*(T), T) + S_x(x^*(T), T)[x(T) - x^*(T)] \quad (2.71) \]

or

\[ S(x^*(T), T) - S(x(T), T) \geq S_x(x^*(T), T)[x(T) - x^*(T)]. \quad (2.72) \]

Integrating both sides of (2.70) from 0 to $T$ and adding (2.72), we have

\[
\left[ \int_0^T F(x^*(t), u^*(t), t) dt + S(x^*(T), T) \right] \\
- \left[ \int_0^T F(x(t), u(t), t) dt + S(x(T), T) \right] \\
\geq [\lambda(T) - S_x(x^*(T), T)] [x(T) - x^*(T)] - \lambda(0)[x(0) - x^*(0)].
\]
Proof for Theorem 2.1 Cont.

- Using the definition of $J(u)$ in (2.3), we get

$$J(u^*) - J(u) \geq [\lambda(T) - S_x(x^*(T), T)][x(T) - x^*(T)] - \lambda(0)[x(0) - x^*(0)],$$

where $J(u)$ is the value of the objective function associated with a control $u$. Since $x^*(0) = x(0) = x_0$, the initial condition, and since $\lambda(T) = S_x(x^*(T), T)$ from the terminal adjoint condition in (2.31), we have

$$J(u^*) \geq J(u).$$

- Thus, $u^*$ is an optimal control. This completes the proof.
Let us show that the problems in Examples 2.2 and 2.3 satisfy the sufficient conditions. From (2.36) and (2.61), we have

\[ H^0 = -x + \lambda u^*, \]

where \( u^* \) is given by (2.37).

Since \( u^* \) is a function of \( \lambda \) only, \( H^0(x, \lambda, t) \) is certainly concave in \( x \) for any \( t \) and \( \lambda \) (and in particular for \( \lambda(t) \) supplied by the maximum principle). Since \( S(x, T) = 0 \), the sufficient conditions hold.
Free and Fixed-end-point problems

- The problems in which the terminal values of the state variables are not constrained are called *free-end-point problems*. All the problems that we considered till now in this chapter can be classified as free-end-point problems.

- The problems in which the terminal values of the state variables are completely specified are termed *fixed-end-point problems*.

- There are problems in between these two extremes.
Fixed-end-point problem

- Suppose the terminal values of the state variable, \( x(T) \), is completely specified, i.e., \( x(T) = k \in E^n \), where \( k \) is a vector of constants. Recall that the proof of the sufficient conditions requires (2.73) to hold, i.e.,

\[
J(u^*) - J(u) \geq [\lambda(T) - S_x(x^*(T), T)][x(T) - x^*(T)] - \lambda(0)[x(0) - x^*(0)].
\]

- In this case, since \( x(T) - x^*(T) = k - k = 0 \), the right-hand side of inequality (2.73) vanishes regardless of the value of \( \lambda(T) \). This means that the sufficiency result would go through for any value of \( \lambda(T) \). Not surprisingly, therefore, the transversality condition (2.30) in the fixed-end-point case changes to

\[
\lambda(T) = \beta,
\]

where \( \beta \in E^n \) is a vector of constants to be determined.
Consider the problem:

\[
\begin{align*}
\max \left\{ J = \int_0^1 -\frac{1}{2} (x^2 + u^2) \, dt \right\} \\
\text{subject to} \\
\dot{x} = -x^3 + u, \ x(0) = 5.
\end{align*}
\]
Solution of Example 2.8

- We form the Hamiltonian
  \[ H = -\frac{1}{2}(x^2 + u^2) + \lambda(-x^3 + u). \]

- The adjoint variable \( \lambda \) satisfies the equation
  \[ \dot{\lambda} = x + 3x^2\lambda, \quad \lambda(1) = 0. \] (2.77)

- Since \( u \) is unconstrained, we set \( H_u = 0 \) to obtain \( u^* = \lambda \). With this, the state equation (2.76) becomes
  \[ \dot{x} = -x^3 + \lambda, \quad x(0) = 5. \] (2.78)

- Thus, the TPBVP is given by the system of equations (2.77) and (2.78).
A simple method to solve the TPBVP uses what is known as the *shooting method*, explained in the flowchart in Figure 2.6.

**Figure 2.6: The Flowchart for Example 2.8**
Substitution of $\triangle x/\triangle t$ for $\dot{x}$ in (2.78) and $\triangle \lambda/\triangle t$ for $\dot{\lambda}$ in (2.77) gives the discrete version of the TPBVP:

$$x(t + \triangle t) = x(t) + [−x(t)^3 + \lambda(t)] \triangle t, \ x(0) = 5,$$

(2.79)

$$\lambda(t + \triangle t) = \lambda(t) + [x(t) + 3x(t)^2\lambda(t)] \triangle t, \ \lambda(1) = 0.$$ (2.80)

Now we will see how we can use excel to solve the above TPBVP problem. For that let us set

$$\Delta t = 0.01, \ \lambda(0) = −0.2 \ \text{and} \ \ x(0) = 5.$$
Solving a TPBVP by Using Excel: Example 2.8 Cont.
Solving a TPBVP by Using Excel: Example 2.8 Cont.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.2</td>
<td>=B1+(-B2*3)+A2^0.01</td>
</tr>
</tbody>
</table>

Formula in cell B1:

```
=B1+(-B2*3)+A2^0.01
```
Solving a TPBVP by Using Excel: Example 2.8 Cont.
Solving a TPBVP by Using Excel: Example 2.8 Cont.
Figure 2.7: Solution of TPBVP by Excel