Chapter 4
The Maximum Principle: Pure State and Mixed Inequality Constraints

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Example 4.1

Consider the problem with a pure state constraint:

\[
\text{max} \left\{ J = \int_{0}^{3} -udt \right\}
\]  \hfill (4.1)

subject to

\[
\dot{x} = u, \quad x(0) = 0, \quad 0 \leq u \leq 3, \quad x - 1 + (t - 2)^2 \geq 0.
\]  \hfill (4.2, 4.3, 4.4)

From (4.1), one can see that it is good to have low values of \( u \).
Solution of Example 4.1

Figure 4.1: Feasible State Space and Optimal State Trajectory for Examples 4.1 and 4.4

$x - 1 + (t - 2)^2 = 0$
Solution of Example 4.1 Cont.

- If we use $u = 0$ to begin with, we see that $x(t) = 0$ as long as $u(t) = 0$.
- At $t = 1$, $x(1) = 0$ and the constraint (4.4) is satisfied with an equality. But continuing with $u(t) = 0$ beyond $t = 1$ is not feasible since $x(t) = 0$ would not satisfy the constraint (4.4) just after $t = 1$.
- In Fig. (4.1), we see that the lowest possible feasible state trajectory from $t = 1$ to $t = 2$ satisfies the state constraint (4.4) with an equality. In order not to violate the constraint (4.4), its first time derivative $u(t) + 2(t - 2)$ must be non-negative. This gives us $u(t) = 2(2 - t)$ to be the lowest feasible value for the control. This value will make the state $x(t)$ ride along the constraint boundary until $t = 2$, at which point $u(2) = 0$; see Fig. (4.1).
- Continuing with $u(t) = 2(2 - t)$ beyond $t = 2$ will make $u(t)$ negative, and violate the lower bound in (4.3). It is easy to see, however, that $u(t) = 0$, $t \geq 2$, is the lowest feasible value, which can be followed all the way to the terminal time $t = 3$. 
Solution of Example 4.1 Cont.

- We restate the values of the state and control variables that we have obtained:

\[ x^*(t) = \begin{cases} 
0, & t \in [0, 1),  \\
1 - (t - 2)^2, & t \in [1, 2), \\
1, & t \in (2, 3], \\
\end{cases} \]

\[ u^*(t) = \begin{cases} 
0, & t \in [0, 1),  \\
2(2 - t), & t \in [1, 2], \\
0, & t \in (2, 3]. \\
\end{cases} \]

(4.5)

- The feedback control \( u^*(x, t) = 0 \) is optimal at any point \((x, t)\) when \( x \geq 1 \) or when \((x, t)\) is on the right-hand side of the parabola in Figure 4.1. Thus, \( V(x, t) = 0 \) on such points.

- On the other hand, when \( x \in [0, 1] \) and it is on the left-hand side of the parabola, then ...
The control is 0 until it hits the trajectory at time \( \tau = 2 - \sqrt{1 - x} \). Then, the control switches to \( 2(2 - s) \) for \( s \in (\tau, 2) \) to climb along the left-hand side of the parabola to reach its peak, and then switches back to 0 on the time interval [2,3]. Thus, in this case,

\[
V(x, t) = -\int_t^\tau 0\,ds - \int_\tau^2 2(2 - s)\,ds - \int_2^3 0\,ds
\]

\[
= \left[ s^2 - 4s \right]_{2-\sqrt{1-x}}^\tau = (x - 1).
\]

Thus, we have the value function

\[
V(x, t) = \begin{cases} 
0, & x \geq 1, \ t \in [0, 3], \\
 x - 1, & x \geq \max\{1 - (t - 2)^2, 0\}, \ t \in [0, 2), \\
 0, & 1 - (t - 2)^2 \leq x \leq 1, \ t \in [2, 3].
\end{cases}
\]
Thus the marginal valuation along the optimal path $x^*(t)$ given in (4.5) as

$$V_x(x^*(t), t) = \begin{cases} 
1, & t \in [0, 2), \\
0, & t \in [2, 3].
\end{cases} \tag{4.6}$$

The marginal valuation is discontinuous at $t = 2$, and it has a downward jump of size 1 at that time.
The Optimal Control Problem with Pure and Mixed Constraints

- Let us consider the problem (3.7) in Chapter 3 along with the pure state variable inequality constraint of type

\[ h(x, t) \geq 0, \quad (4.7) \]

where we assume function \( h : E^n \times E^1 \rightarrow E^p \) to be continuously differentiable in all its arguments. By the definition of function \( h \), (4.7) represents a set of \( p \) constraints \( h_i(x, t) \geq 0, \ i = 1, 2, \ldots, p \).

- The constraint \( h_i \geq 0 \) is called a constraint of \( r \)th order if the \( r \)th time derivative of \( h_i \) is the first time a term in control \( u \) appears in the expression by putting \( f(x, u, t) \) for \( \dot{x} \) after each differentiation.

- The value of \( r \) is referred to as the order of the constraint. If the constraint \( h_i \) is of order \( r \), then we would require \( h_i \) to be \( r \) times continuously differentiable.
In the case of first-order constraints, i.e., $r = 1$, the first time derivative of $h$ has terms in $u$, and it can be defined as follows:

$$h^1(x, u, t) = \frac{dh}{dt} = \frac{\partial h}{\partial x} f + \frac{\partial h}{\partial t}. \quad (4.8)$$

In the important special case of the nonnegativity constraint

$$x(t) \geq 0, \quad t \in [0, T], \quad (4.9)$$

(4.8) is simply $h^1 = f$.

For an upper bound constraint $x(t) \leq M$, written as

$$M - x(t) \geq 0, \quad t \in [0, T], \quad (4.10)$$

(4.8) gives $h^1 = -f$. 
As in Chapter 3, the constraints (4.7) need also to satisfy a full-rank type constraint qualification before a maximum principle can be derived.

W.r.t the $i$th constraint $h_i(x, t) \geq 0$, an interval $(\theta_1, \theta_2) \subset [0, T]$ with $\theta_1 < \theta_2$ is called an *interior interval* if $h_i(x(t), t) > 0, \forall t \in (\theta_1, \theta_2)$.

If the optimal trajectory “hits the boundary,” i.e., satisfies $h_i(x(t), t) = 0$ for $\tau_1 \leq t \leq \tau_2$ for some $i$, then $[\tau_1, \tau_2]$ is called a *boundary interval*. An instant $\tau_1$ is called an *entry time* if there is an interior interval ending at $t = \tau_1$ and a boundary interval starting at $\tau_1$. $\tau_2$ is called an *exit time* if a boundary interval ends and an interior interval starts at $\tau_2$.

If the trajectory just touches the boundary at time $\tau$, i.e., $h(x(\tau), \tau) = 0$ and if the trajectory is in the interior just before and just after $\tau$, then $\tau$ is called a *contact time*. Taken together, entry, exit, and contact times are called *junction times*. 
The full-rank condition on any boundary interval $[\tau_1, \tau_2]$ holds, i.e.,

\[
\begin{bmatrix}
\frac{\partial h_1}{\partial u} \\
\frac{\partial h_2}{\partial u} \\
\vdots \\
\frac{\partial h_{\hat{p}}}{\partial u}
\end{bmatrix}
\]

\[
\text{rank}
\begin{bmatrix}
\frac{\partial h_1}{\partial u} \\
\frac{\partial h_2}{\partial u} \\
\vdots \\
\frac{\partial h_{\hat{p}}}{\partial u}
\end{bmatrix}
= \hat{p},
\]

where for $t \in [\tau_1, \tau_2]$,

\[
h_i(x^*(t), t) = 0, \quad i = 1, 2, \ldots, \hat{p} \leq p
\]

and

\[
h_i(x^*(t), t) > 0, \quad i = \hat{p} + 1, \ldots, p.
\]
Recapitulate the optimal control problem for which we will state a direct maximum principle:

\[
\begin{align*}
\max \left\{ J = \int_0^T F(x(u), t) dt + S[x(T), T] \right\},
\end{align*}
\]

subject to

\[
\begin{align*}
\dot{x} &= f(x, u, t), \quad x(0) = x_0, \\
g(x, u, t) &\geq 0, \\
h(x, t) &\geq 0, \\
a(x(T), T) &\geq 0, \\
b(x(T), T) &= 0.
\end{align*}
\]

(4.11)
The special cases of the mixed constraint \( g(x, u, t) \geq 0 \) are \( u_i \in [0, M] \) for \( M > 0 \) and \( u_i(t) \in [0, x_i(t)] \).

The special cases of the terminal constraints \( a(x(T), T) \geq 0 \) and \( a(x(T), T) = 0 \) are \( x_i(T) \geq k \) and \( x_i(T) = k \), respectively, where \( k \) is a constant.

The special cases of the pure constraints \( h(x, t) \geq 0 \) are \( x_i \geq 0 \) and \( x_i \leq M \), for which \( h_{x_i} = +1 \) and \( h_{x_i} = -1 \), respectively, and \( h_t = 0 \).
To formulate the maximum principle for the problem (4.11), we define the Hamiltonian function $H^d : E^n \times E^m \times E^1 \to E^1$ as

$$H^d = F(x,u,t) + \lambda^d f(x,u,t).$$

The Lagrangian function $L^d : E^n \times E^m \times E^n \times E^q \times E^p \times E^1 \to E^1$ as

$$L^d(x,u,\lambda^d,\mu,\eta^d,t) = H^d(x,u,\lambda^d,t) + \mu g(x,u,t) + \eta^d h(x,t). \quad (4.12)$$

The maximum principle states that the necessary conditions for $u^*$ (with the corresponding state trajectory $x^*$) to be optimal. The conditions are that there exist an adjoint function $\lambda^d$, which may be discontinuous at a time in a boundary interval or a contact time, multiplier functions $\mu, \alpha, \beta, \gamma^d, \eta^d$, and a jump parameter $\zeta^d(\tau)$, at each time $\tau$, where $\lambda^d$ is discontinuous, such that the following (4.13) holds:
Maximum Principle: Direct Method (Necessary Condition)

\[ \dot{x}^* = f(x^*, u^*, t), \quad x^*(0) = x_0, \text{ satisfying constraints} \]
\[ g(x^*, u^*, t) \geq 0, \quad h(x^*, t) \geq 0, \text{ and the terminal constraints} \]
\[ a(x^*(T), T) \geq 0 \text{ and } b(x^*(T), T) = 0; \]
\[ \dot{\lambda}^d = -L_x[x^*, u^*, \lambda^d, \mu, \eta^d, t] \]
with the transversality conditions
\[ \lambda^d(T^-) = S_x(x^*(T), T) + \alpha a_x(x^*(T), T) + \beta b_x(x^*(T), T) \]
\[ + \gamma^d h_x(x^*(T), T), \text{ and} \]
\[ \alpha \geq 0, \quad \alpha a(x^*(T), T) = 0, \quad \gamma^d \geq 0, \quad \gamma^d h(x^*(T), T) = 0; \]
the Hamiltonian maximizing condition
\[ H^d[x^*(t), u^*(t), \lambda^d(t), t] \geq H^d[x^*(t), u, \lambda^d(t), t] \]
at each \( t \in [0, T] \) for all \( u \) satisfying
\[ g[x^*(t), u, t] \geq 0; \]
the jump conditions at any time \( \tau \),
where \( \lambda^d \) is discontinuous, are
\[ \lambda^d(\tau^-) = \lambda^d(\tau^+) + \zeta^d(\tau) h_x(x^*(\tau), \tau) \]
and
\[ H^d[x^*(\tau), u^*(\tau^-), \lambda^d(\tau^-), \tau] = H^d[x^*(\tau), u^*(\tau^+), \lambda^d(\tau^+), \tau] \]
\[ - \zeta^d(\tau) h_t(x^*(\tau), \tau); \]
the Lagrange multipliers \( \mu(t) \) are such that
\[ \partial L^d/\partial u|_{u=u^*(t)} = 0, \quad dH^d/dt = dL^d/dt = \partial L^d/\partial t, \]
and the complementary slackness conditions
\[ \mu(t) \geq 0, \quad \mu(t) g(x^*, u^*, t) = 0, \]
\[ \eta(t) \geq 0, \quad \eta^d(t) h(x^*(t), t) = 0, \text{ and} \]
\[ \zeta^d(\tau) \geq 0, \quad \zeta^d(\tau) h(x^*(\tau), \tau) = 0 \text{ hold}. \]
Terminal Time to be Decided

- $\lambda^d(t)$ has the standard marginal value interpretation with
  
  $$\lambda^d(T) = S_x(x^*(T),T). \quad (4.14)$$

- (4.14) is not needed for the application of the maximum principle (4.13).

- If $T$ is also a decision variable constrained to lie in the interval $[T_1, T_2]$, $0 \leq T_1 < T_2 < \infty$, then in addition to (4.13), if $T^*$ is the optimal terminal time, it must satisfy a condition similar to (3.15) and (3.81), i.e.,

  $$H^d[x^*(T^*), u^*(T^*), \lambda^d(T^*), T^*] + S_T[x^*(T^*), T^*] + \alpha a_T[x^*(T^*), T^*] + \beta b_T[x^*(T^*), T^*] + \gamma^d h_T[x^*(T^*), T^*] \begin{cases} 
  \leq 0 & \text{if } T^* = T_1, \\
  = 0 & \text{if } T^* \in (T_1, T_2), \\
  \geq 0 & \text{if } T^* = T_2. 
\end{cases} \quad (4.15)$$
Remarks 4.1-4.2

- **Remark 4.1**: In most practical examples, $\lambda^d$ and $H^d$ will only jump at junction times. However, in some cases a discontinuity may occur at a time in the interior of a boundary interval, e.g., when a mixed constraint becomes active at that time.

- **Remark 4.2**: It is known that the adjoint function $\lambda^d$ is continuous at a junction time $\tau$, i.e., $\zeta^d(\tau) = 0$, if (i) the entry or exit at time $\tau$ is non-tangential, i.e., if $h^1(x^*(\tau), u^*(\tau), \tau) \neq 0$, or (ii) if the control $u^*$ is continuous at $\tau$ and the

\[
\text{rank} \begin{bmatrix}
\frac{\partial g}{\partial u} & \text{diag}(g) & 0 \\
\frac{\partial h^1}{\partial u} & 0 & \text{diag}(h)
\end{bmatrix} = m + p,
\]

when evaluated at $x^*(\tau)$ and $u^*(\tau)$. 
We will see that the jump conditions on the adjoint variables in (4.13) will give us precisely the jump in Example 4.2, where we will apply the direct maximum principle to the problem in Example 4.1.

The jump condition on $H^d$ in (4.13) requires that the Hamiltonian should be continuous at $\tau$ if $h_t(x^*(\tau), \tau) = 0$. The continuity of the Hamiltonian (in case $h_t = 0$) makes intuitive sense when considered in the light of its interpretation given in Section 2.2.4.
Example 4.2 and its Solution

- Apply the direct maximum principle (4.13) to solve the problem in Example 4.1.

- SOLUTION: Since we already have the optimal $u^*$ and $x^*$ as obtained in (4.5), we can use these in (4.13) to obtain $\lambda^d, \mu_1, \mu_2, \gamma^d, \eta^d,$ and $\zeta^d$. Thus,

$$H^d = -u + \lambda^d u,$$  \hspace{1cm} (4.16)

$$L^d = H^d + \mu_1 u + \mu_2 (3 - u) + \eta^d [x - 1 + (t - 2)^2],$$ \hspace{1cm} (4.17)

$$L_u^d = -1 + \lambda^d + \mu_1 - \mu_2 = 0,$$ \hspace{1cm} (4.18)

$$\dot{\lambda}^d = -L_x^d = -\eta^d, \; \lambda^d (3^-) = \gamma^d,$$ \hspace{1cm} (4.19)

$$\gamma^d [x^*(3) - 1 + (3 - 2)^2] = 0,$$ \hspace{1cm} (4.20)

$$\mu_1 \geq 0, \; \mu_1 u^* = 0, \; \mu_2 \geq 0, \; \mu_2 (3 - u^*) = 0,$$ \hspace{1cm} (4.21)

$$\eta^d \geq 0, \; \eta^d [x^*(t) - 1 + (t - 2)^2] = 0,$$ \hspace{1cm} (4.22)
Solution of Example 4.2

- And if \( \lambda^d \) is discontinuous for some \( \tau \in [1, 2] \), the boundary interval as seen from Figure 4.1, then

\[
\lambda^d(\tau^-) = \lambda^d(\tau^+) + \zeta^d(\tau), \quad \zeta^d(\tau) \geq 0, \tag{4.23}
\]

\[-u^*(\tau^-) + \lambda^d(\tau^-)u^*(\tau^-) = -u^*(\tau^+) + \lambda^d(\tau^+)u^*(\tau^+) - \zeta^d(\tau)2(\tau - 2). \tag{4.24}\]

- Since \( \gamma^d = 0 \) from (4.20), we have \( \lambda^d(3-) = 0 \) from (4.19). Also, we set \( \lambda^d(3) = 0 \) according to (4.14).

- **Interval (2,3]:** \( \eta^d = 0 \) from (4.22) and thus \( \dot{\lambda}^d = 0 \) from (4.19), giving \( \lambda^d = 0 \). From (4.18) and (4.21), we have \( \mu_1 = 1 > 0 \) and \( \mu_2 = 0 \).

- **Interval [1,2]:** \( \mu_1 = \mu_2 = 0 \) from \( 0 < u^* < 3 \) and (4.21). Thus, (4.18) implies \( \lambda^d = 1 \) and (4.19) gives \( \eta^d = -\dot{\lambda}^d = 0 \). Thus \( \lambda^d \) is discontinuous at the exit time \( \tau = 2 \), and we use (4.23) to see that the jump parameter \( \zeta^d(2) = \lambda^d(2^-) - \lambda^d(2^+) = 1 \). Furthermore, it is easy to check that (4.24) also holds at \( \tau = 2 \).
Interval $[0,1)$: Clearly $\mu_2 = 0$ from (4.21). Also $u^* = 0$ would still be optimal if there were no lower bound constraint on $u$ in this interval. This means that the constraint $u \geq 0$ is not binding, giving us $\mu_1 = 0$. Then from (4.18), we have $\lambda^d = 1$. Finally, from (4.19), we have $\eta^d = -\dot{\lambda}^d = 0$.

And the adjoint variable

$$\lambda^d(t) = \begin{cases} 1, & t \in [0,2) \\ 0, & t \in [2,3] \end{cases}$$

(4.25)

is precisely the same as the marginal valuation $V_x(x^*(t), t)$ obtained in (4.6).

Note that $\lambda^d$ is continuous at time $t = 1$ where the entry to the constraint is non-tangential as stated in Remark 4.2.
Define the maximized Hamiltonian

\[ H^{0d}(x, \lambda^d(t), t) = \max_{\{u \mid g(x, u, t) \geq 0\}} H^d(x, u, \lambda^d, t). \]  

(4.26)

**Theorem 4.1:** Let \((x^*, u^*, \lambda^d, \mu, \alpha, \beta, \gamma^d, \eta^d)\) and the jump parameters \(\zeta^d(\tau)\) at each \(\tau\), where \(\lambda^d\) is discontinuous, satisfy the necessary conditions in (4.13). If \(H^{0d}(x, \lambda^d(t), t)\) is concave in \(x\) at each \(t \in [0, T]\), and in (4.11) \(S\) is concave in \(x\), \(g\) is quasiconcave in \((x, u)\), \(h\) and \(a\) are quasiconcave in \(x\), and \(b\) is linear in \(x\), then \((x^*, u^*)\) is optimal.

Theorem 4.1 is written for finite horizon problems. For infinite horizon problems, this theorem remains valid if the transversality condition on the adjoint variable in (4.13) is modified along the lines discussed in Section 3.6.
The main idea underlying the indirect method is that when the pure state constraint (4.7), assumed to be of order one, becomes active, we must require its first derivative $h^1(x, u, t)$ in (4.8) to be nonnegative, i.e.,

$$h^1(x, u, t) \geq 0, \text{ whenever } h(x, t) = 0.$$  \hspace{1cm} (4.27)

To formulate the indirect maximum principle for the problem (4.11), we form the Lagrangian as:

$$L(x, u, \lambda, \mu, \eta, t) = H(x, u, \lambda, t) + \mu g(x, u, t) + \eta h^1(x, u, t),$$  \hspace{1cm} (4.28)

where the Hamiltonian $H = F(x, u, t) + \lambda f(x, u, t)$ as defined in (3.8).
The maximum principle states the necessary conditions for \( u^* \) (with the state trajectory \( x^* \)) to be optimal. These conditions are that there exist an adjoint function \( \lambda \), which may be discontinuous at each entry or contact time, multiplier functions \( \mu, \alpha, \beta, \gamma, \eta \), and a jump parameter \( \zeta(\tau) \) at each \( \tau \), where \( \lambda \) is discontinuous, such that (4.29) on the following page holds.
Maximum Principle: Indirect Method
(Necessary Condition)

\[
\dot{x}^* = f(x^*, u^*, t), \quad x^*(0) = x_0, \text{ satisfying constraints}
\]
\[
g(x^*, u^*, t) \geq 0, \quad h(x^*, t) \geq 0, \text{ and the terminal constraints}
\]
\[
a(x^*(T), T) \geq 0 \text{ and } b(x^*(T), T) = 0;
\]
\[
\dot{\lambda} = -L_x[x^*, u^*, \lambda, \mu, \eta, t] \text{ with the transversality conditions}
\]
\[
\lambda(T^-) = S_x(x^*(T), T) + \alpha a_x(x^*(T), T) + \beta b_x(x^*(T), T)
\]
\[
+ \gamma h_x(x^*(T), T), \text{ and}
\]
\[
\alpha \geq 0, \alpha a(x^*(T), T) = 0, \gamma \geq 0, \gamma h(x^*(T), T) = 0;
\]

the Hamiltonian maximizing condition

\[
H[x^*(t), u^*(t), \lambda(t), t] \geq H[x^*(t), u, \lambda(t), t]
\]

at each \( t \in [0, T] \) for all \( u \) satisfying

\[
g[x^*(t), u, t] \geq 0, \text{ and}
\]
\[
h_i^1(x^*(t), u, t) \geq 0 \text{ whenever } h_i(x^*(t), t) = 0, i = 1, 2, \ldots, p;
\]

the jump conditions at any entry/contact time \( \tau \), where \( \lambda \) is discontinuous, are

\[
\lambda(\tau^-) = \lambda(\tau^+) + \zeta(\tau) h_x(x^*(\tau), \tau) \quad \text{and}
\]
\[
H[x^*(\tau), u^*(\tau^-), \lambda(\tau^-), \tau] = H[x^*(\tau), u^*(\tau^+), \lambda(\tau^+), \tau]
\]
\[
- \zeta(\tau) h_t(x^*(\tau), \tau);
\]

the Lagrange multipliers \( \mu(t) \) are such that

\[
\partial L/\partial u\big|_{u=u^*(t)} = 0, \quad dH/dt = dL/dt = \partial L/\partial t,
\]

and the complementary slackness conditions

\[
\mu(t) \geq 0, \quad \mu(t) g(x^*, u^*, t) = 0,
\]
\[
\eta(t) \geq 0, \quad \eta(t) h(x^*(t), t) = 0, \text{ and}
\]
\[
\zeta(\tau) \geq 0, \quad \zeta(\tau) h(x^*(\tau), \tau) = 0 \text{ hold.}
\]
We now obtain the multipliers of the direct maximum principle from those in the indirect maximum principle. Since the multipliers coincide in the interior, we let $[\tau_1, \tau_2]$ denote a boundary interval:

$$\eta^d(t) = -\dot{\eta}(t), \quad t \in (\tau_1, \tau_2),$$  \hspace{1cm} (4.30)

$$\lambda^d(t) = \lambda(t) + \eta(t)h_x(x^*(t), t), \quad t \in (\tau_1, \tau_2).$$  \hspace{1cm} (4.31)

Note that $\eta^d(t) \geq 0$ in (4.13). Thus, we have $\dot{\eta} \leq 0$, which we shall include in Remark 4.3. The jump parameter at an entry time $\tau_1$, an exit time $\tau_2$, or a contact time $\tau$, respectively, satisfies

$$\zeta^d(\tau_1) = \zeta(\tau_1) - \eta(\tau_1^+), \quad \zeta^d(\tau_2) = \eta(\tau_2^-), \quad \zeta^d(\tau) = \zeta(\tau).$$  \hspace{1cm} (4.32)
By comparing $\lambda^d(T^-)$ in (4.13) and $\lambda(T^-)$ in (4.29) and using (4.31), we have

$$\gamma^d = \gamma + \eta(T^-).$$ \hfill (4.33)

Going the other way, we have

$$\eta(t) = \int_t^{\tau_2} \eta^d(s)ds + \zeta^d(\tau_2), \quad t \in (\tau_1, \tau_2),$$

$$\lambda(t) = \lambda^d(t) - \eta(t)h(x^*(t), t), \quad t \in (\tau_1, \tau_2),$$

$$\zeta(\tau_1) = \zeta^d(\tau_1) + \eta(\tau_1^+), \quad \zeta(\tau_2) = 0, \quad \zeta(\tau) = \zeta^d(\tau),$$

$$\gamma = \gamma^d - \eta(T^-).$$

The multipliers $\mu, \alpha,$ and $\beta$ are the same in both methods.
**Remarks 4.3-4.4**

- **Remark 4.3:** From (4.30), (4.32), and $\eta^d(t) \geq 0$ and $\zeta^d(\tau_1) \geq 0$ in (4.13), we can obtain the conditions

$$\dot{\eta}(t) \leq 0 \quad (4.34)$$

and

$$\zeta(\tau_1) \geq \eta(\tau_1^+) \text{ at each entry time } \tau_1. \quad (4.35)$$

- **Remark 4.4:** In Exercise 4.12, we discuss the indirect method for higher-order constraints.
Consider the problem:

$$\max \left\{ J = \int_0^2 -x \, dt \right\}$$

subject to

$$\dot{x} = u, \quad x(0) = 1, \quad (4.36)$$

$$u + 1 \geq 0, \quad 1 - u \geq 0, \quad (4.37)$$

$$x \geq 0. \quad (4.38)$$

Note that this problem is the same as Example 2.3, except for the nonnegativity constraint (4.38).
The Hamiltonian is

\[ H = -x + \lambda u, \]

which implies the optimal control to have the form

\[ u^*(x, \lambda) = \text{bang}[-1, 1; \lambda], \text{ whenever } x > 0. \tag{4.39} \]

When \( x = 0 \), we impose \( \dot{x} = u \geq 0 \) in order to insure that (4.38) holds. Therefore, the optimal control on the state constraint boundary is

\[ u^*(x, \lambda) = \text{bang}[0, 1; \lambda], \text{ whenever } x = 0. \tag{4.40} \]
Solution of Example 4.3 Cont.

Now we form the Lagrangian

\[ L = H + \mu_1 (u + 1) + \mu_2 (1 - u) + \eta u, \]

where \( \mu_1, \mu_2, \) and \( \eta \) satisfy the complementary slackness conditions

\[
\begin{align*}
\mu_1 &\geq 0, \quad \mu_1 (u + 1) = 0, \quad (4.41) \\
\mu_2 &\geq 0, \quad \mu_2 (1 - u) = 0, \quad (4.42) \\
\eta &\geq 0, \quad \eta x = 0. \quad (4.43)
\end{align*}
\]

Furthermore, the optimal trajectory must satisfy

\[
\frac{\partial L}{\partial u} = \lambda + \mu_1 - \mu_2 + \eta = 0. \quad (4.44)
\]
From the Lagrangian we also get

\[ \dot{\lambda} = -\frac{\partial L}{\partial x} = 1, \lambda(2^-) = \gamma \geq 0, \gamma x(2) = \lambda(2^-)x(2) = 0. \quad (4.45) \]

We guess that the optimal control \( u^* \) will be the one that keeps \( x^* \) as small as possible, subject to the state constraint (4.38). Thus,

\[
  u^*(t) = \begin{cases} 
    -1, & t \in [0, 1), \\
    0, & t \in [1, 2]. 
  \end{cases} \quad (4.46)
\]

This gives

\[
  x^*(t) = \begin{cases} 
    1 - t, & t \in [0, 1), \\
    0, & t \in [1, 2]. 
  \end{cases}
\]
To obtain $\lambda(t)$, let us first try $\lambda(2^-) = \gamma = 0$. Then, since $x^*(t)$ enters the boundary zero at $t = 1$, there are no jumps in the interval $(1, 2]$, and the solution for $\lambda(t)$ is

$$\lambda(t) = t - 2, \quad t \in (1, 2). \tag{4.47}$$

Since $\lambda(t) \leq 0$ and $x^*(t) = 0$ on $(1, 2]$, we have $u^*(t) = 0$ by (4.40), as stipulated. Now let us see what must happen at $t = 1$. We know from (4.47) that $\lambda(1^+) = -1$. To obtain $\lambda(1^-)$, we see that

$$H(1^+) = -x^*(1^+) + \lambda(1^+)u^*(1^+) = 0$$

$$H(1^-) = -x^*(1^-) + \lambda(1^-)u^*(1^-) = -\lambda(1^-).$$

By equating $H(1^-)$ to $H(1^+)$ as required in (4.29), we obtain $\lambda(1^-) = 0$. Using now the jump condition on $\lambda(t)$ in (4.29), we get the value of the jump $\zeta(1) = \lambda(1^-) - \lambda(1^+) = 1 \geq 0$. 

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With $\lambda(1^-) = 0$, we can solve (4.45) to obtain

$$\lambda(t) = t - 1, \quad t \in [0, 1].$$

Since $\lambda(t) \leq 0$ and $x^*(t) = 1 - t > 0$ is positive on $[0,1)$, we can use (4.39) to obtain $u^*(t) = -1$ for $0 \leq t < 1$, which is as stipulated in (4.46).

In the time interval $[0,1)$ by (4.42), $\mu_2 = 0$ since $u^* < 1$, and by (4.43), $\eta = 0$ because $x > 0$.

Therefore, $\mu_1(t) = -\lambda(t) = 1 - t > 0$ for $0 \leq t < 1$, and this with $u^* = -1$ satisfies (4.41).

We calculate the Lagrange multipliers in the interval $[1,2]$. Since $u^*(t) = 0$ on $t \in [1, 2]$, we have $\mu_1(t) = \mu_2(t) = 0$. Then, from (4.44) we obtain $\eta(t) = -\lambda(t) = 2 - t \geq 0$ which, with $x^*(t) = 0$ satisfies (4.43). Thus, our guess $\gamma = 0$ is correct, and we do not need to examine the possibility of $\gamma > 0$. The graphs of $x^*(t)$ and $\lambda(t)$ are shown in Figure 4.2.
Solution of Example 4.3 Cont.

Figure 4.2: State and Adjoint Trajectories in Example 4.3
Remark 4.5: Example 4.3 is a problem instance in which the state constraint is active at the terminal time. In instances where the initial state or the final state or both are on the constraint boundary, the maximum principle may **degenerate** in the sense that there is no nontrivial solution of the necessary conditions, i.e., \( \lambda(t) \equiv 0, \ t \in [0, T] \), where \( T \) is the terminal time.

Remark 4.6: It can easily be seen that Example 4.3 is a problem instance in which multipliers \( \lambda \) and \( \mu_1 \) would not be unique if the jump condition on the Hamiltonian in (4.29) was not imposed. For references dealing with the issue of non-uniqueness of the multipliers and conditions under which the multipliers are unique, see Kurcyusz and Zowe (1979), Maurer (1977, 1979), Maurer and Wiegand (1992), and Shapiro (1997).
Example 4.4 and its Solution

The purpose here is to show that the solution obtained in Example 4.3 satisfies the sufficiency conditions of Theorem 4.1. The direct adjoint variable is:

$$
\lambda^d(t) = \lambda(t) + \eta(t)h_x(x^*(t), t) = \begin{cases} 
  t - 1, & t \in [0, 1), \\
  0, & t \in [1, 2). 
\end{cases}
$$

It is easy to see that

$$
H(x, u, \lambda^d(t), t) = \begin{cases} 
  -x + (t - 1)u, & t \in [0, 1) \\
  -x, & t \in [1, 2] 
\end{cases}
$$

is linear and hence concave in \((x, u)\) at each \(t \in [0, 2]\).
Example 4.4 and its Solution Cont.

- Functions

\[ g(x, u, t) = \begin{pmatrix} u + 1 \\ 1 - u \end{pmatrix} \]

and

\[ h(x) = x \]

are linear and hence quasiconcave in \((x, u)\) and \(x\), respectively.

- Functions \(S \equiv 0, a \equiv 0\) and \(b \equiv 0\) satisfy the conditions of Theorem 4.1 trivially. Thus, the solution obtained for Example 4.3 satisfies all conditions of Theorem 4.3, and is therefore optimal.
Example 4.5 and its Solution

- Consider Example 4.3 with $T = 3$ and the terminal state constraint $x(3) = 1$.

SOLUTION: Clearly, the optimal control $u^*$ will be the one that keeps $x$ as small as possible, subject to the state constraint (4.38) and the boundary condition $x(0) = x(3) = 1$. Thus,

$$u^*(t) = \begin{cases} 
-1, & t \in [0, 1), \\
0, & t \in [1, 2], \\
1, & t \in (2, 3], 
\end{cases} \quad x^*(t) = \begin{cases} 
1 - t, & t \in [0, 1), \\
0, & t \in [1, 2), \\
t - 2, & t \in (2, 3]. 
\end{cases}$$
We will compute the adjoint function and the multipliers that satisfy the optimality conditions. These are

\[
\lambda(t) = \begin{cases} 
  t - 1, & t \in [0, 1], \\
  t - 2, & t \in (1, 3), 
\end{cases}
\] (4.48)

\(\mu_1(t), \mu_2(t), \eta(t), t \in [0, 2]\) remain the same as in Figure 4.2,

\[
\mu_1(t) = 0, \mu_2(t) = t - 2, \eta(t) = 0, t \in (2, 3],
\] (4.49)

\[
\gamma = 0, \beta = \lambda(3^-) = 1, \lambda(3) = 0,
\] (4.50)

and the jump \(\zeta(1) = 1 \geq 0\) so that

\[
\lambda(1^-) = \lambda(1^+) + \zeta(1) \text{ and } H(1^-) = H(1^+).
\] (4.51)
Example 4.6 and its Solution

- Introduce a discount rate $\rho > 0$ in Example 4.1 so that the objective function becomes

\[
\max \left\{ J = \int_0^3 -e^{-\rho t} u dt \right\}
\]  

(4.52)

and re-solve using the indirect maximum principle (4.29).

- SOLUTION: It is obvious that the optimal solution will remain the same as (4.5), shown also in Figure 4.1.
With $u^*$ and $x^*$ as in (4.5), we must obtain $\lambda, \mu_1, \mu_2, \eta, \gamma,$ and $\zeta$ so that the necessary optimality conditions (4.29) hold, i.e.,

$$H = -e^{-\rho t}u + \lambda u,$$

(4.53)

$$L = H + \mu_1 u + \mu_2 (3 - u) + \eta [u + 2(t - 2)],$$

(4.54)

$$L_u = -e^{-\rho t} + \lambda + \mu_1 - \mu_2 + \eta = 0,$$

(4.55)

$$\dot{\lambda} = -L_x = 0, \; \lambda(3^-) = \gamma,$$

(4.56)

$$\gamma[x^*(3) - 1 + (3 - 2)^2] = 0,$$

(4.57)

$$\mu_1 \geq 0, \; \mu_1 u = 0, \; \mu_2 \geq 0, \; \mu_2 (3 - u) = 0,$$

(4.58)

$$\eta \geq 0, \; \eta[x^*(t) - 1 + (t - 2)^2] = 0,$$

(4.59)

and if $\lambda$ is discontinuous at the entry time $\tau = 1$, then

$$\lambda(1^-) = \lambda(1^+) + \zeta(1), \; \zeta(1) \geq 0,$$

(4.60)

$$-e^{-\rho}u^*(1^-) + \lambda(1^-)u^*(1^-) = -e^{-\rho}u^*(1^+) + \lambda(1^+)u^*(1^+) - \zeta(1)(-2).$$

(4.61)
From (4.60), we obtain \( \lambda(1^-) = e^{-\rho} \). This with (4.56) gives

\[
\lambda(t) = \begin{cases} 
  e^{-\rho}, & 0 \leq t < 1, \\
  0, & 1 \leq t \leq 3,
\end{cases}
\]

as shown in Figure 4.3,

\[
\mu_1(t) = \begin{cases} 
  e^{-\rho t} - e^{-\rho}, & 0 \leq t < 1, \\
  0, & 1 \leq t \leq 2, \\
  e^{-\rho t}, & 2 < t \leq 3,
\end{cases}
\]

\[
\mu_2(t) = 0, \quad 0 \leq t \leq 3,
\]

and

\[
\eta(t) = \begin{cases} 
  0, & 0 \leq t < 1, \\
  e^{-\rho t}, & 1 \leq t \leq 2, \\
  0, & 2 < t \leq 3,
\end{cases}
\]
Solution of Example 4.6 Cont.

- \( \lambda \) is continuous at the exit time \( t = 2 \). At the entry time \( \tau_1 = 1 \), \( \zeta(1) = e^{-\rho} \geq \eta(1^+) = e^{-\rho} \), so that (4.35) also holds. Finally, \( \gamma = \eta(3^-) = 0 \).

![Figure 4.3: Adjoint Trajectory for Example 4.4](image)

Figure 4.3: Adjoint Trajectory for Example 4.4
The objective function in the problem (4.11) is replaced by
\[ \max \left\{ J = \int_0^T \phi(x, u) e^{-\rho t} dt + \psi[x(T)] e^{-\rho T} \right\}. \]

With the Hamiltonian \( H \) as defined in (3.35), we can write the Lagrangian as
\[ L[x, u, \lambda, \mu, \eta] := H + \mu g + \eta h^1 = \phi + \lambda f + \mu g + \eta h^1. \]

We can now state the current-value form of the maximum principle, giving the necessary conditions for \( u^* \) (with the state trajectory \( x^* \)) to be optimal. These conditions are that there exist an adjoint function \( \lambda \), which may be discontinuous at each entry or contact time, multiplier functions \( \mu, \alpha, \beta, \gamma, \eta \), and a jump parameter \( \zeta(\tau) \) at each \( \tau \) where \( \lambda^d \) is discontinuous, such that the following (4.62) holds:
\[ \dot{x}^* = f(x^*, u^*, t), \quad x^*(0) = x_0, \text{ satisfying constraints} \]
\[ g(x^*, u^*, t) \geq 0, \quad h(x^*(t), t) \geq 0, \text{ and the terminal constraints} \]
\[ a(x^*(T), T) \geq 0 \quad \text{and} \quad b(x^*(T), T) = 0; \]
\[ \dot{\lambda} = \rho \lambda - L_x \left[ x^*, u^*, \lambda, \mu, \eta, t \right] \]
with the transversality conditions
\[ \lambda(T^-) = \psi(x^*(T), T) + \alpha a_x(x^*(T), T) + \beta b_x(x^*(T), T) + \gamma h_x(x^*(T), T), \]
and
\[ \alpha \geq 0, \quad \alpha a(x^*(T), T) = 0, \quad \gamma \geq 0, \quad \gamma h(x^*(T), T) = 0; \]
the Hamiltonian maximizing condition
\[ H[x^*(t), u^*(t), \lambda(t), t] \geq H[x^*(t), u, \lambda(t), t] \]
at each \( t \in [0, T] \) for all \( u \) satisfying
\[ g[x^*(t), u, t] \geq 0, \quad \text{and} \]
\[ h_i^1(x^*(t), u, t) \geq 0 \quad \text{whenever} \quad h_i(x^*(t), t) = 0, \quad i = 1, 2, \ldots, p; \]
the jump conditions at any entry/contact time \( \tau \),
where \( \lambda \) is discontinous, are
\[ \lambda(\tau^-) = \lambda(\tau^+) + \zeta(\tau) h_x(x^*(\tau), \tau) \quad \text{and} \]
\[ H[x^*(\tau), u^*(\tau^-), \lambda(\tau^-), \tau] = H[x^*(\tau), u^*(\tau^+), \lambda(\tau^+), \tau] - \zeta(\tau) h_t(x^*(\tau), \tau); \]
the Lagrange multipliers \( \mu(t) \) are such that
\[ \partial L / \partial u|_{u = u^*(t)} = 0, \quad dH / dt = dL / dt = \partial L / \partial t + \rho \lambda f, \]
and the complementary slackness conditions
\[ \mu(t) \geq 0, \quad \mu(t) g(x^*, u^*, t) = 0, \]
\[ \eta(t) \geq 0, \quad \eta(t) h(x^*(t), t) = 0, \quad \text{and} \]
\[ \zeta(\tau) \geq 0, \quad \zeta(\tau) h(x^*(\tau), \tau) = 0 \quad \text{hold.} \]
If $T \in [T_1, T_2]$, $0 \leq T_1 < T_2 < \infty$, is also a decision variable, then if $T^*$ is the optimal terminal time, then the optimal solution $x^*, u^*, T^*$ must satisfy (4.62) with $T$ replaced by $T^*$ and the condition

$$H[x^*(T^*), u^*(T^*), \lambda^d(T^* - \eta)] - \rho \psi[x^*(T^*)] + \alpha a_T[x^*(T^*), T^*]$$

$$+ \beta b_T[x^*(T^*), T^*] + \gamma^d h_T[x^*(T^*), T^*] \begin{cases} 
\leq 0 & \text{if } T^* = T_1, \\
= 0 & \text{if } T^* \in (T_1, T_2), \\
\geq 0 & \text{if } T^* = T_2.
\end{cases} \quad (4.63)$$

The infinite horizon problem with pure and mixed constraints can be stated as (3.97) with an additional constraint (4.7). As in Section 3.6, the conditions in (4.62) except the transversality condition on the adjoint variable are still necessary for optimality. As for the sufficiency conditions, an analogue of Theorem 4.1 holds, subject to the discussion on infinite horizon transversality conditions in Section 3.6.
**Remark 4.7:** The current-value version of Remark 4.3 is to replace \( \dot{\eta}(t) \leq 0 \) by \( \dot{\eta}(t) \leq \rho \eta(t) \) in (4.34).

**Remark 4.8:** While various subsets of conditions specified in the maximum principles (4.13), (4.29), or (4.62) have been proved in the literature, proofs of the entire results are still not available. For this reason, Hartl, Sethi, and Vickson (1995) call (4.13), (4.29), or (4.62) as *informal theorems*. Seierstad and Sydsæter (1987) call them *almost necessary conditions* since, very rarely, problems arise where the optimal solution requires more complicated multipliers and adjoint variables than those specified in this chapter.