Applications to Finance

- The Simple Cash Balance Problem
- Optimal Financing of a Corporation: Finite Horizon
- Optimal Financing of a Corporation: Infinite Horizon
- A Numerical Example
- A Stochastic Optimal Consumption - Investment Problem (Chapter 12)
The Simple Cash Balance Problem

- The cash balance problem, in its simplest form, is a problem of controlling the level of a firm’s cash balances to meet its demand for cash at minimum total cost.
- If the firm keeps too much cash, it loses money in terms of opportunity cost, in that it can earn higher returns by buying securities such as bonds.
- Whereas, if the cash balance is too small, the firm has to sell securities to meet the cash demand and thus incur a broker’s commission.
- The problem then is to find the tradeoff between the cash and security balances.
Let us introduce the following notation to formulate the optimal control problem:

\[
T = \text{the time horizon,}
\]

\[
x(t) = \text{the cash balance in dollars at time } t,
\]

\[
y(t) = \text{the security balance in dollars at time } t,
\]

\[
d(t) = \text{the instantaneous rate of demand for cash; } d(t) \text{ can be positive or negative,}
\]

\[
u(t) = \text{the rate of sale of securities in dollars; a negative sales rate means a rate of purchase,}
\]

\[
r_1(t) = \text{the interest rate earned on the cash balance,}
\]

\[
r_2(t) = \text{the interest rate earned on the security balance,}
\]

\[
\alpha = \text{the broker's commission in dollars per dollar's worth of securities bought or sold; } 0 < \alpha < 1.
\]
The state equations are
\[
\dot{x} = r_1 x - d + u - \alpha |u|, \quad x(0) = x_0, \quad (5.1)
\]
\[
\dot{y} = r_2 y - u, \quad y(0) = y_0, \quad (5.2)
\]
and the control constraints are
\[
- U_2 \leq u(t) \leq U_1, \quad (5.3)
\]
where $U_1$ and $U_2$ are nonnegative constants.

The objective function is:
\[
\max \{ J = x(T) + y(T) \} \quad (5.4)
\]
subject to (5.1)-(5.3).

Note that the problem is in the linear Mayer form.
Solution by the Maximum Principle

- The Hamiltonian function

\[ H = \lambda_1 (r_1 x - d + u - \alpha |u|) + \lambda_2 (r_2 y - u). \]  \hfill (5.5)

- The adjoint variables satisfy the differential equations

\[ \dot{\lambda}_1 = -\frac{\partial H}{\partial x} = -\lambda_1 r_1, \quad \lambda_1(T) = 1, \]  \hfill (5.6)

\[ \dot{\lambda}_2 = -\frac{\partial H}{\partial y} = -\lambda_2 r_2, \quad \lambda_2(T) = 1. \]  \hfill (5.7)

- It is easy to solve these, respectively, as

\[ \lambda_1(t) = e^{\int_t^T r_1(\tau) d\tau}, \]  \hfill (5.8)

\[ \lambda_2(t) = e^{\int_t^T r_2(\tau) d\tau}. \]  \hfill (5.9)
\( \lambda_1(t) \) is the future value (at time \( T \)) of one dollar held in the cash account from time \( t \) to \( T \) and, likewise, \( \lambda_2(t) \) is the future value of one dollar invested in securities from time \( t \) to \( T \). Thus, the adjoint variables have natural interpretations as the actuarial evaluations of competitive investments at each point of time.
In order to deal with the absolute value function we write the control variable $u$ as the difference of two nonnegative variables, i.e.,

$$u = u_1 - u_2, \ u_1 \geq 0, \ u_2 \geq 0.$$  \hspace{1cm} (5.10)

In order to make $u = u_1$ when $u_1$ is strictly positive, and $u = -u_2$ when $u_2$ is strictly positive, we also impose the quadratic constraint

$$u_1u_2 = 0,$$  \hspace{1cm} (5.11)

so that at most one of $u_1$ and $u_2$ can be nonzero.

Given (5.10) and (5.11) we can write

$$|u| = u_1 + u_2.$$  \hspace{1cm} (5.12)

Also, since $u \in [-U_2, U_1]$ from (5.3), we must have $u_1 \leq U_1$ and $u_2 \leq U_2$. 
In view of (5.10), the control constraints on the variables $u_1$ and $u_2$ are

$$0 \leq u_1 \leq U_1 \text{ and } 0 \leq u_2 \leq U_2.$$  \hspace{1cm} (5.13)

We can now substitute (5.10) and (5.12) into the Hamiltonian (5.5) and reproduce the part that depends on control variables $u_1$ and $u_2$, and denote it by $W$. Thus,

$$W = u_1[(1 - \alpha)\lambda_1 - \lambda_2] - u_2[(1 + \alpha)\lambda_1 - \lambda_2].$$  \hspace{1cm} (5.14)

Maximizing the Hamiltonian (5.5) with respect to $u \in [-U_1, U_2]$ is same as maximizing $W$ w.r.t. $u_1 \in [0, U_1]$ and $u_2 \in [0, U_2]$. But $W$ is linear in $u_1$ and $u_2$ so that the optimal strategy is bang-bang and is:

$$u^* = u_1^* - u_2^*,$$  \hspace{1cm} (5.15)

$$u_1^* = \text{bang}[0, U_1; (1 - \alpha)\lambda_1 - \lambda_2],$$  \hspace{1cm} (5.16)

$$u_2^* = \text{bang}[0, U_2; -(1 + \alpha)\lambda_1 + \lambda_2].$$  \hspace{1cm} (5.17)
Interpretation of the Optimal Policy

- Since $u_1(t)$ represents the rate of sale of securities, (5.16) says that the optimal policy is: sell at the maximum allowable rate if the future value of a dollar less the broker’s commission (i.e., the future value of $(1 - \alpha)$ dollars) is greater than the future value of a dollar’s worth of securities; and do not sell if these future values are in reverse order.

- In case the future value of a dollar less the commission is exactly equal to the future value of a dollar’s worth of securities, then the optimal policy is undetermined. In fact, we are indifferent as to the action taken, and this is called *singular control*.

- Similarly, $u_2(t)$ represents the purchase of securities. Here we buy, do not buy, or are indifferent, if the future value of a dollar plus the commission is less than, greater than, or equal to the future value of a dollar’s worth of securities, respectively.
Interpretation of the Optimal Policy

Note that if

\[(1 - \alpha) \lambda_1(t) \geq \lambda_2(t),\]

then

\[(1 + \alpha) \lambda_1(t) > \lambda_2(t),\]

so that if \(u_1(t) > 0\), then \(u_2(t) = 0\). Similarly, if

\[(1 + \alpha) \lambda_1(t) \leq \lambda_2(t),\]

then

\[(1 - \alpha) \lambda_1(t) < \lambda_2(t),\]

so that if \(u_2(t) > 0\), then \(u_1(t) = 0\). Hence, with the optimal policy, the relation (5.11) is always satisfied.
Figure 5.1: Optimal Policy Shown in \((\lambda_1, \lambda_2)\) Space

- \((1+\alpha)\lambda_1 = \lambda_2\)
- Lines of singular control
- Path of adjoint variable vector
- Buy securities at maximum rate
- Keep Present Portfolio
- Sell securities at maximum rate

**Figure 5.1: Optimal Policy Shown in \((\lambda_1, \lambda_2)\) Space**
Figure 5.2: Optimal Policy Shown in $(t, \lambda_2/\lambda_1)$ Space
Optimal Financing Model

\[ y(t) = \text{the value of the firm's assets or invested capital at time } t, \]
\[ x(t) = \text{the current earnings rate in dollars per unit time at time } t, \]
\[ u(t) = \text{the external or new equity financing expressed as a multiple of current earnings}; \ u \geq 0, \]
\[ v(t) = \text{the fraction of current earnings retained, i.e., } 1 - v(t) \text{ represents the rate of dividend payout}; \ 0 \leq v(t) \leq 1, \]
\[ 1 - c = \text{the proportional floatation (i.e., transaction) cost for external equity}; \ c \text{ a constant}, \ 0 \leq c < 1, \]
\[ \rho = \text{the continuous discount rate (assumed constant); known commonly as the stockholder's required rate of return, or the cost of capital}, \]
\[ r = \text{the actual rate of return (assumed constant) on the firm's invested capital}; \ r > \rho, \]
\[ g = \text{the upper bound on the growth rate of the firm's assets}, \]
\[ T = \text{the planning horizon}; \ T < \infty \ (T = \infty \text{ in Section 5.2.4}). \]
The Model

- The current earnings rate is \( x = ry \). The rate of change in the current earnings rate is given by
  \[
  \dot{x} = ry = r(cu + v)x, \quad x(0) = x_0. \tag{5.18}
  \]

- The upper bound on the rate of growth of the assets implies the following constraint on the control variables:
  \[
  \frac{\dot{y}}{y} = \frac{(cu + v)x}{x/r} = r(cu + v) \leq g. \tag{5.19}
  \]

- The objective of the firm is to maximize its value, which is taken to be the present value of the future dividend stream accruing to the shares outstanding at time zero. To derive this expression, note that
  \[
  \int_0^T (1 - v)xe^{-\rho t} dt
  \]
  represents the present value of total dividends issued by the firm.
A portion of these dividends go to the new equity, which under the assumption of an efficient market will get a rate of return exactly equal to the discount rate $\rho$. This should therefore be equal to the present value

$$\int_0^T u x e^{-\rho t} dt$$

of the external equity raised over time.

The net present value of the total future dividends that accrue to the initial shares is the difference of the previous two expressions, i.e.,

$$J = \int_0^T e^{-\rho t} (1 - v - u) x dt; \quad (5.20)$$

**Remark 5.1:** An intuitive interpretation of (5.20) is that the value $J$ of the firm is the present value of the cash flows (dividends) going out from the firm to the society less the present value of the cash flows (new equity) coming from the society into the firm.
The optimal control problem is to choose \( u \) and \( v \) over time so as to maximize \( J \) in (5.20) subject to (5.18), the constraints (5.19), \( u \geq 0 \), and \( 0 \leq v \leq 1 \). For convenience, we restate this problem as

\[
\max_{u,v} \left\{ J = \int_0^T e^{-\rho t}(1 - v - u)x \, dt \right\}
\]

subject to

\[
\dot{x} = r(cu + v)x, \quad x(0) = x_0,
\]

and the control constraints

\[
cu + v \leq g/r, \quad u \geq 0, \quad 0 \leq v \leq 1.
\]
Application of the Maximum Principle

- The current-value Hamiltonian is

\[ H = (1 - v - u)x + \lambda r (cu + v)x \]
\[ = [(cr \lambda - 1)u + (r \lambda - 1)v + 1]x, \quad (5.22) \]

- The current-value adjoint variable \( \lambda \) satisfies

\[ \dot{\lambda} = \rho \lambda - (1 - v - u) - \lambda r (cu + v) \quad (5.23) \]

- The transversality condition

\[ \lambda(T) = 0. \quad (5.24) \]
We rewrite the Hamiltonian as

\[ H = [W_1 u + W_2 v + 1] x, \]  

(5.25)

where

\[ W_1 = cr \lambda - 1, \]  

(5.26)

\[ W_2 = r \lambda - 1. \]  

(5.27)

The state variable \( x \) factors out so that the optimal controls are independent of the state variable.

The optimal policy is a combination of generalized bang-bang and singular controls. The characterization of these optimal controls in terms of the adjoint variable \( \lambda \) will require solving a parametric linear programming problem at each instant of time \( t \).
The Hamiltonian maximization problem can be stated as follows:

\[
\begin{align*}
\max_{u,v} \{ W_1 u + W_2 v \} \\
\text{subject to} \\
u \geq 0, \ 0 \leq v \leq 1, \ cu + v \leq g/r.
\end{align*}
\]  

We have two cases:

**Case A:** \( g \leq r \) and **Case B:** \( g > r \),
under each of which, we can solve the linear programming problem (5.28) graphically in a closed form. This is done in Figures 5.3 and 5.4.
Table 5.1: Characterization of Optimal Controls with $c < 1$

<table>
<thead>
<tr>
<th>Conditions on $W_1$, $W_2$</th>
<th>Case A: $g \leq r$</th>
<th>Case B: $g &gt; r$</th>
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<tr>
<td>Subcases</td>
<td>Subcases</td>
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<tr>
<td>(1) $W_2 &lt; 0$</td>
<td>A1</td>
<td>B1</td>
<td>$u^* = 0$, $v^* = 0$</td>
<td>generalized bang-bang</td>
</tr>
<tr>
<td>(2) $W_2 = 0$</td>
<td>A2</td>
<td>B2</td>
<td>$u^* = 0$, $0 \leq v^* \leq \min[1, g/r]$</td>
<td>singular</td>
</tr>
<tr>
<td>(3) $W_2 &gt; 0$</td>
<td>A3</td>
<td>-</td>
<td>$u^* = 0$, $v^* = g/r$</td>
<td>generalized bang-bang</td>
</tr>
<tr>
<td>(4) $W_1 &lt; 0$, $W_2 &gt; 0$</td>
<td>-</td>
<td>B3</td>
<td>$u^* = 0$, $v^* = 1$</td>
<td>generalized bang-bang</td>
</tr>
<tr>
<td>(5) $W_1 = 0$</td>
<td>-</td>
<td>B4</td>
<td>$0 \leq u^* \leq (g - r)/rc$, $v^* = 1$</td>
<td>singular</td>
</tr>
<tr>
<td>(6) $W_1 &gt; 0$</td>
<td>-</td>
<td>B5</td>
<td>$u^* = (g - r)/rc$, $v^* = 1$</td>
<td>generalized bang-bang</td>
</tr>
</tbody>
</table>
Figure 5.3: Case A: $g \leq r$
Figure 5.4: Case B: \( g > r \)
Define the reverse-time variable $\tau$ as

$$\tau = T - t,$$

so that

$$\dot{y} = \frac{dy}{d\tau} = \frac{dy}{dt} \frac{dt}{d\tau} = -\dot{y}. $$

The transversality condition on the adjoint variable

$$\lambda(t = T) = \lambda(\tau = 0) = 0 \quad (5.29)$$

becomes the initial condition in the reverse-time sense.

Parameterize the terminal state by assuming that

$$x(t = T) = x(\tau = 0) = \alpha_A, \quad (5.30)$$

where $\alpha_A$ is a parameter to be determined.

**NOTE:** From now on in this section, *everything is expressed in the reverse-time sense unless otherwise specified.*
Using the definitions of $\dot{x}$ and $\dot{\lambda}$ and the conditions (5.30) and (5.29), we can write reverse-time versions of (5.18) and (5.23) as follows:

$$\dot{x} = -r(cu + v)x, \quad x(0) = \alpha_A,$$  \hspace{1cm} (5.31)

$$\dot{\lambda} = (1 - u - v) - \lambda\{\rho - r(cu + v)\}, \quad \lambda(0) = 0.$$  \hspace{1cm} (5.32)

This is the starting point for our switching point synthesis.
Case A and Subcase A1

- **Case A:** $g \leq r$.

  Note that the constraint $v \leq 1$ is superfluous in this case and the only feasible subcases are A1, A2, and A3. Since $\lambda(0) = 0$, we have $W_1(0) = W_2(0) = -1$ and, therefore, Subcase A1.

- **Subcase A1:** $W_2 = r\lambda - 1 < 0$.

  From Row (1) of Table 5.1, we have $u^* = v^* = 0$, which gives the state equation (5.31) and the adjoint equation (5.32) as
  \begin{equation}
  \dot{x} = 0 \quad \text{and} \quad \dot{\lambda} = 1 - \rho\lambda. \tag{5.33}
  \end{equation}

  With the initial conditions given in (5.29), the solutions for $x$ and $\lambda$ are
  \begin{equation}
  x(\tau) = \alpha_A \quad \text{and} \quad \lambda(\tau) = (1/\rho)[1 - e^{-\rho\tau}]. \tag{5.34}
  \end{equation}

  Since $0 \leq c < 1$, it follows that if $W_2 = r\lambda - 1 < 0$, then $W_1 = cr\lambda - 1 < 0$. To remain in this subcase as $\tau$ increases, $W_2(\tau)$ must remain negative for some time as $\tau$ increases.
Since, we have assumed $r > \rho$, there exists a $\tau_1$ such that
\[ W_2(\tau_1) = (1 - e^{-\rho \tau_1}) \frac{r}{\rho} - 1 = 0. \]
It is easy to compute
\[ \tau_1 = \frac{1}{\rho} \ln \left[ \frac{r}{r - \rho} \right]. \] (5.35)

**Remark 5.2:** When $T$ is not sufficiently large, i.e., when $T \leq \tau_1$ in Case A, the firm stays in Subcase A1. The optimal solution in this case is $u^* = 0$ and $v^* = 0$, i.e., a policy of no investment. From this expression, it is clear that the firm leaves Subcase A1 provided $\tau_1 < T$.

**Remark 5.3:** Note that if we had assumed $r < \rho$, the firm would never have exited from Subcase A1 regardless of the value of $T$. Obviously, there is no use investing if the rate of return is less than the discount rate.
Subcase A2:

Subcase A2: \( W_2 = r\lambda - 1 = 0 \).

\[
\begin{align*}
  u^* &= 0, \quad 0 \leq v^* \leq g/r \\
  \lambda &= (1 - v^*) - \lambda[\rho - rv^*].
\end{align*}
\]  

(5.36)

The optimal controls are obtained by conditions required for sustaining \( W_2 = 0 \) for a finite time interval. This means we must have \( \dot{W}_2 = 0 \), which implies \( \dot{\lambda} = 0 \). To compute \( \dot{\lambda} \), we substitute (5.36) into (5.32) and obtain

\[
\dot{\lambda} = (1 - v^*) - \lambda[\rho - rv^*].
\]  

(5.37)

Substituting \( \lambda = 1/r \), its value at \( \tau_1 \), in (5.37) and equating the right-hand side to zero we obtain

\[
r = \rho.
\]  

(5.38)

But \( r > \rho \) implies \( \lambda(\tau_1) = (1 - \rho/r) > 0 \), and the firm switches to Subcase A3.
Subcase A3

Subcase A3: $W_2 = r\lambda - 1 > 0$.

The optimal controls in this subcase from Row (2) of Table 5.1 are

$$u^* = 0, \quad v^* = g/r.$$ (5.39)

The state and the adjoint equations are

$$\dot{x} = -gx, \quad x(\tau_1) = \alpha_A,$$ (5.40)

$$\dot{\lambda} = (1 - g/r) - \lambda(\rho - g), \quad \lambda(\tau_1) = 1/r,$$ (5.41)

with values at $\tau = \tau_1$ deduced from (5.34) and (5.35).

Since $\dot{\lambda}(\tau_1) > 0$, $\lambda$ is increasing at $\tau_1$ from its value of $1/r$. A further examination of the behavior of $\lambda(\tau)$ as $\tau$ increases will be carried out under two different possible conditions: (i) $\rho > g$ and (ii) $\rho \leq g$. 
(i) $\rho > g$: Under this condition, as $\lambda$ increases, $\dot{\lambda}$ decreases and becomes zero at a value obtained by equating the right-hand side of (5.41) to zero, i.e., at

$$\bar{\lambda} = \frac{1 - g/r}{\rho - g}.$$  \hspace{1cm} (5.42)

This value $\bar{\lambda}$ is, therefore, an asymptote to the solution of (5.41) starting at $\lambda(\tau_1) = 1/r$. Since $r > \rho > g$ in this case,

$$\overline{W}_2 = r\bar{\lambda} - 1 = \frac{r(1 - g/r)}{\rho - g} - 1 = \frac{r - \rho}{\rho - g} > 0,$$  \hspace{1cm} (5.43)

which implies that the firm continues to stay in Subcase A3.
(ii) $\rho \leq g$: Under this condition, as $\lambda(\tau)$ increases, $\hat{\lambda}(\tau)$ increases. So $W_2(\tau) = r\lambda(\tau) - 1$ continues to be greater than zero and the firm continues to remain in Subcase A3.

Since the optimal decisions for $\tau \geq \tau_1$ have been found to be independent of $\alpha_A$ for $T$ sufficiently large, we can sketch the solution for Case A in Figure 5.5 starting with $x_0$. This also gives the value of

$$\alpha_A = x_0 e^{g(T-\tau_1)} = x_0 e^{gT[1 - \rho/r]g/\rho},$$

as shown in Figure 5.5.

**Remark 5.4:** With $\rho \leq g$, note that $\lambda(\tau)$ increases to infinity as $\tau$ increases to infinity. This has important implications later when we deal with the solution of the infinite horizon problem.
Figure 5.5: Optimal Path for Case A: $g \leq r$
We can now express the optimal controls and the optimal state, now in forward time, as

\[ u^*(t) = 0, \ v^*(t) = g/r, \ x^*(t) = x_0e^{gt}, \ t \in [0, T - \tau_1], \] (5.44)

\[ u^*(t) = 0, \ v^*(t) = 0, \ x^*(t) = x_0e^{g(T-\tau_1)}, \ t \in (T - \tau_1, T], \] (5.45)

As for \( \lambda(t) \), from (5.34) we have

\[ \lambda(t) = \frac{1}{\rho} [1 - e^{-\rho(T-t)}], \ t \in (T - \tau_1, T]. \] (5.46)

For \( t \in [0, T - \tau_1] \), we have from (5.41),

\[ \dot{\lambda}(t) = \lambda(\rho - g) - (1 - g/r), \ \lambda(T - \tau_1) = 1/r. \] (5.47)

Following Section A.1, we can solve this equation as

\[ \lambda(t) = \frac{1}{r}e^{-(\rho-g)(T-\tau_1-t)} + \frac{1 - g/r}{\rho - g} [1 - e^{-(\rho-g)(T-\tau_1)}], \ t \in [0, T - \tau_1]. \] (5.48)
The switching time \( t = T - \tau_1 \) has an interesting economic interpretation. It requires at least \( \tau_1 \) time to retain a dollar of earnings to be worthwhile for investment. That means, it pays to invest as much of the earnings as feasible before \( T - \tau_1 \), and it does not pay to invest any earnings after \( T - \tau_1 \). So, \( T - \tau_1 \) is the point of indifference between retaining earnings or paying dividends out of earnings.

Suppose the firm retains one dollar of earnings at \( T - \tau_1 \). Since this is the last time that any of the earnings invested will be worthwhile, it is obvious (because all earnings are paid out) that the dollar just invested at \( T - \tau_1 \) yields dividends at the rate \( r \) from \( T - \tau_1 \) to \( T \). The value of this dividend stream in terms of \( (T - \tau_1) \)-dollars is

\[
\int_0^{\tau_1} re^{-\rho s} ds = \frac{r}{\rho}[1 - e^{-\rho\tau_1}],
\]

which must be equated to one dollar to find the indifference point. Equating (5.49) to 1 yields precisely the value of \( \tau_1 \) given in (5.35).

\begin{equation}
\int_0^{\tau_1} re^{-\rho s} ds = \frac{r}{\rho}[1 - e^{-\rho\tau_1}],
\end{equation}
Switching Time interpretation Cont.

With this interpretation of $\tau_1$, we conclude that enough earnings must be retained so as to make the firm grow exponentially at the maximum rate of $g$ until $t = T - \tau_1$. After this time, all of the earnings are paid out and the firm stops growing. Since $g \leq r$ (assumed for Case A), the growth in the first part of the solution can be financed entirely from retained earnings. Thus, there is no need to resort to more expensive external equity financing. The latter will not be true, however, in Case B when $g > r$, which we now discuss.
Case B & Subcase B1

- **Case B:** \( g > r \).

Since \( g/r > 1 \), the constraint \( v \leq 1 \) in Case B is relevant. The feasible subcases are B1, B2, B3, B4, and B5 shown adjacent to the darkened lines in Figure 5.4.

- **Subcase B1:** \( W_2 = r\lambda - 1 < 0 \).

The analysis of this subcase is the same as Subcase A1. As in that subcase, the firm switches out at time \( \tau = \tau_1 \) to Subcase B2.
Subcase B2

- **Subcase B2:** \( W_2 = r\lambda - 1 = 0. \)

  In this subcase, the optimal controls

  \[ u^* = 0, \quad 0 \leq v^* \leq 1 \quad (5.50) \]

  from Row (3) of Table 5.1 are singular with respect to \( v \).

- As before in Subcase A2, the singular case cannot be sustained for a finite time because of our assumption \( r > \rho \). As in Subcase A2, \( W_2 \) is increasing at \( \tau_1 \) from zero and becomes positive after \( \tau_1 \). Thus, at \( \tau_1^+ \), the firm finds itself in Subcase B3.
Subcase B3

- **Subcase B3:** $W_1 = cr\lambda - 1 < 0$, $W_2 = r\lambda - 1 > 0$.

  The optimal controls in this subcase are

  \[ u^* = 0, \quad v^* = 1, \quad (5.51) \]

  as shown in Row (5) of Table 5.1.

- The state and the adjoint equations are

  \[ \dot{x} = -rx, \quad x(\tau_1) = \alpha_B \quad (5.52) \]

  with $\alpha_B$, a parameter to be determined, and

  \[ \dot{\lambda} = \lambda(r - \rho), \quad \lambda(\tau_1) = 1/r. \quad (5.53) \]

- Since $\lambda(\tau_1) = 1/r$, we have

  \[ \lambda(\tau) = \frac{1}{r} e^{(r-\rho)(\tau-\tau_1)} \quad \text{for} \quad \tau \geq \tau_1. \quad (5.54) \]
As \( \lambda \) increases, \( W_1 \) increases and becomes zero at a time \( \tau_2 \) defined by

\[
W_1(\tau_2) = cr\lambda(\tau_2) - 1 = ce^{(r-\rho)(\tau-\tau_1)} - 1 = 0, \tag{5.55}
\]

which, in turn, gives

\[
\tau_2 = \tau_1 + \frac{1}{r-\rho} \ln \left( \frac{1}{c} \right). \tag{5.56}
\]

At \( \tau_2^+ \), the firm switches to Subcase B4.
Subcase B4

- **Subcase B4:** \( W_1 = cr\lambda - 1 = 0. \)

In Subcase B4, the optimal controls are

\[
0 \leq u^* \leq (g - r)/rc, \quad v^* = 1. \tag{5.57}
\]

- To maintain this singular control over a finite time period, we must keep \( W_1 = 0 \) in the interval. This means we must have \( W_1(\tau_2) = 0 \), which, in turn, implies \( \dot{\lambda}(\tau_2) = 0. \)

- To compute \( \dot{\lambda} \), we substitute (5.57) into (5.32) and obtain

\[
\dot{\lambda} = -u^* - \lambda\{\rho - r(cu^* + 1)\}. \tag{5.58}
\]

- At \( \tau_2 \), \( W_1(\tau_2) = 0 \) gives \( \lambda(\tau_2) = 1/rc. \) With this in (5.58), its right-hand side equals zero only when \( r = \rho. \)
Subcase B4 Cont.

- Since $r > \rho$, a singular path cannot be sustained for $\tau_2 > 0$, and so the firm will not stay in Subcase B4 for a finite amount of time.

- From (5.58), we have

$$\dot{\lambda}(\tau_2) = \frac{r - \rho}{rc} > 0,$$  \hspace{2cm} (5.59)

which implies that $\lambda$ is increasing and therefore, $W_1$ is increasing.

- Thus at $\tau_2^+$, the firm switches to Subcase B5.
Subcase B5

**Subcase B5:** $W_1 = cr \lambda - 1 > 0$.

The optimal controls in this subcase from Row (4) of Table 5.1 are

$$u^* = \frac{g - r}{rc}, \quad v^* = 1.$$  \hspace{1cm} (5.60)

The reverse-time state and the adjoint equations are

$$\dot{x} = -gx,$$  \hspace{1cm} (5.61)

$$\dot{\lambda} = -\left(\frac{g - r}{rc}\right) + \lambda(g - \rho).$$  \hspace{1cm} (5.62)
Subcase B5 Cont. and Remark 5.5

- Since \( \hat{\lambda}(\tau_2) > 0 \) from (5.59), \( \lambda(\tau) \) is increasing at \( \tau_2 \) from its value \( \lambda(\tau_2) = 1/rc > 0 \). We have \( g > r \) in Case B, which together with \( r > \rho \), makes \( g > \rho \). This implies that the second term on the right-hand side of (5.62) is increasing. The second term dominates the first term for \( \tau > \tau_2 \), since \( \lambda(\tau_2) = 1/(rc) > 0 \), and \( r > \rho \) and \( g > r \) imply \( g - \rho > g - r > 0 \). Thus, \( \hat{\lambda}(\tau) > 0 \) for \( \tau > \tau_2 \), and \( \lambda(\tau) \) increases with \( \tau \). Therefore, the firm continues to stay in Subcase B5.

- **Remark 5.5**: Note that \( \lambda(\tau) \) in Case B increases without bound as \( \tau \) becomes large. This will have important implications when dealing with the infinite horizon problem in Section 5.2.4.
Interpretation of Case B

- In this solution for Case B, there are two switching points instead of just one as in Case A. The reason for two switching points becomes quite clear when we interpret the significance of $\tau_1$ and $\tau_2$. It is obvious that $\tau_1$ has the same meaning as before.

- If $\tau_1$ is the remaining time to the horizon, the firm is indifferent between retaining a dollar of earnings or paying it out as dividends.

- Since external equity is more expensive than retained earnings as a source of financing, investment financed by external equity requires more time to be worthwhile. That is,

$$\tau_2 - \tau_1 = \frac{1}{r - \rho} \ln \left( \frac{1}{c} \right) > 0 \quad (5.63)$$

should be the time required to compensate for the floatation cost of external equity.
Figure 5.6: Optimal Path for Case B: $g > r$
Interpretation of Case B Cont.

- When the firm issues a dollar’s worth of stock at time $t = T - \tau_2$, it incurs a future dividend obligation in the amount of one $(T - \tau_2)$-dollar, even though the capital acquired is only $c$ dollars because of the floatation cost $(1 - c)$.

- Since we are attempting to find the break-even time for external equity, it is obvious that retaining all of the earnings for investment is still profitable. Thus, there is no dividend from $(T - \tau_2)$ to $(T - \tau_1)$, and the firm grows at the rate $r$. Therefore, this investment of $c$ dollars at time $(T - \tau_2)$ grows into $ce^{r(\tau_2 - \tau_1)}$ dollars at time $(T - \tau_1)$.

- From the point of view of a buyer of the stock at time $(T - \tau_2)$, since no dividend is paid until time $(T - \tau_1)$ and since the stockholder’s required rate of return is $\rho$, the firm’s future dividend obligation at time $(T - \tau_1)$ is $e^{\rho(\tau_2 - \tau_1)}$ in terms of $(T - \tau_1)$-dollars. But then we must have

$$e^{\rho(\tau_2 - \tau_1)} = ce^{r(\tau_2 - \tau_1)},$$

which can be rewritten precisely as (5.63).
Moreover, the firm is marginally indifferent between investing any costless retained earnings at time \((T - \tau_1)\) or paying it all out as dividends. This also means that the firm will be indifferent between having the new available capital of \(ce^{r(\tau_2 - \tau_1)}\) dollars at time \((T - \tau_1)\) as a result of issuing a dollar’s worth of stock at time \((T - \tau_2)\), or not having it.

Thus, we can conclude that the firm is indifferent between issuing a dollar’s worth of stock at time \((T - \tau_2)\) or not issuing it. This means that before time \((T - \tau_2)\), it pays to issue stocks at as large a rate as feasible, and after time \((T - \tau_2)\), it does not pay to issue any external equity at all.

We have now concluded that all earnings must be retained from time \((T - \tau_2)\) to \((T - \tau_1)\). Because \(r > \rho\), it follows that the excess return on the proceeds \(c\) from the new stock issue is \(ce^{r(\tau_2 - \tau_1)} - ce^{\rho(\tau_2 - \tau_1)}\) at time \((T - \tau_1)\).
When discounted this amount back to time \((T - \tau_2)\), we can use (5.63) or (5.64) to see that

\[
\left[ ce^{r(\tau_2 - \tau_1)} - ce^{\rho(\tau_2 - \tau_1)} \right] e^{-\rho(\tau_2 - \tau_1)} = ce^{\ln(1/c)} - c = 1 - c.
\]

Thus, the excess return from time \((T - \tau_2)\) to \((T - \tau_1)\) recovers precisely the floatation cost.

**Remark 5.6:** When \(T\) is not sufficiently large, i.e., when \(T < \tau_2\) in Case B, the optimal solution is the same as in Remark 5.1 when \(T \leq \tau_1\). If \(\tau_1 < T \leq \tau_2\), then the optimal solution is \(u^* = 0\) and \(v^* = 1\) until \(t = T - \tau_1\). For \(t > T - \tau_1\), the optimal solution is \(u^* = 0\) and \(v^* = 0\).
For the infinite horizon case, the transversality condition must be changed to
\[
\lim_{t \to \infty} e^{-\rho t} \lambda(t) = 0.
\] (5.65)

This condition may no longer be a necessary condition. It is a sufficient condition for optimality however, in conjunction with the other sufficiency conditions stated in Theorem 2.1.

A common method of solving an infinite horizon problem is to take the limit as \(T \to \infty\) of the finite horizon solution and then prove that the limiting solution obtained solves the infinite horizon problem.

We now analyze the infinite horizon case following the above procedure. We start with Case A.
Case A: $g \leq r$.

Let us first consider the case $\rho > g$ and examine the solution in forward time obtained in (5.44)-(5.48) as $T$ goes to infinity. Clearly (5.45) and (5.46) disappear, and (5.44) and (5.48) can be written as

$$u^*(t) = 0, \quad v^*(t) = \frac{g}{r}, \quad x^*(t) = x_0 e^{gt}, \quad t \geq 0,$$

(5.66)

$$\lambda(t) = \frac{1 - g/r}{\rho - g} = \lambda, \quad t \geq 0.$$

(5.67)

Clearly $\lambda(t)$ satisfies (5.65). Furthermore,

$$W_2(t) = r\bar{\lambda} - 1 = \frac{r - \rho}{\rho - g} > 0, \quad t \geq 0,$$

which implies that the firm is in Subcase A3 for $t \geq 0$. The maximum principle holds, and (5.66) and (5.67) represent an optimal solution for the infinite horizon problem.
Note that the assumption $\rho > g$ together with our overall assumption that $\rho < r$ gives $g < r$ so that $1 - v^* > 0$, which means a constant fraction of earnings is being paid as dividends.

The value of the adjoint variable $\bar{\lambda}$ in this case is a constant and its form is reminiscent of Gordon’s classic formula.

In the control theory framework, the value of $\bar{\lambda}$ represents the marginal worth per additional unit of earnings. Obviously, a unit increase in earnings will mean an increase of $1 - v^*$ or $1 - g/r$ units in dividends. This, of course, should be capitalized at a rate equal to the discount rate less the growth rate (i.e., $\rho - g$), which is precisely Gordon’s formula.
For $\rho \leq g$, it is clear from (5.48) that $\lambda(t)$ does not satisfy (5.65). A moment’s reflection shows that for $\rho \leq g$, the objective function can be made infinite. For example, any control policy with earnings growing at rate $q$, $\rho \leq q \leq g$, coupled with a partial dividend payout, i.e., a constant $v$ such that $0 < v < 1$, gives an infinite value for the objective function. That is, with $u^* = 0, v^* = q/r < 1$, we have

$$J = \int_0^\infty e^{-\rho t} (1 - u^* - v^*)x^* dt = \int_0^\infty e^{-\rho t} (1 - q/r)x_0 e^{qt} dt = \infty.$$ 

If $g < r$, then the rate of growth $q$ may be chosen in the closed interval $[\rho, g]$; if $g = r$, then $q$ may be chosen in the half-open interval $[\rho, r)$. In either case, the choice of a low rate of growth (i.e., a high proportional dividend payout) would mean a higher dividend rate (in dollars per unit time) early in time, but a lower dividend rate later in time because of the slower growth rate.
To conclude, we note that for $\rho \leq g$ in Case A, the limiting solution of the finite case is an optimal solution for the infinite horizon problem in the sense that the objective function becomes infinite. However, this will not be the situation in Case B.

**Case B: $g > r$.**

The limit of the finite horizon optimal solution is to grow at the maximum allowable growth rate with

$$u = \frac{g - r}{rc} \text{ and } v = 1$$

Since $\tau_1$ disappears in the limit, the stockholders will never collect dividends. The firm has become an infinite sink for investment.

The limiting solution is a *pessimal* solution because the value of the objective function associated with it is zero. From the point of view of optimal control theory, this can be explained as before in Case A when $\rho \leq g$. In Case B, we have $g > r$ so that (since $r > \rho$ throughout the chapter) we have $\rho < g$. 

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Solution for the Infinite Horizon Problem Cont.

- \( \lambda(\tau) \) increases without bound as \( \tau \) increases and, therefore, (5.64) does not have a solution.

- As in Case A with \( \rho < g \), any control policy with earnings growing at rate \( q \in [\rho, g] \) coupled with a constant \( v, 0 < v < 1 \), has an infinite value for the objective function.

- We note that the only nondegenerate case in the infinite horizon problem is when \( \rho > g \). In this case, which occurs only in Case A, the policy of maximum allowable growth is optimal. On the other hand, when \( \rho \leq g \), whether in Case A or B, the infinite horizon problem has nonunique policies with infinite values for the objective function.
Remark 5.7

Let \((u^*_T, v^*_T)\) denote the optimal control for the finite horizon problem in Case B. Let \((u^*_\infty, v^*_\infty)\) denote any optimal control for the infinite horizon problem in Case B. We already know that \(J(u^*_\infty, v^*_\infty) = \infty\). Define an infinite horizon control \((u_\infty, v_\infty)\) by extending \((u^*_T, v^*_T)\) as follows:

\[
(u_\infty, v_\infty) = \lim_{T \to \infty} (u^*_T, v^*_T).
\]

We now note that for our model in Case B, we have

\[
\lim_{T \to \infty} J(u^*_T, v^*_T) = \infty \text{ and } J(\lim_{T \to \infty} (u^*_T, v^*_T)) = J(u_\infty, v_\infty) = 0.
\]

(5.68)

Obviously \((u_\infty, v_\infty)\) is \textit{not} an optimal control for the infinite horizon problem. Since the two terms in (5.68) are not equal, we can say in technical terms that \(J(u, v)\), regarded as a mapping, is not a \textit{closed} mapping.
If we introduce a salvage value $Bx(T)$, $B > 0$, for the finite horizon problem, then the new objective function,

$$J_B(u, v) = \begin{cases} 
\int_0^T e^{-\rho t}(1 - u - v)x dt + Bx(T)e^{-\rho T}, & \text{if } T < \infty, \\
\int_0^\infty e^{-\rho t}(1 - u - v)x dt + \lim_{T \to \infty} \{Bx(T)e^{-\rho T}\}, & \text{if } T = \infty,
\end{cases}$$

is a closed mapping in the sense that

$$\lim_{T \to \infty} J_B(u_T^*, v_T^*) = \infty \text{ and } J_B(\lim_{T \to \infty} (u_T^*, v_T^*)) = J_B(u_\infty, v_\infty) = \infty$$

for the modified model.
We will now assign numbers to the various parameters in the optimal financing problem in order to compute the optimal solution. Let

\[ x_0 = \frac{1000}{\text{month}}, \quad T = 60 \text{ months}, \]
\[ r = 0.15, \quad \rho = 0.10, \quad g = 0.05, \quad c = 0.98. \]

**SOLUTION:** Since \( g \leq r \), the problem belongs to Case A. We compute

\[ \tau_1 = \frac{1}{\rho} \ln\left[\frac{r}{(r - \rho)}\right] = 10 \ln 3 \approx 11 \text{ months}. \]

The optimal controls for the problem are

\[ u^* = 0, \quad v^* = \frac{g}{r} = \frac{1}{3}, \quad t \in [0, 49), \]
\[ u^* = 0, \quad v^* = 0, \quad t \in [49, 60], \]
Solution of Example 5.1

- The optimal state trajectory is
  \[ x(t) = \begin{cases} 
  1000e^{0.05t}, & t \in [0, 49), \\
  1000e^{2.45}, & t \in [49, 60].
\end{cases} \]

- The value of the objective function is
  \[
  J^* = \int_0^{49} e^{-0.1t}(1 - 1/3)(1000)e^{0.05t} dt + \int_{49}^{60} 1000e^{2.45} \cdot e^{-0.1t} dt
  = 12,578.75.
  \]

- Note that the infinite horizon problem is well defined in this case, since \( g < \rho \) and \( g < r \). The optimal controls are
  \[ u^* = 0, \quad v^* = g/r = 1/3, \]
  and
  \[
  J = \int_0^{\infty} e^{-0.1t}(2/3)(1000)e^{0.05t} dt = 2000/0.15 = 13,333\frac{1}{3}.
  \]