Applications to Marketing

- **State Equation:** Rate of Sales expressed in terms of advertising, which is a control variable.
- **Objective:** Profit maximization.
- **Constraints:** Advertising rate to be non-negative.
The Nerlove-Arrow Advertising Model

- The belief that advertising expenditures by a firm affect its present and future sales, and hence its present and future net revenues, has led a number of economists including Nerlove and Arrow (1962) to treat advertising as an investment in building up some sort of advertising capital, usually called *goodwill*.

- Let $G(t) \geq 0$ denote the stock of goodwill at time $t$. It is assumed that the stock of goodwill depreciates over time at a constant proportional rate $\delta$, so that

$$\dot{G} = u - \delta G, \quad G(0) = G_0,$$

where $u = u(t) \geq 0$ is the advertising effort at time $t$ measured in dollars per unit time.

- In economic terms, equation (7.1) states that the net investment in goodwill is the difference between gross investment $u(t)$ and depreciation $\delta G(t)$.
To formulate the optimal control problem for a monopolistic firm, assume that the rate of sales $S(t)$ depends on the stock of goodwill $G(t)$, the price $p(t)$, and other exogenous factors $Z(t)$, such as consumer income, population size, etc. Thus,

$$S = S(p, G; Z). \quad (7.2)$$

Assuming the rate of total production cost is $c(S)$, we can write the total revenue net of production cost as

$$R(p, G; Z) = pS(p, G; Z) - c(S(p, G; Z)). \quad (7.3)$$

The revenue net of advertising expenditure is therefore $R(p, G; Z) - u$.

We assume that the firm wants to maximize the present value of net revenue streams discounted at a fixed rate $\rho$, i.e.,

$$\max_{u \geq 0, p \geq 0} \left\{ J = \int_0^\infty e^{-\rho t}[R(p, G; Z) - u] \, dt \right\} \quad (7.4)$$
Since the only place that $p$ occurs is in the integrand, we can maximize $J$ by first maximizing $R$ with respect to price $p$ while holding $G$ fixed, and then maximize the result with respect to $u$. Thus,

$$\frac{\partial R}{\partial p} = S + p \frac{\partial S}{\partial p} - c'(S) \frac{\partial S}{\partial p} = 0,$$

which implicitly gives the optimal price $p^*(t) = p(G(t); Z(t))$.

Defining $\eta = -(p/S)(\partial S/\partial p)$ as the elasticity of demand with respect to price, we can rewrite condition (7.5) as

$$p^* = \frac{\eta c'(S)}{\eta - 1},$$

which is the usual price formula for a monopolist, known sometimes as the Amoroso-Robinson relation.

In words, the relation means that the marginal revenue $(\eta - 1)p/\eta$ must equal the marginal cost $c'(S)$. See, e.g., Cohen and Cyert (1965, p.189).
Defining $\Pi(G; Z) = R(p^*, G; Z)$, the objective function in (7.4) can be rewritten as

$$\max_{u \geq 0} \left\{ J = \int_0^\infty e^{-\rho t} [\Pi(G; Z) - u] \, dt \right\}.$$ 

We assume $Z$ to be a given constant. Thus, we can define $\pi(G) = \Pi(G; Z)$ and restate the optimal control problem which we have just formulated:

$$\begin{cases} 
\max_{u \geq 0} \left\{ J = \int_0^\infty e^{-\rho t} \pi(G) - u \right\} \\
\text{subject to} \\
\dot{G} = u - \delta G, \quad G(0) = G_0.
\end{cases}$$

(7.7)

It is reasonable to assume the functions introduced in (7.2) and (7.3) to satisfy conditions so that $\pi(G)$ is increasing and concave in goodwill $G$. More specifically, we assume that $\pi'(G) \geq 0$ and $\pi''(G) < 0$. 

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We form the current-value Hamiltonian

\[ H = \pi(G) - u + \lambda[u - \delta G], \]  

(7.8)

where the current-value adjoint variable \( \lambda \) satisfies the differential equation

\[ \dot{\lambda} = \rho \lambda - \frac{\partial H}{\partial G} = (\rho + \delta)\lambda - \frac{d\pi}{dG} \]  

(7.9)

with the boundary condition

\[ \lim_{t \to +\infty} e^{-\rho t} \lambda(t) = 0. \]  

(7.10)

The adjoint variable \( \lambda(t) \) is the shadow price associated with the goodwill at time \( t \). Thus, the Hamiltonian in (7.8) can be interpreted as the dynamic profit rate which consists of two terms:

i) the current net profit rate \( \pi(G) - u \) and

ii) the value \( \lambda \dot{G} = \lambda[u - \delta G] \) of the goodwill rate \( \dot{G} \) created by advertising at rate \( u \).
Equation (7.9) corresponds to the usual equilibrium relation for investment in capital goods; see Arrow and Kurz (1970) and Jacquemin (1973). It states that the marginal opportunity cost \( \lambda (\rho + \delta) dt \) of investment in goodwill, by spending on advertising, should equal the sum of the marginal profit \( \pi'(G) dt \) from the increased goodwill due to that investment and the capital gain \( d\lambda := \dot{\lambda} dt \) on the unit price of goodwill.

Use (3.108) to obtain the optimal long-run stationary equilibrium or turnpike \( \{\bar{G}, \bar{u}, \bar{\lambda}\} \). That is, \( \lambda = \bar{\lambda} = 1 \) from (7.8) by using \( \partial H/\partial u = 0 \). Then set \( \lambda = \bar{\lambda} = 1 \) and \( \dot{\lambda} = 0 \) in (7.9) to obtain

\[
\pi'(\bar{G}) = \rho + \delta. \tag{7.11}
\]

We may assume \( \pi'(0) > \rho + \delta \) and \( \pi'(\infty) < \rho + \delta \) to obtain a strictly positive equilibrium goodwill level \( \bar{G} \).
Defining $\beta = (G/S)(\partial S/\partial G)$ as the elasticity of demand w.r.t. goodwill and using (7.3), (7.5), (7.6), and (7.9) with $\dot{\lambda} = 0$ and $\bar{\lambda} = 1$ to derive

$$\frac{\bar{G}}{pS} = \frac{\beta}{\eta(\rho + \delta)}.$$  \hfill (7.12)

The property of $\bar{G}$ is that the optimal policy is to go to $\bar{G}$ as fast as possible.

If $G_0 < \bar{G}$, it is optimal to jump instantaneously to $\bar{G}$ by applying an appropriate impulse at $t = 0$ and then set $u^*(t) = \bar{u} = \delta \bar{G}$ for $t > 0$.

If $G_0 > \bar{G}$, the optimal control $u^*(t) = 0$ until the stock of goodwill depreciates to the level $\bar{G}$, at which time the control switches to $u^*(t) = \delta \bar{G}$ and stays at this level to maintain the level $\bar{G}$ of goodwill.

This optimal policy is graphed in Figure 7.1 for the above two different initial conditions.
Solution by the Maximum Principle Cont.

Case 1: $G_0 \geq \bar{G}$

$u^* = 0$

Case 2: $G_0 < \bar{G}$

$u^* = I(G_0, \bar{G}; 0)$

$u^* = \bar{u} = \delta \bar{G}$

Figure 7.1: Optimal Policies in the Nerlove-Arrow Model
Suppose, instead, that we need to spend $m$ dollars on current advertising to increase goodwill by one unit. Then, the new optimal control problem is:

$$\max_{0 \leq u \leq M} \left\{ J = \int_0^\infty e^{-\rho t} [\pi(G) - mu]dt \right\}. \quad (7.13)$$

The equilibrium goodwill level formula (7.11) changes to

$$\pi'(\bar{G}) = (\rho + \delta)m. \quad (7.14)$$

With $\bar{G}$ thus defined, the optimal solution is as shown in Figure 7.1 with the dotted curve representing the solution in Case 2: $G_0 < \bar{G}$. 
Figure 7.2: A Case of a Time-Dependent Turnpike and the Nature of Optimal Control
Time-dependent Turnpike

- For a time-dependent $Z$, however, $\tilde{G}(t) = G(Z(t))$ will be a function of time. To maintain this level of $\tilde{G}(t)$, the required control is $\tilde{u}(t) = \delta \tilde{G}(t) + \dot{G}(t)$.

- If $\tilde{G}(t)$ is decreasing sufficiently fast, then $\tilde{u}(t)$ may become negative and thus infeasible. If $\tilde{u}(t) \geq 0$ for all $t$, then the optimal policy is as before.

- However, suppose $\tilde{u}(t)$ is infeasible in the interval $[t_1, t_2]$ shown in Figure 7.2. In such a case, it is feasible to set $u(t) = \tilde{u}(t)$ for $t \leq t_1$; at $t = t_1$ (which is point $A$ in Figure 7.2) we can no longer stay on the turnpike and must set $u(t) = 0$ until we hit the turnpike again (at point $B$ in Figure 7.2). However, such a policy is not necessarily optimal.
Time-dependent Turnpike

For instance, suppose we leave the turnpike at point C anticipating the infeasibility at point A. The new path CDEB may be better than the old path CAB. Roughly the reason this may happen is that path CDEB is “nearer” to the turnpike than CAB. The picture in Figure 7.2 illustrates such a case. The optimal policy is the one that is “nearest” to the turnpike. This discussion will become clearer in Section 7.2.2, when a similar situation arises in connection with the Vidale-Wolfe model. For further details; see Sethi (1977b) and Breakwell (1968).
Convex Advertising Cost and Relaxed Controls

- Let \( c(u) \) be a strictly concave advertising cost function with \( c(0) = 0 \), \( c'(u) > 0 \) and \( c''(u) < 0 \) for \( 0 \leq u \leq M \), where \( M > 0 \) denotes an upper bound on the advertising rate.

- Let us also consider \( T > 0 \) to be the fixed terminal time. Then, our problem is the following modification of problem (7.7):

\[
\begin{cases}
\max_{0 \leq u \leq M} \left\{ J_1 = \int_0^T e^{-\rho t} [\pi(G') - c(u)] \, dt \right\} \\
\text{subject to} \\
\dot{G} = u - \delta G, \quad G(0) = G_0.
\end{cases}
\]  

(7.15)

- Note that with concave \( c(u) \), the profit rate \( \pi(G') - c(u) \) is convex in \( u \). Thus, its maximum over \( u \) would occur at the boundary 0 or \( M \) of the set \([0, M]\). It should be clear that if we replace \( c(u) \) by the linear function \( mu \) with \( m = c(M)/M \), then

\[
\pi(G') - c(u) < \pi(G) - mu, \quad u \in (0, M).
\]  

(7.16)
This means that if problem (7.15) with \( mu \) in place of \( c(u) \), i.e., the problem

\[
\begin{align*}
\max_{0 \leq u \leq M} \left\{ J_2 = \int_0^T e^{-\rho t} [\pi(G) - mu] \, dt \right\} \\
\text{subject to} \\
\dot{G} = u - \delta G, \quad G(0) = G_0,
\end{align*}
\]

has only the bang-bang solution, then the solution of problem (7.17) would also be the solution of the convex problem (7.15), for a sufficiently small value of \( T \). However, if \( T \) is large or infinity, then the solution of (7.17) will have a singular portion, and it will not solve (7.17).

In particular, let us consider problems (7.15) and (7.17) when \( T = \infty \) and \( G_0 < \bar{G} \).
Note that problem (7.17) is the same as the problem in Exercise 7.4, and its optimal solution is as shown in Figure 7.1 with $\bar{G}$ given by (7.14) and the optimal trajectory given by the dotted line followed by the solid horizontal line representing the singular part of the solution.

Let $u_2^*$ denote the optimal control of problem (7.17). Since the singular control is in the open interval $(0, M)$, then in view of (7.16),

$$J_1(u_2^*) < J_2(u_2^*).$$ (7.18)

Thus, for sufficiently small $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, we can “chatter” between $G_1 = (\bar{G} + \varepsilon_1)$ and $G_2 = (\bar{G} - \varepsilon_2)$ by using controls $M$ and 0 alternately, as shown in Figure 7.3, to obtain a near-optimal control of problem (7.15). Clearly, in the limit as $\varepsilon_1$ and $\varepsilon_2$ go to 0, the objective function of problem (7.15) will converge to $J_2(u_2^*)$. 
Figure 7.3: A Near-Optimal Control of Problem (7.15)
The theory of relaxed controls manifests itself for our problem is to provide a probability measure on the boundary values \( \{0, M\} \). Thus, let \( v \) be the probability that control \( M \) is used, so that the probability of using control 0 is \( (1 - v) \). With this, we transform problem (7.15) with \( T = \infty \) as follows:

\[
\max_{v \in [0,1]} \left\{ J_3 = \int_0^\infty e^{-\rho t} \left[ \pi(G) - vc(M) \right] dt \right\}
\]

subject to

\[
\dot{G} = vM - \delta G, \quad G(0) = G_0.
\]

We can now use the maximum principle to solve problem (7.19).
The Hamiltonian

\[ H = \pi G - vc(M) + \lambda(vM - \delta G) \]

The adjoint equation as defined by (7.9) and (7.10). The optimal control is given by

\[ v^* = \text{bang}[0, 1; \lambda M - c(M)]. \] (7.20)

The singular control is given by

\[ \bar{\lambda} = m, \quad \pi'(\bar{G}) = (\rho + \delta)m, \quad \bar{v} = \delta \bar{G}/M. \] (7.21)
A Nonlinear extension

The way we interpret this control is by use of a biased coin with the probability of heads being $\bar{v}$. We flip this coin infinitely fast, and use the maximum control $M$ when heads comes up and the minimum control 0 when tails comes up. Because the control will chatter infinitely fast according to the outcome of the coin tosses, such a control is also referred to as a *chattering control*.

While such a chattering control cannot be implemented, it can be arbitrarily approximated by using alternately $u^* = M$ for $\tau \bar{v}$ periods and $u^* = 0$ for $\tau (1 - \bar{v})$ periods for a small $\tau > 0$. With reference to Figure 7.3 and with $G_1$ and $G_2$ to be determined for the given $\tau$, this approximate policy of rapidly switching the control between $M$ and 0 can be said to begin at time $t_1$, when the goodwill reaches $G_2$. After that goodwill goes up to $G_1$ and then back down to $G_2$, and so on. See Exercise 7.8.

In marketing parlance, advertising rates that alternate between maximum and zero are known as a *pulsing policy*. 
The Vidale-Wolfe Advertising Model

- We now present the analysis of the Vidale-Wolfe advertising model which, in contrast to the Nerlove-Arrow model, does not make use of the idea of advertising goodwill, instead the model exploits the closely related notion that the effect of advertising tends to persist, but diminishes over subsequent time periods.

- Vidale and Wolfe argued that changes in the rate of sales of a product depend on two effects:
  1. the action of advertising (via the response constant $a$) on the unsold portion of the market and
  2. the loss of sales (via the decay constant $b$) from the sold portion of the market.

- Let $M(t)$, known as the saturation level or market potential, denote the maximum potential rate of sales at time $t$. Let $S(t)$ be the actual rate of sales at time $t$.

- Then, the Vidale-Wolfe model for a monopolistic firm can be stated as

$$
\dot{S} = au(1 - \frac{S}{M}) - bS.
$$

(7.22)
To maximize a certain objective function over the horizon $T$, while also attaining a terminal sales target, it is convenient to transform (7.22) by making the change of variable

$$x = \frac{S}{M}. \quad (7.23)$$

Thus, $x$ represents the market share (or more precisely, the rate of sales expressed as a fraction of the saturation level $M$). Furthermore, we define

$$r = \frac{a}{M}, \quad \delta = b + \frac{\dot{M}}{M}. \quad (7.24)$$

Now we can rewrite (7.22) as

$$\dot{x} = ru(1 - x) - \delta x, \quad x(0) = x_0. \quad (7.25)$$

From now on we assume $M$, and hence $\delta$ and $r$, to be positive constants.
The optimal control problem can be stated as follows:

$$
\begin{align*}
\max \left\{ J = \int_0^T e^{-\rho t}(\pi x - u)dt \right\} \\
\text{subject to} \\
\dot{x} = ru(1 - x) - \delta x, \ x(0) = x_0, \\
\text{the terminal state constraint} \\
x(T) = x_T, \\
\text{and the control constraint} \\
0 \leq u \leq Q.
\end{align*}
$$

(7.26)

where $\pi$ denote the maximum sales revenue corresponding to $x = 1$, with $\pi x$ denoting the revenue function for $x \in [0, 1]$, $Q$ be the maximum allowable rate of advertising expenditure and $\rho$ denote the continuous discount rate. Here $Q$ can be finite or infinite and the target market share $x_T$ is in $[0, 1]$. 
To make use of Green’s theorem, it is convenient to consider times \( \tau \) and \( \theta \), where \( 0 \leq \tau < \theta \leq T \), and the problem:

\[
\max \left\{ J(\tau, \theta) = \int_{\tau}^{\theta} e^{-\rho t} (\pi x - u) dt \right\}
\]

subject to

\[
\dot{x} = ru(1 - x) - \delta x, \quad x(\tau) = A, \quad x(\theta) = B,
\]

\( 0 \leq u \leq Q. \)

To change the objective function in (7.27) into a line integral along any feasible arc \( \Gamma_1 \) from \((\tau, A)\) to \((\theta, B)\) in \((t, x)\)-space as shown in Figure 7.4, we multiply (7.28) by \( dt \) and obtain the formal relation

\[
u dt = \frac{dx + \delta x dt}{r(1 - x)},
\]

which we substitute into the objective function (7.27).
Solution Using Green’s Theorem when $Q$ is Large Cont.

Thus,

$$J_{\Gamma_1} = \int_{\Gamma_1} \left\{ \pi x - \frac{\delta x}{r(1-x)} \right\} e^{-\rho t} dt - \frac{1}{r(1-x)} e^{-\rho t} dx \right\}. $$

Figure 7.4: Feasible Arcs in $(t, x)$-Space
Consider another feasible arc $\Gamma_2$ from $(\tau, A)$ to $(\theta, B)$ lying above $\Gamma_1$ as shown in Figure 7.4.

Let $\Gamma = \Gamma_1 - \Gamma_2$, where $\Gamma$ is a simple closed curve traversed in the counter-clockwise direction. That is, $\Gamma$ goes along $\Gamma_1$ in the direction of its arrow and along $\Gamma_2$ in the direction opposite to its arrow. We now have

$$J_\Gamma = J_{\Gamma_1 - \Gamma_2} = J_{\Gamma_1} - J_{\Gamma_2}. \quad (7.30)$$

Since $\Gamma$ is a simple closed curve, we can use Green's theorem to express $J_\Gamma$ as an area integral over the region $R$ enclosed by $\Gamma$. Thus, treating $x$ and $t$ as independent variables, we can write

$$J_\Gamma = \oint_{\Gamma} \left\{ \pi x - \frac{\delta x}{r(1-x)} \right\} e^{-\rho t} \, dt - \frac{1}{r(1-x)} e^{-\rho t} \, dx$$

$$= \iint_R \left\{ \frac{\partial}{\partial t} \left[ \frac{-e^{-\rho t}}{r(1-x)} \right] - \frac{\partial}{\partial x} \left[ (\pi x - \frac{\delta x}{r(1-x)}) e^{-\rho t} \right] \right\} \, dt \, dx$$
Solution Using Green’s Theorem when \( Q \) is Large Cont.

\[
= \int \int_R \left[ \frac{\delta}{(1 - x)^2} + \frac{\rho}{(1 - x)} - \pi r \right] e^{-\rho t} \frac{r}{t} dt dx. \tag{7.31}
\]

- Denote the term in brackets of the integrand of (7.31) by
  \[
  I(x) = \frac{\delta}{(1 - x)^2} + \frac{\rho}{(1 - x)} - \pi r. \tag{7.32}
  \]
- Note that the sign of the integrand is the same as the sign of \( I(x) \).
Lemma 7.1

- **Lemma 7.1**: (Comparison Lemma). Let $\Gamma_1$ and $\Gamma_2$ be the lower and upper feasible arcs as shown in Figure 7.4. If $I(x) \geq 0$ for all $(x, t) \in R$, then the lower arc $\Gamma_1$ is at least as profitable as the upper arc $\Gamma_2$. Analogously, if $I(x) \leq 0$ for all $(x, t) \in R$, then $\Gamma_2$ is at least as profitable as $\Gamma_1$.

- **Proof**: If $I(x) \geq 0$ for all $(x, t) \in R$, then $J_{\Gamma} \geq 0$ from (7.31) and (7.32). Hence from (7.30), $J_{\Gamma_1} \geq J_{\Gamma_2}$. The proof of the other statement is similar. □
To make use of this lemma to find the optimal control for the problem stated in (7.26), we need to find regions where $I(x)$ is positive and where it is negative.

For this, note first that $I(x)$ is an increasing function of $x$ in $[0, 1]$. Solving $I(x) = 0$ will give that value of $x$, above which $I(x)$ is positive and below which $I(x)$ is negative. Since $I(x)$ is quadratic in $1/(1 - x)$, we can use the quadratic formula (see Exercise 7.16) to get

$$x = 1 - \frac{2\delta}{-\rho \pm \sqrt{\rho^2 + 4\pi r \delta}}. \quad (7.33)$$

To keep $x$ in the interval $[0, 1]$, we must choose the positive sign before the radical. The optimal $x$ must be nonnegative so we have

$$x_s = \max \left\{ 1 - \frac{2\delta}{-\rho + \sqrt{\rho^2 + 4\pi r \delta}}, 0 \right\},$$  

where the superscript $s$ is used because this will turn out to be a singular trajectory.
Since \( x^s \) is nonnegative, the control

\[
  u^s = \frac{\delta x^s}{r(1 - x^s)}
\]

(7.35)

corresponding to (7.34) will always be nonnegative. Also since \( Q \) is assumed to be large, \( u^s \) will always be feasible.

- Note that \( x^s = 0 \) and \( u^s = 0 \) if, and only if, \( \pi r \leq \delta + \rho \).

- To obtain the optimal solution for (7.26) when \( Q \) is assumed to be sufficiently large, i.e., \( Q \geq u^s \), where \( u^s \) is given in (7.35), we state these in the form of two theorems: Theorem 7.1 refers to the case in which \( T \) is large; Theorem 7.2 refers to the case in which \( T \) is small.

- To define these concepts, let \( t_1 \) be the shortest time to go from \( x_0 \) to \( x^s \) and similarly let \( t_2 \) be the shortest time to go from \( x^s \) to \( x_T \).

Then, we say \( T \) is large if \( T \geq t_1 + t_2 \); otherwise \( T \) is small.

Figures 7.5-7.8 show cases for which \( T \) is large, while Figures 7.10-7.11 show cases for which \( T \) is small.
**Theorem 7.1 and its proof**

- **Theorem 7.1:** Let $T$ be large and let $x_T$ be reachable from $x_0$. For the Cases 1-4 of inequalities relating $x_0$ and $x_T$ to $x^s$, the optimal trajectories are given in Figures 7.5-7.8, respectively.

- **Proof:** Figure 7.9 shows the optimal trajectory for Figure 7.5 together with an arbitrarily chosen feasible trajectory, shown dotted. It should be clear that the dotted trajectory cannot cross the arc $x_0$ to $C$, since $u = Q$ on that arc. Similarly, the dotted trajectory cannot cross the arc $G$ to $x_T$, because $u = 0$ on that arc.

- We subdivide the interval $[0, T]$ into subintervals over which the dotted arc is either above, below, or identical to the solid arc.

- In Figure 7.9 these subintervals are $[0, d]$, $[d, e]$, $[e, f]$, and $[f, T]$. Because $I(x)$ is positive for $x > x^s$ and $I(x)$ is negative for $x < x^s$, the regions enclosed by the two trajectories have been marked with a + or − sign depending on whether $I(x)$ is positive or negative on the regions, respectively.
By Lemma 7.1, the solid arc is better than the dotted arc in the subintervals \([0, d], [d, e], \) and \([f, T]\); in interval \([e, f]\), they have identical values. Hence the dotted trajectory is inferior to the solid trajectory. This proof can be extended to any (countable) number of crossings of the trajectories; see Sethi (1977b).

Figures 7.5-7.8 are drawn for the situation when \(T > t_1 + t_2\). In Exercise 7.25, you are asked to consider the case when \(T = t_1 + t_2\). The following theorem deals with the case when \(T < t_1 + t_2\).
Figure 7.5: Optimal Trajectory for Case 1: $x_0 \leq x^s$ and $x_T \leq x^s$
Figure 7.6: Optimal Trajectory for Case 2: $x_0 < x^s$ and $x_T > x^s$
Figure 7.7: Optimal Trajectory for Case 3: $x_0 > x^s$ and $x_T < x^s$. 
Figure 7.8: Optimal Trajectory for Case 4: \( x_0 > x^s \) and \( x_T > x^s \)
Theorem 7.2 and its proof

**Theorem 7.2:** Let $T$ be small, i.e., $T < t_1 + t_2$, and let $x_T$ be reachable from $x_0$. For the two possible Cases 1 and 2 of inequalities relating $x_0$ and $x_T$ to $x^s$, the optimal trajectories are given in Figures 7.10 and 7.11, respectively.

**Proof:** The requirement of feasibility when $T$ is small rules out cases where $x_0$ and $x_T$ are on opposite sides of or equal to $x^s$. The proofs of optimality of the trajectories shown in Figures 7.10 and 7.11 are similar to the proofs of the parts of Theorem 7.1, and are left as Exercise 7.25. In Figures 7.10 and 7.11, it is possible to have either $t_1 \geq T$ or $t_2 \geq T$. Try sketching some of these special cases. □

We had assumed that $Q$ was finite and sufficiently large, but we can easily extend this to the case when $Q = \infty$. This possibility makes the arcs in Figures 7.5-7.10, corresponding to $u^* = Q$, become vertical line segments corresponding to impulse controls. For example, Figure 7.6 becomes Figure 7.12 when $Q = \infty$ and we apply the impulse control $\text{imp}(x_0, x^s; 0)$ when $x_0 < x^s$. 

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Figure 7.9: Optimal Trajectory (Solid Lines)
Optimal Trajectory when $T$ is Small in Case 1

Figure 7.10: Optimal Trajectory when $T$ is Small in Case 1: $x_0 < x^s$ and $x_T > x^s$
Optimal Trajectory when $T$ is Small in Case 2

Figure 7.11: Optimal Trajectory when $T$ is Small in Case 2: $x_0 > x^s$ and $x_T > x^s$
Optimal Trajectory for Case 2 of Theorem 7.1 for $Q = \infty$

Figure 7.12: Optimal Trajectory for Case 2 of Theorem 7.1 for $Q = \infty$
Next we compute the cost of $\text{imp}(x_0, x^s; 0)$ by assessing its effect on the objective function of (7.26). For this, we integrate the state equation in (7.26) from 0 to $\varepsilon$ with the initial condition $x_0$ and $u$ treated as constant. By using (A.7), we can write the solution as

$$x(\varepsilon) = x_0 e^{-(\delta+ru)\varepsilon} + \int_0^\varepsilon e^{(\delta+ru)(\tau-\varepsilon)} ru d\tau$$

$$= \left( x_0 - \frac{ru}{\delta + ru} \right) e^{-(\delta+ru)\varepsilon} + \frac{ru}{\delta + ru}.$$

We must choose $u(\varepsilon)$ so that $x(\varepsilon)$ is $x^s$. It should be clear that $u(\varepsilon) \to \infty$ as $\varepsilon \to 0$. With $F(x, u, \tau) = \pi x(\tau) - u(\tau)$ and $t = 0$ in (1.23), we have the impulse

$$I = \text{imp}(x_0, x^s; 0) = \lim_{\varepsilon \to 0} [-u(\varepsilon)\varepsilon].$$
It is possible to solve for $I$ by letting $\varepsilon \to 0$, $-u(\varepsilon)\varepsilon \to I$, $u(\varepsilon) \to \infty$, and $x(\varepsilon) = x^s$ in the expression for $x(\varepsilon)$ obtained above. This gives

$$x(0+) = e^{rI}(x_0 - 1) + 1.$$ 

Therefore,

$$\text{imp}(x_0, x^s; 0) = -\frac{1}{r} \ln \left[ \frac{1 - x_0}{1 - x^s} \right]. \quad (7.36)$$

Note that this formula holds for any time $t$, as well as $t = 0$. Hence it can also be used at $t = T$ to compute the impulse at the end of the period.
Solution when $Q$ is Small

- When $Q$ is small, it is not possible to go along the turnpike $x^s$, so the arguments based on Green’s theorem become difficult to apply and so we return to the maximum principle approach to analyze the problem. By “$Q$ is small” we mean $Q < u^s$, where $u^s$ is defined in (7.35).
- The current-value Hamiltonian and Lagrangian function is defined as

$$H = \pi x - u + \lambda [ru(1 - x) - \delta x]$$
$$= \pi x - \delta \lambda x + u [-1 + r\lambda(1 - x)], \quad (7.37)$$

$$L = H + \mu (Q - u). \quad (7.38)$$

- The adjoint variable $\lambda$ and the Lagrange multiplier $\mu$ in (7.38), respectively, satisfies

$$\dot{\lambda} = \rho \lambda - \frac{\partial L}{\partial x} = \rho \lambda + \lambda (ru + \delta) - \pi,$$ \quad (7.39)

$$\mu \geq 0, \quad \mu (Q - u) = 0. \quad (7.40)$$

where $\lambda(T)$ is a constant.
Solution when \( Q \) is Small Cont.

- From (7.37) we notice that the Hamiltonian is linear in the control. So the optimal control is

\[
    u^*(t) = \text{bang}[0, Q; W(t)],
\]

(7.41)

where

\[
    W(t) = W(x(t), \lambda(t)) = r\lambda(t)(1 - x(t)) - 1.
\]

(7.42)

- When \( W = 0 \), we have the possibility of a singular control, provided we can maintain this equality over a finite time interval. For the case when \( Q \) is large, we showed in the previous section that the optimal trajectory contains a segment on which \( x = x^s \) and \( u^* = u^s \), where \( 0 \leq u^s \leq Q \).

- A complete solution of problem (7.26) when \( Q \) is small requires a lengthy switching point analysis. Interested reader can find the details in Sethi (1973a).
We formulate the infinite horizon version of (7.26):

\[
\max \left\{ J = \int_0^\infty e^{-\rho t} (\pi x - u) \, dt \right\}
\]
subject to
\[
\dot{x} = ru(1 - x) - \delta x, \quad x(0) = x_0, \\
0 \leq u \leq Q.
\]

We divide the analysis of this problem into the same two cases defined as before, namely, “\(Q\) is large” and “\(Q\) is small”.

(7.43)
Solution when $T$ is Infinite Cont.

- When $Q$ is large, the results of Theorem 7.1 suggest the solution when $T$ is infinite. Because of the discount factor, the ending parts of the solutions shown in Figures 7.5-7.8 can be shown to be irrelevant (i.e., the discounted profit accumulated during the interval $(T - t_2, T)$ goes to 0 as $T$ goes to $\infty$). Therefore, we only have two cases: (a) $x_0 \leq x^s$, and (b) $x_0 \geq x^s$.

- The optimal control in Case (a) is to use $u^* = Q$ in the interval $[0, t_1)$ and $u^* = u^s$ for $t \geq t_1$. Similarly, the optimal control in Case (b) is to use $u^* = 0$ in the interval $[0, t_1)$ and $u^* = u^s$ for $t \geq t_1$. 
When $Q$ is small, i.e., $Q < u^s$, it is not possible to follow the turnpike $x = x^s$, because that would require $u = u^s$, which is not a feasible control. Intuitively, it seems clear that the “nearest” stationary path to $x^s$ that we can follow is the path obtained by setting $\dot{x} = 0$ and $u = Q$, the largest possible control, in the state equation of (7.43).

This gives

$$\bar{x} = \frac{rQ}{rQ + \delta},$$

(7.44)

and correspondingly we obtain

$$\bar{\lambda} = \frac{\pi}{\rho + \delta + rQ},$$

(7.45)

by setting $u = Q$ and $\dot{\lambda} = 0$ in (7.39) and solving for $\lambda$. 
Solution when $T$ is Infinite Cont.

- To find an optimal solution from any given initial $x_0$, the approach we take is to find a feasible path that is “nearest” to $x^s$; See Sethi (1977b) for further discussion.

- For $x_0 < x^s$, such a path is obtained by using the maximum possible control $Q$ all the way. For $x_0 > x^s$, the situation is more difficult.

- The following two theorems give the turnpike as well as the optimal path starting from any initial $x_0$. Let us define $\hat{x}$ and $\bar{\mu}$ such that
  \[ W(\hat{x}, \bar{\lambda}) = r\bar{\lambda}(1 - \hat{x}) - 1 = 0 \]
  \[ L_u(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu}) = W(\bar{x}, \bar{\lambda}) - \bar{\mu} = 0. \]
  Thus,
  \[ \hat{x} = 1 - 1/r\bar{\lambda}, \quad (7.46) \]
  \[ \bar{\mu} = r\bar{\lambda}(1 - \bar{x}) - 1. \quad (7.47) \]
Theorem 7.3 and its proof

- **Theorem 7.3**: When $Q$ is small, the quadruple $\{\bar{x}, Q, \bar{\lambda}, \bar{\mu}\}$ forms a turnpike.

- **Proof**: We show that the turnpike conditions (3.107) hold for the quadruple. The first two conditions of (3.107) are (7.44) and (7.45). By Exercise 7.31 we know $\bar{x} \leq \hat{x}$, which, from definitions (7.46) and (7.47), implies $\bar{\mu} \geq 0$. Furthermore $\bar{u} = Q$, so (7.40) holds and the third condition of (3.107) also holds. Finally because $W = \bar{\mu}$ from (7.42) and (7.47), it follows that $W \geq 0$, so the Hamiltonian maximizing condition of (3.107) holds with $\bar{u} = Q$. □
Theorem 7.4 and its proof

Theorem 7.4: When $Q$ is small, the optimal control at any time $\tau \geq 0$ is given by: (a) If $x(\tau) \leq \hat{x}$, then $u^*(\tau) = Q$.
(b) If $x(\tau) > \hat{x}$, then $u^*(\tau) = 0$.

Proof: (a) We set $\lambda(t) = \bar{\lambda}$ for all $t \geq \tau$ and note that $\lambda$ satisfies the adjoint equation (7.39) and the transversality condition (3.99).

By Exercise 7.31 and the assumption that $x(\tau) \leq \hat{x}$, we know that $x(t) \leq \hat{x}$ for all $t$. The proof that (7.40) and (7.41) hold for all $t \geq \tau$ relies on the fact that $x(t) \leq \hat{x}$ and on an argument similar to the proof of the previous theorem.

Figure 7.13 shows the optimal trajectories when $x_0 < \hat{x}$ for two different starting values of $x_0$, one above and the other below $\bar{x}$. Note that in this figure we are always in Case (a) since $x(\tau) \leq \hat{x}$ for all $\tau \geq 0$. 
Figure 7.13: Optimal Trajectories for $x(0) < \hat{x}$
Proof to Theorem 7.4 Cont.

(b) Assume $x_0 > \hat{x}$. In this case we show that the optimal trajectory is as shown in Figure 7.14, which is obtained by applying $u = 0$ until $x = \hat{x}$ and $u = Q$ thereafter. Using this policy we can find the time $t_1$ at which $x(t_1) = \hat{x}$, by solving the state equation in (7.43) with $u = 0$. This gives

$$t_1 = \frac{1}{\delta} \ln \frac{x_0}{\hat{x}}. \quad (7.48)$$

For $t \geq t_1$, the policy $u = Q$ is optimal because Case (a) applies. We now consider the interval $[0, t_1]$, where we set $u = 0$. Let $\tau$ be any time in this interval as shown in Figure 7.14, and let $x(\tau)$ be the corresponding value of the state variable. Then $x(\tau) = x_0 e^{-\delta \tau}$.

With $u = 0$ in (7.39), the adjoint equation on $[0, t_1]$ becomes

$$\dot{\lambda} = (\rho + \delta) \lambda - \pi.$$
Proof to Theorem 7.4 Cont.

- We know that $x(t_1) = \hat{x}$. Thus, Case (a) applies at time $t_1$, and we would like to have $\lambda(t_1) = \bar{\lambda}$. So, we solve the adjoint equation with $\lambda(t_1) = \bar{\lambda}$ and obtain

\[
\lambda(\tau) = \frac{\pi}{\rho + \delta} + \left(\bar{\lambda} - \frac{\pi}{\rho + \delta}\right)e^{(\rho + \delta)(\tau - t_1)}, \quad \tau \in [0, t_1]. \tag{7.49}
\]

- Now, with the values of $x(\tau)$ and $\lambda(\tau)$ in hand, we can use (7.42) to obtain the switching function value $W(\tau)$.

- In Exercise 7.34, you are asked to show that the switching function $W(\tau)$ is negative for each $\tau$ in the interval $[0, t_1)$ and $W(t_1) = 0$. Therefore by (7.41), the policy $u = 0$ used in deriving (7.48) and (7.49) satisfies the maximum principle. This policy "joins" the optimal policy after $t_1$ because $\lambda(t_1) = \bar{\lambda}$. 
Figure 7.14: Optimal Trajectory for $x(0) > \hat{x}$