Abstract

We consider a production planning problem for a jobshop with unreliable machines producing a number of products. There are upper and lower bounds on intermediate parts and an upper bound on finished parts. The machine capacities are modelled as finite state Markov chains. The objective is to choose the rate of production so as to minimize the total discounted cost of inventory and production. This is a class of difficult problems. In this paper, instead of searching for exact optimal control policies, we derive an asymptotic approximation by letting the rates of change of the machine states approach infinity. The asymptotic analysis leads to a limiting problem in which the stochastic machine capacities are replaced by their equilibrium mean capacities. The value function for the original problem is shown to converge to the value function of the limiting problem. The convergence rate of the value function together with the error estimate for the constructed asymptotic optimal production policies are established.

Keywords: optimal production policy, manufacturing systems, stochastic dynamic programming, discounted cost, asymptotic analysis
1 Introduction

We consider a manufacturing system producing a variety of products in demand using machines in a general network configuration, which generalizes both the parallel and tandem machine models. Each product follows a process plan that specifies the sequence of machines it must visit and the operations performed by them. A process plan may call for multiple visits to a given machine, as is the case in semiconductor manufacturing (Lou and Kager (1989)). Often the machines are unreliable. Over time they break down and must be repaired. A manufacturing system so described will be termed a dynamic jobshop. The term will be made mathematically precise in the next section.

It is in the nature of such a dynamic jobshop that the inventory of intermediate parts must remain nonnegative. In reality, the buffer capacities are also limited imposing upper bounds on both intermediate as well as finished parts.

The problem under consideration is to control the production rates of intermediate parts and of finished parts in a manufacturing system consisting of a network of failure-prone machines. The objective is to meet the demand for finished products at the minimum possible discounted cost of production, inventories, and backlogs. The problem is not easy to solve, particularly because of the state constraints associated with the intermediate and finished parts. Certainly, no explicit solution, unlike in a single machine system without any state constraints (Akella and Kumar (1988)), is available for our problem.

Recognizing the complexity of the problem, Sethi and Zhou (1994) develop a hierarchical approach for approximately solving the stochastic optimal production planning problem of a jobshop with a discount cost criterion, when the rates of changes in machine states are much larger than the rate of discounting of costs. They show that the problem can be approximated by a deterministic optimization problem. Then by using the optimal control of the deterministic problem, they construct a production policy for the original stochastic problem, which is asymptotically optimal as the rate of changes in machine states approach infinity. The hierarchical production controls for the other systems with the discounted cost criterion, such as single or parallel machine systems and flowshops, can be found in Sethi and Zhang (1994). However, this paper did not consider the upper bound constraints associated with intermediate and finished parts. Moreover, their method of constructing asymptotic optimal production policies do not apply to systems with upper bound state constraints.

Sethi, Zhang, and Zhang (2000) consider an N-machine flowshop with limited buffers. They develop a method of shrinking, entire lifting and modification to construct asymptotic optimal open-loop production policy for the N-machine flowshop from a near-optimal production policy for the corresponding limiting problem. Here “shrinking” means to find a production policy that uses a little less than the full machine capacities at a time. Based on this shrinking, the entire lifting procedure involves appropriate increase in production rates on the intervals so as to make the inventory levels in the internal buffers to be positive. After shrinking and entire lifting, they obtain a near-optimal production policy for the limiting problem. Using this near-optimal production policy, they construct a production policy for the original problem, which they further modify to make it also a near-optimal admissible production policy for the original problem.
The purpose of this paper is to obtain asymptotic optimal policies for a jobshop with finite buffers. Owing to the fact that more than one machine can feed a given buffer, the shrinking, entire lifting, and modification method of Sethi, Zhang, and Zhang (2000) is no longer sufficient in constructing asymptotic optimal production policies. To overcome this difficulty, we introduce a scheme of parameter distribution of the machine capacities method. We also show that the control constructed by this technique is indeed near optimal. The error estimate we obtain is of the same order as the one obtained in Sethi, Zhang, and Zhang (2000).

The plan of our paper is as follows. In Section 2 we introduce the problem and specify the required assumptions. Section 3 is devoted to formulating the limiting control problem. In Section 4, we establish the convergence of the minimum discounted expected cost for the original problem to the minimum discounted cost for the limiting problem. Finally in Section 5, we conclude the paper.

2 Problem Formulation

We begin with a manufacturing system that consists of $m_c$ failure-prone machines and $n$ buffers including $m$ internal buffers. We use the notation of the jobshop model developed in Sethi and Zhou (1994). Then we give a simple example to illustrate the model.

Let $k(\varepsilon, t) = (k_1(\varepsilon, t), \ldots, k_{mc}(\varepsilon, t))$ denote a stochastic process defined on the standard probability space $(\Omega, \mathcal{F}, \Pr)$ with $k_\ell(\varepsilon, t)$ representing the capacity of the $\ell$th machine at time $t$, $\ell = 1, \ldots, m_c$, where $\varepsilon$ is a small parameter to be precisely specified later.

We denote the surplus at time $t$ in buffer $j$ by $x_j(\varepsilon, t)$. Write it in vector form as $x(\varepsilon, t) = (x_1(\varepsilon, t), \ldots, x_n(\varepsilon, t))'$. Note that $x_j(\varepsilon, t), j = 1, 2, \ldots, m$, is called a work-in-process at time $t$ and $x_j(\varepsilon, t), j = m + 1, \ldots, n$, is called a surplus of the finished product $j$ at time $t$.

Let $(V, A)$ denote a manufacturing digraph where $V$ is a finite nonempty set of vertices and $A$ is the collection of ordered arcs. Let us now suppose that $(V, A)$ contains a total of $(n_0 + n + 1)$ vertices including $n_0$ sources, the sink, $m$ internal buffers, and $(n - m)$ external buffers for some integer $m$ and $n$ with $0 \leq m \leq n - 1$ and $n \geq 1$. Let $\mathcal{K} = \{K_1, K_2, \ldots, K_{mc}\}$ denote the corresponding placement, i.e., $\mathcal{K}$ is a partition of $B = \{(i, j) \in A : i \leq m\}$, namely, $\emptyset \neq K_j \subset B$, $K_j \cap K_\ell = \emptyset$ for $j \neq \ell$, and $\bigcup_{j=1}^{mc} K_j = B$; see Sethi and Zhou (1994) (or Sethi and Zhang (1994)) for further details. The controls $u_{i,j}(\varepsilon, t)$ with $(i, j) \in K_\ell$, $\ell = 1, \ldots, m_c$, $t \geq 0$, must satisfy the following constraints:

$$0 \leq \sum_{(i,j) \in K_\ell} u_{i,j}(\varepsilon, t) \leq k_\ell(\varepsilon, t) \text{ for all } t \geq 0, \; \ell = 1, \ldots, m_c. \quad (1)$$

Then the dynamics of the system are given by

$$\frac{d}{dt} x_j(\varepsilon, t) = \sum_{\ell=-n_0+1}^{j-1} u_{\ell,j}(\varepsilon, t) - \sum_{\ell=j+1}^{n} u_{j,\ell}(\varepsilon, t), \quad 1 \leq j \leq m,$$

$$\frac{d}{dt} x_j(\varepsilon, t) = \sum_{\ell=-n_0+1}^{m} u_{\ell,j}(\varepsilon, t) - z_j, \quad m + 1 \leq j \leq n, \quad (2)$$
with $x(\varepsilon, 0) = (x_1(\varepsilon, 0), ..., x_n(\varepsilon, 0))' = (x_1, ..., x_n)' = x$, where $u_{j, \ell}(\varepsilon, t) = 0$ if $(j, \ell) \notin A$. The state constraints are

$$
0 \leq x_j(\varepsilon, t) \leq H_j, \quad t \geq 0, \quad j = 1, ..., m,
$$

$$
-\infty < x_j(\varepsilon, t) \leq H_j, \quad t \geq 0, \quad j = m + 1, ..., n,
$$

where $H_j$ is the capacity of buffer $j$. Note that if $x_j(\varepsilon, t) > 0$, $j = 1, ..., n$, we have an inventory in buffer $j$, and if $x_j(\varepsilon, t) < 0$, $j = m + 1, ..., n$, we have a shortage of finished product $j$.

**Example.** To illustrate the model, let us consider a simple jobshop given in Fig. 1.

![Fig. 1. A Typical Dynamic Jobshop](attachment:image1.png)

Here we have four machines $M_1, M_2, M_3$ and $M_4$, and seven buffers $b_0, b_1, ..., b_6$ (or simply buffers 0, 1, ..., 6). Machine $M_i$ ($i = 1, 2, 3, 4$) has the production capacity of $k_i(\varepsilon, t)$ at time $t$. Buffers $b_0$ and $b_6$ are fictitious in the sense that $b_0$ is an infinite source containing all required raw materials and $b_6$ is a sink with no constraints. Buffers $b_4$ and $b_5$ are external buffers corresponding to specific final products and the remaining buffers are internal buffers corresponding to specific intermediate products. Quantities $z_4$ and $z_5$ represent the rates of demand of the final products, $u_{i,j}(t)$ is the rate of conversion from the material in buffer $i$ to the material in buffer $j$ by using the machine on the arc $(i, j)$, $i = 0, 1, 2, 3$ and $j = 1, 2, 3, 4, 5$. Let $x_i(t)$, $i = 1, ..., 5$, be the contents of buffer $b_i$ at time $t$. Clearly $x_i(t)$ for $i = 1, 2, 3$, cannot become negative, and for $i = 4, 5$ it defines the inventory/shortage of final product $i$ and may take negative and positive values.

The system dynamics and the constraints are stated below.

**System dynamics:**

$$
\dot{x}_1(t) = u_{01}(t) - u_{12}(t) - u_{14}(t), \quad \dot{x}_2(t) = u_{12}(t) - u_{23}(t), \quad \dot{x}_3(t) = u_{23}(t) - u_{34}(t),
$$

$$
\dot{x}_4(t) = u_{14}(t) + u_{34}(t) - z_4, \quad \dot{x}_5(t) = u_{05}(t) - z_5;
$$

(4)

**State constraints:**

$$
0 \leq x_i(t) \leq H_i, \quad i = 1, 2, 3 \quad \text{and} \quad -\infty < x_i(t) \leq H_i, \quad i = 4, 5;
$$

(5)
Control constraints:
\[\begin{align*}
&u_{01}(t) \leq k_1(\varepsilon, t), \\
&u_{12}(t) + u_{34}(\varepsilon, t) \leq k_3(\varepsilon, t), \\
&u_{23}(t) \leq k_2(\varepsilon, t), \\
&u_{05}(t) + u_{14}(t) \leq k_4(\varepsilon, t),
\end{align*}\]

where \(H_i\) is the size of the buffer \(i, i = 1, 2, 3, 4, 5\). In this example, the placements for the job-shop in Figure 1 are \(K_1 = \{(0, 1)\}, \ K_2 = \{(2, 3)\}, \ K_3 = \{(3, 4), (1, 2)\}, \ K_4 = \{(0, 5), (1, 4)\}\).

It is convenient to write the control in a vector form. To do this, for \(\ell = -n_0 + 1, \ldots, 0\), and \(j = 1, \ldots, m\), let
\[\begin{align*}
&u_{\ell}(\varepsilon, t) = \begin{pmatrix} u_{\ell,1}(\varepsilon, t) \\ \vdots \\ u_{\ell,n}(\varepsilon, t) \end{pmatrix}, \\
&u_{j}(\varepsilon, t) = \begin{pmatrix} u_{j,1}(\varepsilon, t) \\ \vdots \\ u_{j,n}(\varepsilon, t) \end{pmatrix},
\end{align*}\]

Then the system equation in (2) can be written in the following vector form:
\[\frac{d}{dt}x(\varepsilon, t) = (A_{-n_0+1}, \ldots, A_{m+1}) \begin{pmatrix} u(\varepsilon, t) \\ z \end{pmatrix}, \quad x(\varepsilon, 0) = x,\]

with suitable choice of \((A_{-n_0+1}, \ldots, A_{m+1})\), which is an \(n \times \tilde{n}\) matrix with \(\tilde{n} = n_0n + (n - m) + \sum_{\ell=1}^{m}(n - \ell)\). Let \(\mathcal{I}\) denote the state space, i.e., \(\mathcal{I} = \Pi_{\ell=1}^{m}[0, H_\ell] \times \Pi_{\ell=m+1}^{n}(-\infty, H_\ell]\). Furthermore, we introduce a linear operator \(L\) from \(\mathbb{R}_{\tilde{n}}^{+}\) to \(\mathcal{I}\):
\[L(u(\varepsilon, t), z) = (A_{-n_0+1}, \ldots, A_{m+1}) \begin{pmatrix} u(\varepsilon, t) \\ z \end{pmatrix}.\]

Then, (9) can be written as
\[\frac{d}{dt}x(\varepsilon, t) = L(u(\varepsilon, t), z), \quad x(\varepsilon, 0) = x.\]

We are now in the position to formulate our stochastic optimal control problem for the jobshop defined by (1), (2) and (3). For \(k = (k_1, \ldots, k_{m_c})\), let
\[\mathcal{U}(k) = \left\{ (u_{i,\ell}) : (u_{i,\ell}) \in \mathcal{U}, \ 0 \leq \sum_{(i,\ell) \in K_j} u_{i,\ell} \leq k_j, \ 1 \leq j \leq m_c \right\}.\]

By (7) and (8), for each \((u_{i,j}) \in \mathcal{U}(k)\), we can generate a unique nonnegative \(\tilde{n}\)-dimensional vector \(u\) in the rest of the paper, we use \(u\) and \((u_{i,j}) \in \mathcal{U}(k)\) interchangeably. For \(x \in \mathcal{I}\) and
corresponding state process taken to be the one that satisfies

\[ \gamma \epsilon_{t} \in M \subset \mathcal{P} \]

\( H_{j} \) denotes the finite state Markov chain with the infinitesimal generator

\[ \gamma Q \]

\( \gamma \) and \( c(\cdot) \) denote the equilibrium distribution of \( Q^{(2)} \), that is, \( \gamma \) is the only nonnegative solution to the equation

\[
\gamma Q^{(2)} = 0 \text{ and } \sum_{i=1}^{p} \gamma_{i} = 1.
\]

\( A(2) \) \( h(\cdot) \) and \( c(\cdot) \) are non-negative convex functions. For all \( x, x' \in \mathcal{I} \) and \( u, u' \in U(k^{\ell}), j = 1, ..., p \), there exist constants \( C_{0} \) and \( K_{h} \geq 1 \) such that

\[
|h(x) - h(x')| \leq C_{0}(1 + |x|^{K_{h}} + |x'|^{K_{h}})|x - x'|,
\]

and

\[
|c(u) - c(u')| \leq C_{0}|u - u'|.
\]

We use \( \mathcal{A}(x, k) \) to denote the set of all admissible controls with respect to \( x \in \mathcal{I} \) and \( k(0) = k \). We use \( \mathcal{P}^{\varepsilon} \) \( J^{\varepsilon}(x, u(\varepsilon, \cdot), k) = E \int_{0}^{\infty} e^{-\rho t} [h(x(\varepsilon, t)) + c(u(\varepsilon, t))] dt \)

\[
\mathcal{P}^{\varepsilon}: \quad \begin{cases}
\minimize J^{\varepsilon}(x, u(\varepsilon, \cdot), k) = E \int_{0}^{\infty} e^{-\rho t} [h(x(\varepsilon, t)) + c(u(\varepsilon, t))] dt, \\
\subject{\begin{aligned}
\dot{x}(\varepsilon, t) &= \mathcal{L}(u(\varepsilon, t), x), \\
\mathcal{L}(u(\varepsilon, t), x(\varepsilon, 0) &= x, \\
u(\varepsilon, \cdot) &\in \mathcal{A}(x, k) \\
\end{aligned}}
\end{cases}
\]

value function \( V^{\varepsilon}(x, k) = \inf_{u(\varepsilon, \cdot) \in \mathcal{A}(x, k)} J^{\varepsilon}(x, u(\varepsilon, \cdot), k) \).
3 The Limiting Control Problem

In this section we derive the limiting control problem as $\varepsilon \to 0$. Intuitively, as the rates of the machine breakdown and repair approach infinity, the problem $\mathcal{P}^\varepsilon$, which is termed the original problem, can be approximated by a simpler problem called the limiting problem, where the stochastic machine capacity process $k(\varepsilon, t)$ is replaced by a weighted form.

The Hamilton-Jacobi-Bellman equation in the directional derivative sense with the discounted optimal control problem in $\mathcal{P}^\varepsilon$ as shown in Sethi, Zhang, and Zhang (1999), takes the form

$$V^\varepsilon(x, k^r) = \inf_{u \in U(x, k^r)} \left\{ \frac{\partial V^\varepsilon(x, k^r)}{\partial L(u, z)} + c(u) + \left( Q^{(1)} + \frac{1}{\varepsilon} Q^{(2)} \right) V^\varepsilon(x, \cdot)(k^r) \right\} + h(x) + (Q^1 + 1) \varepsilon Q^2(1), \quad (12)$$

where $\frac{\partial V^\varepsilon(x, k^r)}{\partial L(u, z)}$ denotes the directional derivative of $V^\varepsilon(x, k^r)$ along the direction $L(u, z)$, and $Qf(\cdot)(k^r) := \sum_{i \neq r} q_{ir}(f(k^i) - f(k^r))$ for any function $f(\cdot)$ on $\mathcal{M}$. Moreover, Sethi, and Zhang (1997) show that $V^\varepsilon(x, k^r)$ is a solution of (12).

The limiting problem can be formulated as follows. Consider an augmented control $U(\cdot) = (u^1(\cdot), \ldots, u^p(\cdot))$, where $u^i(t) \in \mathcal{U}(k^i)$ and $u^i(t), \ t \geq 0$, is a deterministic process.

**Definition 3.1** For $x \in \mathcal{I}$, let $\mathcal{A}^0(x)$ denote the set of the following measurable controls $U(\cdot) = (u^1(\cdot), \ldots, u^p(\cdot))$ such that the solution of

$$\frac{d}{dt}x(t) = L \left( \sum_{i=1}^{p} \gamma_i u^i(t), z \right), \quad x(0) = x,$$

satisfies $x(t) \in \mathcal{I}$ for all $t \geq 0$.

The objective of this problem is to choose a control $U(\cdot) \in \mathcal{A}^0(x)$ that minimizes

$$J(x, U(\cdot)) = \int_0^\infty e^{-\rho t} \left[ h(x(s)) + \sum_{i=1}^{p} \gamma_i c(u^i(s)) \right] ds.$$

We use $\mathcal{P}^0$ to denote this problem, known as the limiting control problem, and restate it as follows:

$$\mathcal{P}^0: \begin{cases} 
\text{minimize} & J(x, U(\cdot)) = \int_0^\infty e^{-\rho t} \left[ h(x(s)) + \sum_{i=1}^{p} \gamma_i c(u^i(s)) \right] ds, \\
\text{subject to} & \frac{d}{dt}x(t) = L \left( \sum_{i=1}^{p} \gamma_i u^i(t), z \right), \ x(0) = x, \ U(\cdot) \in \mathcal{A}^0(x), \\
\text{value function} & V(x) = \inf_{U(\cdot) \in \mathcal{A}^0(x)} J(x, U(\cdot)).
\end{cases} \quad (13)$$
The HJBDD associated with $P^0$ is

$$V(x) = \inf_{u \in U^0(x)} \left\{ \frac{\partial V(x)}{\partial \mathcal{L}(\sum_{i=1}^p \gamma_i u^i, z)} + \sum_{i=1}^p \gamma_i c(u^i) \right\} + h(x), \quad (14)$$

where

$$U^0(x) = \left\{ (u^1, \ldots, u^p) : u^i \in U(k^i) \text{ and } x_j = 0 \right\}.$$

4 Convergence of Value Functions

In this section we consider the convergence of the value function (minimum expected cost) $V^\varepsilon(x, k)$ as $\varepsilon$ goes to zero, and establish its convergence rate. First we give without proof the following lemma similar to Lemma C.3 of Sethi and Zhang (1994).

**Lemma 4.1** Let $P(\varepsilon, t)$ denote the transition matrix of the Markov process $k(\varepsilon, \cdot)$. Then

$$|P(\varepsilon, t) - \bar{P}| \leq C_1(\varepsilon + e^{-\eta t/\varepsilon}),$$

for some positive constant $C_1$ and $\eta_0$, where $\bar{P} = (\gamma_11, \ldots, \gamma_p1)$ with $1 = (1, \ldots, 1)'$ and $\gamma = (\gamma_1, \ldots, \gamma_p)$ given in (10). Moreover, for all $k^r \in \mathcal{M}$ and $t \geq 0$,

$$|\Pr\{k(\varepsilon, t) = k^r\} - \gamma_r| \leq C_1(\varepsilon + e^{-\eta t/\varepsilon}).$$

The next result we require is as follows:

**Lemma 4.2** Let

$$\Phi(\varepsilon, t) = \Phi(k(\varepsilon, t)) = (I^1_{k(\varepsilon, t) = k^1}, \ldots, I^p_{k(\varepsilon, t) = k^p})'.$$

Then for any bounded deterministic measurable process $\beta(\cdot)$, $\delta \in [0, \frac{1}{4}]$, and $\tau$, which is a Markov time with respect to $k(\varepsilon, \cdot)$, there exist positive constants $C_2$ and $\kappa_1$ such that

$$\Pr\left( \sup_{0 \leq t \leq T} \left| \int_\tau^{t+\delta} [\Phi(\varepsilon, s) - \gamma'] \beta(s) \, ds \right| \geq \varepsilon^\delta \right) \leq C_2e^{\kappa_1\varepsilon^{-(1/2-\delta)(1+T)^{-3}},$$

for all $T \geq 0$ and sufficiently small $\varepsilon$. 

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Proof. The proof is similar to the one given by Sethi, Zhang, and Zhou (1992) or Yin and Zhang (1998). Here we omit the proof. □

In order to get the required convergence result, we need the following auxiliary lemma, which is a key step towards our main result.

Lemma 4.3 For $\delta \in [0, \frac{1}{2})$ and $x = (x_1, ..., x_n) \in \mathcal{I}$, there exist a positive constant $C > 0$, $x(\delta) = (x_1(\delta), ..., x_n(\delta)) \in \mathcal{I}$, and

$$U(\delta, \varepsilon, \cdot) = (u^1(\delta, \varepsilon, \cdot), ..., u^p(\delta, \varepsilon, \cdot)) \in \mathcal{A}_0^0(x),$$

such that

$$|x - x(\delta)| \leq \varepsilon \delta,$$  \hspace{1cm} (15)

$$\inf_{0 \leq s < \infty} x_j(\delta, \varepsilon, s) \geq \varepsilon \delta, \hspace{1cm} j = 1, \ldots, m,$$  \hspace{1cm} (16)

$$\sup_{0 \leq s < \infty} x_j(\delta, \varepsilon, s) \leq H_j - \varepsilon \delta, \hspace{1cm} j = 1, \ldots, n,$$  \hspace{1cm} (17)

and

$$V(x) + C \varepsilon \delta > \int_0^\infty \left[ h(x(\delta, \varepsilon, t)) + \sum_{i=1}^p \gamma_i c(u^i(\delta, \varepsilon, t)) \right] dt,$$  \hspace{1cm} (18)

where $x(\delta, \varepsilon, t)$ is the trajectory under $U(\delta, \varepsilon, t)$ with $x(\delta, \varepsilon, 0) = x(\delta)$.

Proof. For each fixed $\varepsilon > 0$ and $x \in \mathcal{I}$, we select

$$\tilde{U}(\cdot) = (\tilde{u}^1(\cdot), ..., \tilde{u}^p(\cdot)) \in \mathcal{A}_0^0(x)$$  \hspace{1cm} (19)

to be an $\varepsilon$-optimal control for $\mathcal{P}^0$, i.e.,

$$\left| \int_0^\infty e^{-\rho t} \left[ h(\tilde{x}(t)) + \sum_{i=1}^p \gamma_i c(\tilde{u}^i(t)) \right] dt - V(x) \right| \leq \varepsilon,$$  \hspace{1cm} (20)

where $\tilde{x}(t)$ is the solution of

$$\frac{d}{dt} x(t) = \mathcal{L} \left( \sum_{i=1}^p \gamma_i \tilde{u}^i(t), z \right), \hspace{1cm} x(0) = x.$$

Furthermore, let

$$a(H) = \max_{1 \leq j \leq n} \left\{ \frac{H_j}{H_j - 2\varepsilon \delta} \right\}.$$  \hspace{1cm} (21)

Define

$$\tilde{u}^i(t) = \frac{\tilde{u}^i(t)}{a(H)} \hspace{1cm} i = 1, \ldots, p,$$
Let 
\[
\hat{\mathbf{x}}_j(t) = \frac{x_j}{a(H)} + \int_0^t \left[ \sum_{\ell=-n_0+1}^{j-1} \hat{u}_{\ell,j}(s) - \sum_{\ell=j+1}^{n} \hat{u}_{j,\ell}(s) \right] ds, \quad j = 1, \ldots, m, \\
\hat{\mathbf{x}}_j(t) = \frac{x_j}{a(H)} + \int_0^t \left[ \sum_{\ell=-n_0+1}^{m} \hat{u}_{\ell,j}(s) - z_j \right] ds, \quad j = m + 1, \ldots, n.
\]

Then,
\[
\hat{\mathbf{x}}_j(t) = \tilde{\mathbf{x}}_j(t) a(H), \quad j = 1, \ldots, m, \tag{22}
\]
\[
\hat{\mathbf{x}}_j(t) = \tilde{\mathbf{x}}_j(t) a(H) + \left( \frac{1}{a(H)} - 1 \right) z_j t, \quad j = m + 1, \ldots, n. \tag{23}
\]

Thus, in view of \(\tilde{\mathbf{x}}(t) \in \mathcal{I}\) and \(a(H) > 1\), we get
\[
\hat{U}(\cdot) = (\hat{\mathbf{u}}^1, \ldots, \hat{\mathbf{u}}^p(\cdot)) \in \mathcal{A}(\tilde{\mathbf{x}}), \tag{24}
\]
where \(\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n)\) with \(\tilde{x}_\ell = x_\ell/a(H)\), \(\ell = 1, \ldots, n\). Furthermore, from the definition of \(a(H)\), we have that for \(\ell = 1, \ldots, m,\)
\[
\tilde{x}_\ell(t) = \frac{\tilde{x}_\ell(t)}{a(H)} \leq \frac{H_\ell}{a(H)} \leq \frac{H_\ell}{H_\ell/(H_\ell - 2\varepsilon\delta)} = H_\ell - 2\varepsilon\delta, \tag{25}
\]
and for \(j = m + 1, \ldots, n,\)
\[
\tilde{x}_\ell(t) = \frac{\tilde{x}_\ell(t)}{a(H)} + \left( \frac{1}{a(H)} - 1 \right) z_j t \leq \frac{H_\ell}{a(H)} \leq \frac{H_\ell}{H_\ell/(H_\ell - 2\varepsilon\delta)} = H_\ell - 2\varepsilon\delta. \tag{26}
\]

On the other hand, it follows from (22) and (23) that
\[
|\tilde{x}_j(t) - \tilde{\mathbf{x}}_j(t)| = \left| \frac{\tilde{x}_j(t)}{a(H)} - \tilde{\mathbf{x}}_j(t) \right| \leq \frac{2\varepsilon\delta}{\min_{1 \leq i \leq n} \{H_i\}} \cdot \tilde{x}_j(t), \quad j = 1, \ldots, m, \tag{27}
\]
and
\[
|\tilde{x}_j(t) - \tilde{\mathbf{x}}_j(t)| = \left| \frac{\tilde{x}_j(t)}{a(H)} + \left( \frac{1}{a(H)} - 1 \right) z_j t - \tilde{\mathbf{x}}_j(t) \right| \leq \frac{2\varepsilon\delta}{\min_{1 \leq i \leq n} \{H_i\}} \cdot (|\tilde{x}_j(t)| + z_j t), \quad j = m + 1, \ldots, n. \tag{28}
\]
Hence,

\[ |\tilde{x}(t) - \bar{x}(t)| \leq \frac{2\varepsilon\delta}{\min_{1 \leq i \leq n} \{H_i\}} \cdot (|\tilde{x}(t)| + |z|). \]  

(29)

From the definition of \( \tilde{u}(\cdot) \), we have

\[ |\tilde{\mu}^j(t) - \hat{\mu}^j(t)| = \left| 1 - \frac{1}{a(H)} \right| \cdot |\tilde{\mu}^j(t)| \leq \frac{2\varepsilon\delta}{\min_{1 \leq i \leq n} \{H_i\}} \cdot |\tilde{\mu}^j(t)|. \]  

(30)

Based on the difference between \( \hat{x}(t) \) and \( \tilde{x}(t) \) given in (29) and the difference between \( \hat{U}(t) \) and \( \tilde{U}(t) \) given in (30), we next estimate the difference between \( J(x, \tilde{U}(\cdot)) \) and \( J(\hat{x}, \hat{U}(\cdot)) \). First we have

\[
|J(x, \tilde{U}(\cdot)) - J(\hat{x}, \hat{U}(\cdot))| = \left| \int_0^\infty e^{-\rho t} \left[ h(\tilde{x}(t)) + \sum_{i=1}^p \gamma_i c(\tilde{\mu}^i(t)) \right] \, dt - \int_0^\infty e^{-\rho t} \left[ h(\hat{x}(t)) + \sum_{i=1}^p \gamma_i c(\hat{\mu}^i(t)) \right] \, dt \right| \\
\leq \int_0^\infty e^{-\rho t} |h(\tilde{x}(t)) - h(\hat{x}(t))| \, dt \\\n+ \int_0^\infty e^{-\rho t} \left| \sum_{i=1}^p \gamma_i c(\tilde{\mu}^i(t)) - \sum_{i=1}^p \gamma_i c(\hat{\mu}^i(t)) \right| \, dt. 
\]  

(31)

Assumption (A2) and (29) imply that

\[
\int_0^\infty e^{-\rho t} |h(\tilde{x}(t)) - h(\hat{x}(t))| \, dt \\
\leq C_0 \int_0^\infty e^{-\rho t} \left[ 1 + |\tilde{x}(t)|^{K_h} + |\hat{x}(t)|^{K_h} \right] \cdot |\tilde{x}(t) - \hat{x}(t)| \, dt \\
\leq C_0 \varepsilon \delta \int_0^\infty e^{-\rho t} \left[ 1 + |\tilde{x}(t)|^{K_h} + |\hat{x}(t)|^{K_h} \right] \cdot (|\tilde{x}(t)| + |zt|) \, dt \\
\leq C_1 \varepsilon \delta, 
\]  

(32)

for some \( C_1 > 0 \). In the same way, Assumption (A2) and (30) imply that

\[
\int_0^\infty \sum_{i=1}^p \gamma_i e^{-\rho t} \left| c(\tilde{\mu}^i(t)) - c(\hat{\mu}^i(t)) \right| \, dt \leq C_2 \varepsilon \delta, 
\]  

(33)

for some \( C_2 > 0 \). Combining (31)–(33), we get

\[
|J(x, \tilde{U}(\cdot)) - J(\hat{x}, \hat{U}(\cdot))| \leq C_3 \varepsilon \delta, 
\]  

(34)
for some $C_3 > 0$. Consequently, (20) gives

$$V(x) + (C_3 + 1)\varepsilon^\delta \geq \int_0^\infty e^{-\rho t} \left[ h(\hat{x}(t)) + \sum_{i=1}^p \gamma_i c(\hat{u}^i(t)) \right] dt. \quad (35)$$

Finally, we let

$$x(\delta) = \tilde{x} + (1, \ldots, 1)'\varepsilon^\delta,$$
$$u^j(\delta, \varepsilon, t) = \tilde{u}^j(t), \quad j = 1, \ldots, p,$$
$$U(\delta, \varepsilon, \cdot) = (u^1(\delta, \varepsilon, \cdot), \ldots, u^p(\delta, \varepsilon, \cdot)).$$

Furthermore, let

$$\frac{d}{dt} x(\delta, \varepsilon, t) = L(U(\delta, \varepsilon, t), z).$$

Then, (27) and (28) imply that for $j = 1, \ldots, m$,

$$x_j(\delta, \varepsilon, t) = \varepsilon^\delta + \hat{x}_j(t) \leq H_j - \varepsilon^\delta,$$

and for $j = m + 1, \ldots, n$

$$x_j(\delta, \varepsilon, t) = \varepsilon^\delta + \hat{x}_j(t) \leq H_j - \varepsilon^\delta.$$

Clearly, for $j = 1, \ldots, m$,

$$x_j(\delta, \varepsilon, t) = \varepsilon^\delta + \hat{x}_j(t) \geq \varepsilon^\delta.$$

Thus,

$$U(\delta, \varepsilon, \cdot) \in A(x(\delta)),$$

and (16) and (17) hold. Note that

$$|x(\delta, \varepsilon, t) - \tilde{x}(t)| \leq \varepsilon^\delta.$$

Similar to (34), there exists a positive constant $C_4$ such that

$$|J(x(\delta), U(\delta, \varepsilon, \cdot)) - J(x(\delta), \hat{U}(\cdot))| \leq C_4 \varepsilon^\delta. \quad (36)$$

Consequently, (22) follows from (35) and (36).

With Lemmas 4.1, 4.2 and 4.3, we can show our main result.

**Theorem 4.1** Let Assumptions (A.1) and (A.2) hold. Then, for any $\delta \in [0, \frac{1}{2})$, there exists a constant $C > 0$ such that for all sufficiently small $\varepsilon > 0$,

$$|V^\varepsilon(x, k) - V(x)| \leq C(1 + |x|^{K_h}) \varepsilon^\delta. \quad (37)$$

This implies in particular that $\lim_{\varepsilon \to 0} V^\varepsilon(x, k) = V(x)$. 

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Remark 4.1 Theorem 4.1 says that the problem $\mathcal{P}^0$ is indeed a limiting problem in the sense that the $V^\varepsilon(x, k)$ of $\mathcal{P}^\varepsilon$ converges to $V(x)$ of $\mathcal{P}^0$. Moreover, it gives the corresponding convergence rate.

Proof. We outline the major steps in the proof. First we prove $V^\varepsilon(x, k) < V(x) + C(1 + |x|^K)\varepsilon^\delta$ by constructing an admissible control $U(\varepsilon, t) \in A^\varepsilon(x, k)$ of $\mathcal{P}^\varepsilon$ from the near-optimal control of the limiting problem $\mathcal{P}^0$, and by estimating the difference between the state trajectories corresponding to these two controls. Then we establish the opposite inequality, namely, $V^\varepsilon(x, k) > V(x) - C(1 + |x|^K)\varepsilon^\delta$, by constructing a control of the limiting problem $\mathcal{P}^0$ from a near-optimal control of $\mathcal{P}^\varepsilon$ and using Assumption (A.2).

In order to show that $V^\varepsilon(x, k) - V(x) \leq C(1 + |x|^K)\varepsilon^\delta$, (38) we can choose, in view of Lemma 4.3, $x(\delta) \in I$ and

$$U(\delta, \varepsilon, t) = (u^1(\delta, \varepsilon, t), \ldots, u^p(\delta, \varepsilon, t)) \in A(x(\delta)),$$

such that

$$|x(\delta) - x| \leq \varepsilon^\delta,$$  \hspace{1cm} (39)

$$\inf_{0 \leq s < \infty} x_j(\delta, \varepsilon, s) \geq \varepsilon^\delta, \quad j = 1, \ldots, m,$$  \hspace{1cm} (40)

$$\sup_{0 \leq s < \infty} x_j(\delta, \varepsilon, s) \leq H_j - \varepsilon^\delta, \quad j = 1, \ldots, n,$$  \hspace{1cm} (41)

and

$$v(x) + C_1\varepsilon^\delta \geq J(x(\delta), U(\delta, \varepsilon, \cdot))$$

$$= \int_0^\infty e^{-\rho t} \left[ h(x(\delta, \varepsilon, t)) + \sum_{j=1}^p \gamma_j c(u^j(\delta, \varepsilon, t)) \right] dt,$$  \hspace{1cm} (42)

where $x(\delta, \varepsilon, t)$ is the state trajectory under the control $U(\delta, \varepsilon, t)$ with $x(\delta, \varepsilon, 0) = x(\delta)$. Let

$$\tilde{u}^j(\varepsilon, t) = I_{\{k(\varepsilon, t) = k^j\}} u^j(\delta, \varepsilon, t), \quad \tilde{u}(\varepsilon, t) = \sum_{j=1}^p I_{\{k(\varepsilon, t) = k^j\}} u^j(\delta, \varepsilon, t),$$

and

$$\frac{d}{dt} \tilde{x}(\varepsilon, t) = \mathcal{L}(\tilde{u}(\varepsilon, t), z), \quad \tilde{x}(\varepsilon, 0) = x(\delta).$$

Generally, the control $\tilde{u}(\varepsilon, t)$ may not be admissible. We need to make it admissible and still satisfy (38). This modification will be done in two steps. First we modify $\tilde{u}(\varepsilon, t)$ such that the works-in-process of its state trajectory are nonnegative. That this can be done is asserted in the following lemma. Its proof is given in Appendix.
Lemma 4.4 There is $\bar{u}(\varepsilon, t)$ such that for any $(i, j) \in A$ and $i \neq m + 1, \ldots, n$,

$$u_{i,j}(\varepsilon, t) \leq \bar{u}_{i,j}(\varepsilon, t),$$

$$\pi_{\ell}(\varepsilon, t) \geq 0, \quad \ell = 1, \ldots, m,$$

$$E \int_0^\infty e^{-\rho t} |u_{i,j}(\varepsilon, t) - \bar{u}_{i,j}(\varepsilon, t)| \, dt \leq C_2 \exp \left\{ -\kappa_1 \varepsilon \frac{1-2\delta}{4} \right\},$$

$$\forall (i, j) \in A, \ i \neq m + 1, \ldots, n,$$  \hspace{1cm} (43)

and

$$E \int_0^\infty e^{-\rho t} \Pr \left( \pi_{\ell}(\varepsilon, t) \geq H_{\ell} - \left( \frac{1}{2} - \sum_{j=1}^{\ell} \frac{1}{2j+1} \right) \varepsilon \delta \right) \, dt \leq C_2 \exp \left\{ -\kappa_1 \varepsilon \frac{1-2\delta}{4} \right\}, \quad \ell = 1, \ldots, n,$$  \hspace{1cm} (44)

for some $C_2 > 0$ and $\kappa_1 > 0$, where

$$\frac{d}{dt} \pi(\varepsilon, t) = L(\bar{u}(\varepsilon, t), z), \quad \pi(\varepsilon, 0) = \pi(\delta).$$

Because $\bar{u}(\varepsilon, t)$ is not admissible, it may violate the upper bound. So we need to modify it further. This modification provides a feasible control as stated in the following lemma. We relegate its proof to the Appendix.

Lemma 4.5 There exists a control $u(\varepsilon, t) \in \mathcal{A}(x(\delta), k)$ such that for $(i, j) \in A$ and $i \neq m + 1, \ldots, n$,

$$u_{i,j}(\varepsilon, t) \leq \bar{u}_{i,j}(\varepsilon, t)$$

and

$$E \int_0^\infty e^{-\rho t/2} |u_{i,j}(\varepsilon, t) - \bar{u}_{i,j}(\varepsilon, t)| \, dt \leq C_3 \exp \left\{ -\kappa_2 \varepsilon \frac{1-2\delta}{4} \right\},$$

$$\forall (i, j) \in A, \ i \neq m + 1, \ldots, n,$$  \hspace{1cm} (46)

for some $C_3 > 0$ and $\kappa_2 > 0$.

Based on Lemmas 4.4 and 4.5, we can estimate

\begin{align*}
J^c(\pi(\delta), k, u(\varepsilon, \cdot)) - J(\pi(\delta), U(\delta, \varepsilon, \cdot)) &= E \int_0^\infty e^{-\rho t} [h(\pi(\varepsilon, t)) - h(\bar{\pi}(\varepsilon, t))] \, dt + E \int_0^\infty e^{-\rho t} [c(u(\varepsilon, t)) - c(\bar{u}(\varepsilon, t))] \, dt \\
&\quad + E \int_0^\infty e^{-\rho t} [h(\bar{\pi}(\varepsilon, t)) - h(\bar{x}(\varepsilon, t))] \, dt + E \int_0^\infty e^{-\rho t} [c(\bar{u}(\varepsilon, t)) - c(\bar{u}(\varepsilon, t))] \, dt \\
&\quad + E \int_0^\infty e^{-\rho t} \left[ c(\bar{u}(\varepsilon, t)) - \sum_{\ell=1}^p \gamma_{\ell} c(u^\ell(\delta, \varepsilon, t)) \right] \, dt.
\end{align*}

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By the Lipschitz property of \(h(\cdot)\), there is \(C_4 > 0\) and \(\varepsilon_0\) such that for \(\varepsilon \in (0, \varepsilon_0)\),

\[
E \int_0^\infty e^{-pt} [h(x(\varepsilon, t)) - h(x(\varepsilon_0, t))] \, dt \leq C_0 \left(1 + |x(\delta)|^{K_h}\right) E \int_0^\infty e^{-pt} (1 + t^{K_h}) |x(\varepsilon, t) - x(\varepsilon_0, t)| \, dt \\
\leq C_4 (1 + |x(\delta)|^{K_h}) E \int_0^\infty e^{-pt/2} \left(\sum_{(i,j) \in A} |\mathbf{u}_{i,j}(\varepsilon, s) - \mathbf{u}_{i,j}(\varepsilon_0, s)|\right) \, ds \, dt \\
= 2C_4 (1 + |x(\delta)|^{K_h}) \rho^{-1} E \int_0^\infty e^{-pt/2} \left(\sum_{(i,j) \in A} |\mathbf{u}_{i,j}(\varepsilon, s) - \mathbf{u}_{i,j}(\varepsilon_0, s)|\right) \, dt \\
\leq 2C_4 (1 + |x(\delta)|^{K_h}) \rho^{-1} \varepsilon^\delta \quad \text{(by (46)).} \tag{48}
\]

In the same way, we get

\[
E \int_0^\infty e^{-pt} [h(x(\varepsilon, t)) - h(\hat{x}(\varepsilon, t))] \, dt \leq C_5 \varepsilon^\delta, \tag{49}
\]

\[
E \int_0^\infty e^{-pt} [c(u(\varepsilon, t)) - c(x(\varepsilon_0, t))] \, dt \leq C_5 \varepsilon^\delta, \tag{50}
\]

and

\[
E \int_0^\infty e^{-pt} [c(x(\varepsilon, t)) - c(\hat{x}(\varepsilon, t))] \, dt \leq C_5 \varepsilon^\delta, \tag{51}
\]

for some \(C_5 > 0\). Similar to (48), there is a positive constant \(C_6 > 0\) such that

\[
E \int_0^\infty e^{-pt} [h(\hat{x}(\varepsilon, t)) - h(x(\delta, \varepsilon, t))] \, dt \\
\leq C_6 (1 + |x(\delta)|^{K_h}) \int_0^\infty e^{-pt/2} E[\hat{x}(\varepsilon, t) - x(\delta, \varepsilon, t)] \, dt. \tag{52}
\]

The definition of \(\hat{x}(\varepsilon, t)\) and Lemma 4.2 imply

\[
E[\hat{x}(\varepsilon, t) - x(\delta, \varepsilon, t)] \leq \sum_{\ell=1}^p E \left| \int_0^t \left[I_{\{k(\varepsilon, s) = k^\ell\}} - \gamma_\ell \right] u^\ell(\delta, \varepsilon, s) \, ds \right| \\
\leq C_7 (1 + t^2)^{1/2} \varepsilon^{1/2}, \tag{53}
\]

for some \(C_7 > 0\). It follows from (52)–(53) that

\[
E \int_0^\infty e^{-pt} [h(\hat{x}(\varepsilon, t)) - h(x(\delta, \varepsilon, t))] \, dt \leq C_8 \varepsilon^{1/2}, \tag{54}
\]

for some \(C_8 > 0\). Furthermore, from Lemma 4.1 and the boundedness of \(c(u^\ell(\delta, \varepsilon, t))\),

\[
E \int_0^\infty e^{-pt} \left[c(\hat{u}(\varepsilon, t)) - \sum_{\ell=1}^p \gamma_\ell c(u^\ell(\delta, \varepsilon, t))\right] \, dt \\
\leq E \int_0^\infty e^{-pt} \sum_{\ell=1}^p \left[I_{\{k(\varepsilon, t) = k^\ell\}} - \gamma_\ell \right] c(u^\ell(\delta, \varepsilon, t)) \, dt \\
\leq C_9 \varepsilon, \tag{55}
\]
for some $C_9 > 0$. Therefore, it follow from (47)-(55) that there is a positive constant $C_{10} > 0$ such that

$$J^\varepsilon(x(\delta), k, u(\varepsilon, \cdot)) - J(x(\delta), U(\delta, \varepsilon, \cdot)) \leq C_{10}\varepsilon^\delta.$$  \hspace{1cm} (56)

On the other hand, from the Lipschitz continuity of $V^\varepsilon(x, k)$, (39) implies

$$|V^\varepsilon(x, k) - V^\varepsilon(x(\delta), k)| \leq C_{11}(1 + |x|^K)\varepsilon^\delta,$$  \hspace{1cm} (57)

for some $C_{11} > 0$. In view of $J^\varepsilon(x(\delta), k, u(\varepsilon, \cdot)) \geq V^\varepsilon(x(\delta), k)$, (38) follows from (42) and (56)-(57).

We now show the opposite inequality, that is,

$$V^\varepsilon(x, k) - V(x) \geq C(1 + |x|^K)\varepsilon^\delta.$$  \hspace{1cm} (58)

Similar to Lemma 4.3, we can prove that there exist $x(\delta) \in I$ and a control $u(\varepsilon, \cdot) \in A^\varepsilon(x(\delta), k)$ such that

$$|x(\delta) - x| \leq \varepsilon^\delta,$$  \hspace{1cm} (59)

$$\min_{1 \leq k \leq m} \inf_{0 \leq s < \infty} E[x_k(\varepsilon, t)] \geq \varepsilon^\delta,$$  \hspace{1cm} (60)

$$\sup_{0 \leq s < \infty} E[x_k(\varepsilon, s)] \leq H_k - \varepsilon^\delta, \quad k = 1, \ldots, n,$$  \hspace{1cm} (61)

and

$$E \int_0^\infty e^{-t}[h(x(\varepsilon, t)) + c(u(\varepsilon, t))] dt \leq V^\varepsilon(x, k) + \tilde{C}_1\varepsilon^\delta,$$  \hspace{1cm} (62)

for some $\tilde{C}_1 > 0$, where $x(\varepsilon, t)$ is the state trajectory under the control $u(\varepsilon, t)$ with the initial condition $x(\varepsilon, 0) = x(\delta)$. Now we choose $\tilde{U}(\varepsilon, \cdot) = (\tilde{u}^1(\varepsilon, t), \ldots, \tilde{u}^p(\varepsilon, t))$ defined by

$$\tilde{u}^j(\varepsilon, t) = E\left[ u(\varepsilon, t) \middle| k(\varepsilon, t) = k_j \right], \quad j = 1, \ldots, p.$$  

Then we have

$$E[x_k(\varepsilon, t)] = x_k(\delta) + \int_0^t \left[ \sum_{j=1}^p \Pr(k(\varepsilon, t) = k_j) \left( \sum_{i=-n+1}^{k-1} \tilde{u}^{ij}_i(\varepsilon, s) - \sum_{i=1}^m \tilde{u}^{ij}_{i,k}(\varepsilon, s) \right) ds, \right.$$

$$k = 1, \ldots, m,$$

$$E[x_k(\varepsilon, t)] = x_k(\delta) + \int_0^t \left[ \sum_{j=1}^p \Pr(k(\varepsilon, t) = k_j) \left( \sum_{i=-n+1}^m \tilde{u}^{ij}_{i,k}(\varepsilon, s) - z_k \right) ds, \right.$$  

$$k = m + 1, \ldots, n.$$

Define

$$\tilde{x}_k(\varepsilon, t) = x_k(\delta) + \int_0^t \left[ \sum_{j=1}^p \gamma_j \sum_{i=-n+1}^{k-1} \tilde{u}^{ij}_{i,k}(\varepsilon, s) - \sum_{j=1}^p \gamma_j \sum_{i=k+1}^m \tilde{u}^{ij}_{i,k}(\varepsilon, s) \right] ds,$$

$$k = 1, \ldots, m,$$

$$x_k(\varepsilon, t) = x_k(\delta) + \int_0^t \left[ \sum_{j=1}^p \gamma_j \sum_{i=-n+1}^m \tilde{u}^{ij}_{i,k}(\varepsilon, s) - z_k \right] ds, \quad k = m + 1, \ldots, n.
Using Lemma 4.2, we have

\[
\left| E[u(\varepsilon, t)] - \sum_{\ell=1}^{p} \gamma_{\ell} \tilde{u}_{\ell}(\varepsilon, t) \right| = \left| \sum_{\ell=1}^{p} \left[ \Pr(k(\varepsilon, t) = k^\ell) - \gamma_{\ell} \right] \tilde{u}_{\ell}(\varepsilon, t) \right|
\leq \hat{C}_2(\varepsilon + e^{-K_1\varepsilon^{-1}t}),
\]  

(63)

for some \( \hat{C}_2 > 0 \). Then for \( k = 1, \ldots, m, \)

\[
\left| \tilde{x}_k(\varepsilon, t) - E[x_k(\varepsilon, t)] \right| = \left| \int_{0}^{t} \left\{ \sum_{\ell=1}^{p} \gamma_{\ell} \left( \sum_{i=-n_0+1}^{k-1} \tilde{u}_{i,k}(\varepsilon, s) - \sum_{i=k+1}^{n} \tilde{u}_{i,k}(\varepsilon, s) \right) - E \left[ \sum_{i=-n_0+1}^{k-1} u_{i,k}(\varepsilon, s) - \sum_{i=k+1}^{n} u_{i,k}(\varepsilon, s) \right] \right\} ds \right|
\leq \hat{C}_3\varepsilon(1 + t),
\]  

(64)

and for \( k = m + 1, \ldots, n, \)

\[
\left| \tilde{x}_k(\varepsilon, t) - E[x_k(\varepsilon, t)] \right| \leq \left| \int_{0}^{t} \left[ \sum_{\ell=1}^{p} \gamma_{\ell} \left( \sum_{i=-n_0+1}^{m} \tilde{u}_{i,k}(\varepsilon, s) - \sum_{i=-n_0+1}^{m} u_{i,k}(\varepsilon, s) \right) \right] ds \right|
\leq \hat{C}_3(1 + t)\varepsilon,
\]  

(65)

for some \( \hat{C}_3 > 0 \). According to Lemma 5.4 of Sethi, Zhang, and Zhou (1992), there exists \( \tau_\varepsilon > 0 \) such that

\[
\hat{C}_3\varepsilon(1 + \tau_\varepsilon) \leq \varepsilon^\delta
\]  

(66)

and

\[
\int_{\tau_\varepsilon}^{\infty} e^{-\rho t} (1 + t^{K_2+1}) dt \leq \hat{C}_4\varepsilon^\delta,
\]  

(67)

for some \( \hat{C}_4 > 0 \). Therefore, if one defines

\[
\tilde{u}_{i,k}(\varepsilon, t) = \begin{cases} 
\tilde{u}_{i,k}(\varepsilon, t), & 0 \leq t \leq \tau_\varepsilon, \\
0, & t > \tau_\varepsilon,
\end{cases}
\]
for $\ell = 1, \ldots, p$, and $(i, k) \in A$ with $i \neq m + 1, \ldots, n$, and lets $\hat{x}(\varepsilon, t)$ be the state trajectory under the control $U(\varepsilon, t) = (\hat{u}^1(\varepsilon, t), \ldots, \hat{u}^p(\varepsilon, t))$ with $\hat{x}(0) = x(\delta)$, then (64)--(66) imply

$$U(\varepsilon, t) = (\hat{u}^1(\varepsilon, t), \ldots, \hat{u}^p(\varepsilon, t)) \in A(\delta).$$

It follows from (67) and Assumption (A.2) that

$$E = 1 + \left| \sum_{j=1}^{p} \gamma_j c(\hat{u}^j(\varepsilon, t)) \right| dt$$

for some $\tilde{C}_5 > 0$. In view of the convexity and the local Lipschitz continuity of $h(\cdot)$, Jensen’s inequality and (64)-(65) yield

$$E[h(x(\varepsilon, t)]) \geq h(E[x(\varepsilon, t)])$$

$$= h(\tilde{x}(\varepsilon, t)) + |h(E[x(\varepsilon, t)]) - h(\tilde{x}(\varepsilon, t))|$$

$$\geq h(\tilde{x}(\varepsilon, t)) - C_0 \left( 1 + |E[x(\varepsilon, t)]|^{K_h} + |\tilde{x}(\varepsilon, t)|^{K_h} \right) |E[x(\varepsilon, t)] - \tilde{x}(\varepsilon, t)|$$

$$\geq h(\tilde{x}(\varepsilon, t)) - \tilde{C}_6 \varepsilon (1 + |\tilde{x}|^{K_h})(1 + t^{K_h+1})(1 + t), \quad \text{(69)}$$

for some $\tilde{C}_6 > 0$. In the same way, using Lemma 4.2, we can establish

$$E[c(u(\varepsilon, t))] = \sum_{j=1}^{p} \Pr(k(\varepsilon, t) = k^j) E[c(U(\varepsilon, t))|k^c(t) = k^j]$$

$$\geq \sum_{j=1}^{p} \Pr(k(\varepsilon, t) = k^j) c(\hat{u}^j(\varepsilon, t))$$

$$\geq \sum_{j=1}^{p} \gamma_j c(\hat{u}^j(\varepsilon, t)) - \tilde{C}_7 \varepsilon^{e^{-\kappa_1 t/\varepsilon}}, \quad \text{(70)}$$

for some positive $\tilde{C}_7$. By combining (69) and (70), we obtain

$$E \int_0^\infty e^{-\rho t} [h(x(\varepsilon, t)) + c(U(\varepsilon, t))] dt$$

$$\geq \int_0^\infty e^{-\rho t} \left[ h(\tilde{x}(\varepsilon, t)) + \sum_{j=1}^{p} \gamma_j c(\hat{u}^j(\varepsilon, t)) \right] dt - \tilde{C}_8 \varepsilon,$$
for some positive constant $\hat{C}_8$. Thus, in view of $V(x(\delta)) \leq J(x(\delta), \hat{U}(\varepsilon, \cdot))$, (68) gives that there is a positive constant $\hat{C}_9$ such that
\[
\mathbb{E} \int_0^\infty e^{-\rho t} [h(x(\varepsilon, t)) + c(u(\varepsilon, t))] \, dt \geq V(x(\delta)) - \hat{C}_9 (1 + |x(\delta)|^{K_h}) \varepsilon^\delta.
\] (71)
On the other hand, the Lipschitz continuity of $V(x)$ and (59) imply
\[
|V(x) - V(x(\delta))| \leq \hat{C}_{10} (1 + |x|^{K_h}) \varepsilon^\delta.
\]
Consequently, the inequality (71) implies (58).

## 5 Asymptotic Optimal Control

In this section, based on the proof of Theorem 4.1, we supply a procedure to construct an asymptotic optimal control.

**Construction of an Asymptotic Optimal Control**

**Step I:** Pick an $\varepsilon$-optimal control $U(\varepsilon, \cdot) = (u^1(\varepsilon, \cdot), \ldots, u^p(\varepsilon, \cdot)) \in A^0(x)$ for $P^0$, i.e.,
\[
\int_0^\infty e^{-\rho t} \left[ h(x(\varepsilon, t)) + \sum_{j=1}^p \gamma_j c(u^j(\varepsilon, t)) \right] \, dt < V(x) + \varepsilon,
\]
where $x(\varepsilon, t)$ is the state trajectory under the control $U(\varepsilon, t)$ with $x(\varepsilon, 0) = x$. Let
\[
a(H) = \max_{1 \leq j \leq n} \left\{ \frac{H_j}{H_j - 2\varepsilon^\delta} \right\}.
\]
Define
\[
\hat{u}_{i,j}(\varepsilon, t) = \frac{u_{i,j}(\varepsilon, t)}{a(H)}, \quad (i, j) \in A, \quad i \neq m + 1, \ldots, n.
\]

**Step II:** Set
\[
\hat{u}(\varepsilon, t) = \sum_{\ell=1}^p I_{\{k(\varepsilon, t) = k^\ell\}} \hat{u}^\ell(\varepsilon, t)
\]
and
\[
\frac{d}{dt} \hat{x}(\varepsilon, t) = \mathcal{L}(\hat{u}(\varepsilon, t), x), \quad \hat{x}(\varepsilon, 0) = x + (1, \ldots, 1)' \varepsilon^\delta.
\]
Set
\[
\pi_{i,j}(\varepsilon, t) = \hat{u}_{i,j}(\varepsilon, t), \quad i = -n_0 + 1, \ldots, 0,
\]
and
\[
\pi_{i,j}(\varepsilon, t), j = i + 1, \ldots, n, \quad i = 1, \ldots, \ell - 2.
\]
Sub-step $\ell (\ell = 2, \ldots, m)$: Set
\[
B^\varepsilon_{\ell - 1} = \left\{ t : \hat{x}^{\ell - 2}_{\ell - 1}(t) - 0 \wedge \left( \inf_{0 \leq s \leq t} \hat{x}^{\ell - 2}_{\ell - 1}(s) \right) = 0 \text{ and } \hat{x}^{\ell - 2}_{\ell - 1}(t) < 0 \right\},
\]
where
\[
\hat{x}^{\ell - 2}_{\ell - 1}(t) = x_{\ell - 1} + \varepsilon^\delta + \int_0^t \left[ \sum_{i=-n_0+1}^{\ell - 2} \varpi_{i,\ell - 1}(\varepsilon, s) - \sum_{i=\ell}^{n} \tilde{u}_{\ell - 1,i}(\varepsilon, s) \right] ds.
\]
For $t \in B^\varepsilon_{\ell - 1}$ and $i = \ell, \ldots, n$,
\[
\bar{u}_{\ell - 1,i}(\varepsilon, t) \leq \hat{u}_{\ell - 1,i}(\varepsilon, t)
\]
and
\[
\sum_{i=\ell}^{n} \bar{u}_{\ell - 1,i}(\varepsilon, t) = \sum_{i=-n_0+1}^{\ell - 2} \varpi_{i,\ell - 1}(\varepsilon, t).
\]
Let
\[
\varpi_{\ell - 1,i}(\varepsilon, t) = \begin{cases} 
\bar{u}_{\ell - 1,i}(\varepsilon, t), & \text{if } t \in B^\varepsilon_{\ell - 1}, \\
\hat{u}_{\ell - 1,i}(\varepsilon, t), & \text{if } t \notin B^\varepsilon_{\ell - 1}.
\end{cases}
\]
Then we get
\[
\varpi_{i,j}(\varepsilon, t), \quad (i, j) \in A.
\]

Step III: Set
\[
\frac{d}{dt} \varphi(\varepsilon, t) = \mathcal{L}(\varpi(\varepsilon, t), z), \quad \varphi(\varepsilon, 0) = x + (1, \ldots, 1)' \varepsilon^\delta.
\]
For $j = m + 1, \ldots, n$, define
\[
\hat{B}^\varepsilon_j = \{ t : (H_j - \varphi_j(\varepsilon, t)) - 0 \wedge \left( \inf_{0 \leq s \leq t} \{ H_j - \varphi_j(\varepsilon, s) \} \right) = 0 \}.
\]
For $j = m + 1, \ldots, n$, choose
\[
\bar{u}_{i,j}(\varepsilon, t) \leq \varpi_{i,j}(\varepsilon, t),
\]
\[
\sum_{i=-n_0+1}^{j-1} \bar{u}_{i,j}(\varepsilon, t) = z_j.
\]
Define
\[
u_{i,j}(\varepsilon, t) = \begin{cases} 
\varpi_{i,j}(\varepsilon, t), & t \notin \hat{B}^\varepsilon_j, \\
\bar{u}_{i,j}(\varepsilon, t), & t \in \hat{B}^\varepsilon_j.
\end{cases}
\]
Sub-step $j(j = m, \ldots, 1)$: Set

$$
\hat{B}_j^\varepsilon = \{ t : (H_j - \tilde{x}_j^\varepsilon(t)) - 0 \leq \inf_{0 \leq s \leq t} \{ H_j - \tilde{x}_j^\varepsilon(s) \} = 0 \},
$$

where

$$
\tilde{x}_j^\varepsilon(t) = x_j + \varepsilon \delta + \int_0^t \left[ \sum_{i = -n_0 + 1}^{j-1} \pi_{i,j}(\varepsilon, s) - \sum_{i = j+1}^{n} u_{j,i}(\varepsilon, s) \right] ds.
$$

For $t \in \hat{B}_j^\varepsilon$ and $i = -n_0 + 1, \ldots, j-1$, choose

$$
\tilde{u}_{i,j}(\varepsilon, t) \leq \pi_{i,j}(\varepsilon, t)
$$

$$
\sum_{i = -n_0 + 1}^{j-1} \tilde{u}_{i,j}(\varepsilon, t) = \sum_{i = -n_0 + 1}^{j-1} \pi_{i,j}(\varepsilon, t).
$$

For $i = -n_0 + 1, \ldots, j-1$, define

$$
u_{i,j}(\varepsilon, t) = \begin{cases} \tilde{u}_{i,j}(\varepsilon, t), & \text{if } t \in \hat{B}_j^\varepsilon, \\ \pi_{i,j}(\varepsilon, t), & \text{if } t \notin \hat{B}_j^\varepsilon. \end{cases}
$$

Then we get

$$
u_{i,j}(\varepsilon, t), \quad (i,j) \in A.
$$

6 Concluding Remarks

In this paper, we have considered a hierarchical production control of a jobshop with a discounted cost criterion. We have constructed near optimal control policies based on the corresponding limiting problem simpler than the original problem. The main advantage of our approach is the reduction of the system complexity. It would be of interest to consider hierarchical production controls for long-run average cost objective. This is a topic of future research.

7 Appendix

In this appendix, we provide proofs of Lemmas 4.4 and 4.5.

**Proof of Lemma 4.4.** First we estimate the probability that $x_k^\varepsilon(t) \notin (\varepsilon^\delta / 2, H_k - \varepsilon^\delta / 2)$ ($k = 1, \ldots, m$). By (40) and (41),

$$
\Pr \left( \tilde{x}_k(\varepsilon, t) \leq \varepsilon^\delta / 2 \text{ or } \tilde{x}_k(\varepsilon, t) \geq H_k - \varepsilon^\delta / 2 \right)
\leq \Pr \left( |\tilde{x}_k(\varepsilon, t) - x_k(\varepsilon, t)| \geq \varepsilon^\delta / 2 \right)
\leq \Pr \left( \sum_{t=1}^{p} \left( I_{\{k(s) = k\}} - \gamma_t \right) \sum_{i=-n_0+1}^{k-1} u_{i,k}(s) - \sum_{i=k+1}^{n} u_{k,i}(s) \right) ds \geq \varepsilon^\delta / 2.
$$
Lemma 4.2 implies that there is a positive constant $C_1$ such that for $k = 1, \ldots, m$,
\[
\Pr \left( \hat{x}_k(\varepsilon, t) \leq \varepsilon^\delta/2 \text{ or } \hat{x}_k(\varepsilon, t) \geq H_k - \varepsilon^\delta/2 \right) \\
\leq C_1 \left( e^{-\kappa_1 \varepsilon^{1+3t}} + e^{-\kappa_2 \varepsilon^{1-2\delta}}(1+t)^3 \right).
\]
(72)
Thus, for $k = 1, \ldots, m$,
\[
\int_0^{\infty} e^{-\rho t} \Pr \left( \hat{x}_k(\varepsilon, t) \leq \varepsilon^\delta/2 \text{ or } \hat{x}_k(\varepsilon, t) \geq H_k - \varepsilon^\delta/2 \right) dt \\
\leq C_2 \exp \left\{ -\kappa_3 \varepsilon^{1-2\delta} \right\},
\]
(73)
for some $C_2 > 0$ and $\kappa_3 > 0$. In the same way, for $k = m + 1, \ldots, n$,
\[
\int_0^{\infty} e^{-\rho t} \Pr \left( \hat{x}_k(\varepsilon, t) \geq H_k - \varepsilon^\delta/2 \right) dt \leq C_2 \exp \left\{ -\kappa_3 \varepsilon^{1-2\delta} \right\}.
\]
(74)
Let
\[
B^*_1 = \left\{ t : \hat{x}_1(\varepsilon, t) - 0 \land \left( \inf_{0 \leq s \leq t} \hat{x}_1(\varepsilon, s) \right) = 0 \text{ and } \hat{x}_1(\varepsilon, t) < 0 \right\}.
\]
Then, for $t \in B^*_1$,
\[
\sum_{i = -n_0 + 1}^{0} \hat{u}_{i,1}(\varepsilon, t) \leq \sum_{i = 2}^{n} \hat{u}_{1,i}(\varepsilon, t).
\]
(75)
For $t \in B^*_1$ and $i = 2, \ldots, n$, choose
\[
\hat{u}_{1,i}(\varepsilon, t) \leq \hat{u}_{i,1}(\varepsilon, t)
\]
and
\[
\sum_{i = 2}^{n} \hat{u}_{1,i}(\varepsilon, t) = \sum_{i = 2}^{n} \hat{u}_{i,1}(\varepsilon, t).
\]
Define for $i = -n_0 + 1, \ldots, 0$,
\[
\pi_{i,1}(\varepsilon, t) = \hat{u}_{i,1}(\varepsilon, t)
\]
and for $j = 2, \ldots, n$,
\[
\pi_{1,j}(\varepsilon, t) = \begin{cases} 
\hat{u}_{1,j}(\varepsilon, t), & t \notin B^*_1, \\
\hat{u}_{i,j}(\varepsilon, t), & t \in B^*_1.
\end{cases}
\]
Thus,
\[
\hat{x}_1(\varepsilon, t) = x_1(\delta) + \int_0^t \left[ \sum_{j = -n_0 + 1}^{0} \pi_{j,1}(\varepsilon, s) - \sum_{j = 2}^{n} \pi_{1,j}(\varepsilon, s) \right] ds \geq 0.
\]
Furthermore,
\[
E \int_0^\infty e^{-\rho t} |\pi_{1,j}(\varepsilon, t) - \hat{u}_{1,j}(\varepsilon, t)| \, dt \\
= E \int_0^\infty e^{-\rho t} |\pi_{1,j}(\varepsilon, t) - \hat{u}_{1,j}(\varepsilon, t)| I_{\{t \in B_1\}} \, dt \\
\leq E \int_0^\infty e^{-\rho t} |\pi_{1,j}(\varepsilon, t) - \hat{u}_{1,j}(\varepsilon, t)| I_{\{\varepsilon(t) < 0\}} \, dt \\
\leq \max_{1 \leq i \leq p} \{k_{i,j}\} E \int_0^\infty e^{-\rho t} I_{\{\varepsilon(t) < 0\}} \, dt \\
= \max_{1 \leq i \leq p} \{k_{i,j}\} \int_0^\infty e^{-\rho t} \Pr (\hat{\varepsilon}_1(t) \leq 0) \, dt \\
\leq \max_{1 \leq i \leq p} \{k_{i,j}\} \int_0^\infty e^{-\rho t} \left( \frac{\hat{\varepsilon}_1(t) - \varepsilon}{2} \right) \, dt \\
= C_3 \exp \left\{ -\bar{\kappa}_{31} \varepsilon^{\frac{1-2d}{4}} \right\} \quad \text{(by (73))},
\]
for some \(C_3 > 0\). Using this, we get
\[
\int_0^\infty e^{-\rho t} \Pr \left( \pi_1(\varepsilon, t) \geq H_1 - \left( \frac{1}{2} - \frac{1}{22} \right) \varepsilon \right) \, dt \\
\leq \int_0^\infty e^{-\rho t} \Pr \left( \hat{\varepsilon}_1(\varepsilon, t) \geq H_1 - \frac{1}{2} \varepsilon \right) \, dt \\
+ \int_0^\infty e^{-\rho t} \Pr \left( \pi_1(\varepsilon, t) \geq H_1 - \left( \frac{1}{2} - \frac{1}{22} \right) \varepsilon \right) \text{ and } \hat{\varepsilon}_1(\varepsilon, t) \leq H_1 - \frac{1}{2} \varepsilon \right) \, dt \\
\leq \int_0^\infty e^{-\rho t} \Pr \left( \hat{\varepsilon}_1(\varepsilon, t) \geq H_1 - \frac{1}{2} \varepsilon \right) \, dt \\
+ \int_0^\infty e^{-\rho t} \Pr \left( |\pi_1(\varepsilon, t) - \hat{\varepsilon}_1(\varepsilon, t)| \geq \frac{1}{22} \varepsilon \right) \, dt \\
\leq \int_0^\infty e^{-\rho t} \Pr \left( \hat{\varepsilon}_1(\varepsilon, t) \geq H_1 - \frac{1}{2} \varepsilon \right) \, dt \\
+ \frac{\varepsilon^2}{\varepsilon^2} \int_0^\infty e^{-\rho t} E \int_0^t \left| \sum_{j=2}^n \pi_{1,j}(\varepsilon, s) - \sum_{j=2}^n \hat{u}_{1,j}(\varepsilon, s) \right| \, ds \, dt.
\]
Using the method given in Sethi, Zhang, and Zhou (1992), we can show, in view of (76), that there are positive constants \(C_4\) and \(\bar{\kappa}_{31} > 0\) such that
\[
\int_0^\infty e^{-\rho t} \Pr \left( \pi_1(\varepsilon, t) \geq H_1 - \left( \frac{1}{2} - \frac{1}{22} \right) \varepsilon \right) \, dt \leq C_4 \exp \left\{ -\bar{\kappa}_{31} \varepsilon^{\frac{1-2d}{4}} \right\}. \quad (77)
\]
Now we consider the system
\[
\begin{align*}
\mathbf{x}_j^1(\varepsilon, t) &= x_j(\delta) + \int_0^t \left( \frac{\partial}{\partial s} \mathbf{u}_{1,j}(\varepsilon, s) + \sum_{i \neq 1, \varepsilon = -n_0+1}^{j-1} \mathbf{u}_{i,j}(\varepsilon, s) - \sum_{i=j+1}^n \mathbf{u}_{i,j}(\varepsilon, s) \right) ds, \\
& \text{for } j = 2, \ldots, m, \\
\mathbf{x}_j^1(\varepsilon, t) &= x_j(\delta) + \int_0^t \left( \frac{\partial}{\partial s} \mathbf{u}_{1,j}(\varepsilon, s) + \sum_{i \neq 1, \varepsilon = -n_0+1}^m \mathbf{u}_{i,j}(\varepsilon, s) - z_j \right) ds, \\
& \text{for } j = m+1, \ldots, n.
\end{align*}
\] (78)

Note that
\[
\Pr \left( \mathbf{x}_1^1(\varepsilon, t) \leq (\frac{1}{2} - \frac{1}{2^2})\varepsilon^\delta \text{ or } \mathbf{x}_2^1(\varepsilon, t) \geq H_2 \left( \frac{1}{2} - \frac{1}{2^2} \right)\varepsilon^\delta \right) \\
\leq \Pr \left( \mathbf{x}_2^1(\varepsilon, t) \leq \varepsilon^\delta/2 \text{ or } \mathbf{x}_2^1(\varepsilon, t) \geq H_2 - \varepsilon^\delta/2 \right) \\
+ \Pr \left( \mathbf{x}_2^1(\varepsilon, t) \leq (\frac{1}{2} - \frac{1}{2^2})\varepsilon^\delta \text{ or } \mathbf{x}_2^1(\varepsilon, t) \geq H_2 \left( \frac{1}{2} - \frac{1}{2^2} \right)\varepsilon^\delta \right) \\
\cap \left\{ \mathbf{x}_1^1(\varepsilon, t) \in (\varepsilon^\delta/2, H_2 - \varepsilon^\delta/2) \right\} \\
\leq \Pr \left( \mathbf{x}_2^1(\varepsilon, t) \leq \varepsilon^\delta/2 \text{ or } \mathbf{x}_2^1(\varepsilon, t) \geq H_2 - \varepsilon^\delta/2 \right) \\
+ \Pr \left( \mathbf{x}_2^1(\varepsilon, t) \leq (\frac{1}{2} - \frac{1}{2^2})\varepsilon^\delta \text{ or } \mathbf{x}_2^1(\varepsilon, t) \geq H_2 \left( \frac{1}{2} - \frac{1}{2^2} \right)\varepsilon^\delta \right) \\
\leq \Pr \left( \mathbf{x}_2^1(\varepsilon, t) \leq \varepsilon^\delta/2 \text{ or } \mathbf{x}_2^1(\varepsilon, t) \geq H_2 - \varepsilon^\delta/2 \right) \\
+ \Pr \left( \int_0^t |\mathbf{u}_{1,2}(\varepsilon, s) - \mathbf{u}_{1,2}(\varepsilon, s)| ds \geq \varepsilon^\delta/2 \right).
\]
Repeating the procedure of the modification for \( \{ \hat{u}_{i,j}(\varepsilon, t), j = 3, \ldots, n \} \) and \( \{ \hat{u}_{j,2}(\varepsilon, t), j = -n_0 + 1, \ldots, 2, 0 \} \) on the system (78), suppose that the following modification

\[
\hat{u}_{i,j}(\varepsilon, t), \quad -n_0 + 1 \leq i \leq j - 1,
\]

\[
\hat{u}_{j,i}(\varepsilon, t), \quad i = j + 1, \ldots, n
\]

for \( j = 1, \ldots, r \) is done. That is, we get for \( j = 1, \ldots, r \),

\[
\widetilde{u}_{i,j}(\varepsilon, t), \quad -n_0 + 1 \leq i \leq j - 1,
\]

\[
\widetilde{u}_{j,i}(\varepsilon, t), \quad i = j + 1, \ldots, n
\]

such that for \( j = 1, \ldots, r \),

\[
\widetilde{u}_{i,j}(\varepsilon, t) \leq \hat{u}_{i,j}(\varepsilon, t), \quad -n_0 + 1 \leq i \leq j - 1,
\]

\[
\widetilde{u}_{j,i}(\varepsilon, t) \leq \hat{u}_{j,i}(\varepsilon, t), \quad i = j + 1, \ldots, n
\]

and

\[
\begin{align*}
\mathcal{P}_j(\varepsilon, t) &= x_j(\delta) + \int_0^\delta \left[ \sum_{\ell=-n_0+1}^{n} \tilde{u}_{\ell,j}(\varepsilon, s) - \sum_{\ell=-j+1}^{n} \tilde{u}_{j,\ell}(\varepsilon, s) \right] ds, \\
& \quad j = 1, \ldots, r,
\end{align*}
\]

\[
\begin{align*}
\mathcal{P}_j(\varepsilon, t) &= x_j(\delta) + \int_0^\delta \left[ \sum_{\ell=-n_0+1}^{n} \tilde{u}_{\ell,j}(\varepsilon, s) + \sum_{\ell=j+1}^{n} \tilde{u}_{j,\ell}(\varepsilon, s) \right] ds, \\
& \quad j = r + 1, \ldots, m,
\end{align*}
\]

\[
\begin{align*}
\mathcal{P}_j(\varepsilon, t) &= x_j(\delta) + \int_0^\delta \left[ \sum_{\ell=-n_0+1}^{n} \tilde{u}_{\ell,j}(\varepsilon, s) + \sum_{\ell=1}^{m} \tilde{u}_{\ell,j}(\varepsilon, s) - z_j \right] ds, \\
& \quad j = m + 1, \ldots, n.
\end{align*}
\]

Furthermore, for \( j = 1, \ldots, r \),

\[
\begin{align*}
E \int_0^\infty e^{-\rho t} |\tilde{u}_{\ell,j}(\varepsilon, t) - \tilde{u}_{\ell,j}(\varepsilon, t)| dt & \leq C_6 \exp \left\{ -\kappa_3 \varepsilon^{-\frac{1-2t}{4}} \right\}, \\
& \ell = -n_0 + 1, \ldots, j - 1,
\end{align*}
\]

\[
\begin{align*}
E \int_0^\infty e^{-\rho t} |\tilde{u}_{j,\ell}(\varepsilon, t) - \tilde{u}_{j,\ell}(\varepsilon, t)| dt & \leq C_6 \exp \left\{ -\kappa_3 \varepsilon^{-\frac{1-2t}{4}} \right\}, \\
& \ell = j + 1, \ldots, n,
\end{align*}
\]

\[
\begin{align*}
E \int_0^\infty e^{-\rho t} \Pr \left( \mathcal{P}_j(\varepsilon, t) \geq H_j - \left( \frac{1}{2} - \sum_{i=1}^{r} \frac{1}{2i+1} \right) \varepsilon^\delta \right) dt & \leq C_6 \exp \left\{ -\kappa_3 \varepsilon^{-\frac{1-2t}{4}} \right\},
\end{align*}
\]

for \( j = r + 1, \ldots, m \),

\[
\begin{align*}
E \int_0^\infty e^{-\rho t} \Pr \left( \mathcal{P}_j(\varepsilon, t) \leq \left( \frac{1}{2} - \sum_{i=1}^{r} \frac{1}{2i+1} \right) \varepsilon^\delta, \\
\quad \mathcal{P}_j(\varepsilon, t) \notin \left[ H_j - \left( \frac{1}{2} - \sum_{i=1}^{r} \frac{1}{2i+1} \right) \varepsilon^\delta \right] \right) dt & \leq C_6 \left\{ \exp \left\{ -\kappa_3 \varepsilon^{-\frac{1-2t}{4}} \right\} + \exp \left\{ -\kappa_3 \varepsilon^{-\frac{1-2t}{4}} \right\} \right\},
\end{align*}
\]
and for \( j = m + 1, \ldots, n, \)
\[
E \int_0^\infty e^{-\rho t} \Pr \left\{ \bar{\mathbf{x}}_j^r(\epsilon, t) \notin \left( -\infty, \frac{1}{2} - \frac{1}{2^{r+1}} \right] \epsilon^j \right\} dt \\
\leq C_6 \left( \exp \left\{ -\kappa_3 \epsilon^{1-2\delta} \right\} + \exp \left\{ -\bar{\kappa}_3 \epsilon^{1-2\delta} \right\} \right),
\]
for some \( C_6 > 0, \kappa_3, \) and \( \bar{\kappa}_3. \) Now we modify
\[
\tilde{u}_{\ell, r+1}(\epsilon, t), \quad \ell = -n_0 + 1, \ldots, 0,
\]
\[
\tilde{u}_{r+1, \ell}(\epsilon, t), \quad \ell = r + 2, \ldots, n,
\]
for the system (82). To do this, we define
\[
B^\varepsilon_{r+1} = \left\{ t : \bar{\mathbf{x}}^r_{r+1}(\epsilon, t) - 0 \wedge \left( \inf_{0 \leq s \leq t} \bar{\mathbf{x}}^r_{r+1}(\epsilon, s) \right) = 0 \text{ and } \bar{\mathbf{x}}^r_{r+1}(\epsilon, t) < 0 \right\}.
\]
Then, we have that for \( t \in B^\varepsilon_{r+1}, \)
\[
\sum_{\ell = r+2}^n \tilde{u}_{r+1, \ell}(\epsilon, t) \geq \sum_{\ell = -n_0 + 1}^0 \tilde{u}_{\ell, r+1}(\epsilon, t) + \sum_{\ell = 1}^r \tilde{u}_{\ell, r+1}(\epsilon, t).
\]
(88)
For \( \ell = r + 2, \ldots, n \) and \( t \in B^\varepsilon_{r+1}, \) we choose
\[
\hat{u}_{r+1, \ell}(\epsilon, t) \leq \tilde{u}_{r+1, \ell}(\epsilon, t)
\]
and
\[
\sum_{\ell = r+2}^n \hat{u}_{r+1, \ell}(\epsilon, t) = \sum_{\ell = -n_0 + 1}^0 \hat{u}_{\ell, r+1}(\epsilon, t) + \sum_{\ell = 1}^r \hat{u}_{\ell, r+1}(\epsilon, t).
\]
Define
\[
\mathfrak{u}_{\ell, r+1}(\epsilon, t) = \hat{u}_{\ell, r+1}(\epsilon, t), \quad \ell = -n_0 + 1, \ldots, 0,
\]
\[
\mathfrak{u}_{r+1, \ell}(\epsilon, t) = \begin{cases} 
\hat{u}_{r+1, \ell}(\epsilon, t), & \text{if } t \notin B^\varepsilon_{r+1}, \\
\tilde{u}_{r+1, \ell}(\epsilon, t), & \text{if } t \in B^\varepsilon_{r+1}.
\end{cases}
\]
By the definition of \( B^\varepsilon_{r+1}, \) we know that
\[
\bar{\mathbf{x}}^r_{r+1}(\epsilon, t) = x_{r+1}(\delta) + \int_0^t \left[ \sum_{\ell = -n_0 + 1}^n \mathfrak{u}_{\ell, r+1}(\epsilon, s) - \sum_{\ell = r+2}^n \mathfrak{u}_{r+1, \ell}(\epsilon, s) \right] ds \geq 0.
\]
Furthermore, for \( j = r + 2, \ldots, n, \)
\[
E \int_0^\infty e^{-\rho t} |\mathfrak{u}_{r+1, j}(\epsilon, t) - \tilde{u}_{r+1, j}(\epsilon, t)| dt \\
\leq \max_{1 \leq \ell \leq p} \left\{ k_\ell^j \right\} E \int_0^\infty e^{-\rho t} \Pr \left( \bar{\mathbf{x}}^r_{r+1}(\epsilon, t) \leq 0 \right) dt \\
\leq \max_{1 \leq \ell \leq p} \left\{ k_\ell^j \right\} E \int_0^\infty e^{-\rho t} \Pr \left( \bar{\mathbf{x}}^r_{r+1}(\epsilon, t) \leq \left( \frac{1}{2} - \frac{1}{2^{r+1}} \right) \epsilon^j \right) dt.
\]
Consequently, by (86),
\[
\mathbb{E} \int_0^{\infty} e^{-\rho t} |\pi_{r+1,j}(\varepsilon, t) - \tilde{u}_{r+1,j}(\varepsilon, t)| \, dt \leq C_7 \exp \left\{ -\kappa_{3,r+1} \varepsilon^{-\frac{1-2\delta}{4}} \right\},
\]
for some $C_7 > 0$ and $\kappa_{3,r+1} > 0$. Similar to (77), we use (83) and (85) to obtain
\[
\mathbb{E} \int_0^{\infty} e^{-\rho t} \Pr \left\{ \pi_{r+1}(\varepsilon, t) \geq H_{r+1} - \left( \frac{1}{2} - \sum_{j=1}^{r+1} \frac{1}{2j+1} \varepsilon^\delta \right) \right\} \, dt
\leq C_8 \exp \left\{ -\tilde{\kappa}_{3,r+1} \varepsilon^{-\frac{1-2\delta}{4}} \right\},
\]
for some $C_8 > 0$ and $\tilde{\kappa}_{3,r+1} > 0$. Similar to (79), we can see from (83)–(85) that there is a positive constant $C_9 > 0$ such that for $j = r + 2, \ldots, m$,
\[
\mathbb{E} \int_0^{\infty} e^{-\rho t} \Pr \left\{ \pi_j^{r+1}(\varepsilon, t) \notin \left( \frac{1}{2} - \sum_{j=1}^{r+1} \frac{1}{2j+1} \varepsilon^\delta \right) \right\} \, dt
\leq C_9 \left( \exp \left\{ -\kappa_{3,r+1} \varepsilon^{-\frac{1-2\delta}{4}} \right\} + \exp \left\{ -\tilde{\kappa}_{3,r+1} \varepsilon^{-\frac{1-2\delta}{4}} \right\} \right).
\]
Let $r = m$, and we get Lemma 4.4.

**Proof of Lemma 4.5.** For $j = m + 1, \ldots, n$, let
\[
\tilde{B}_j^\varepsilon = \{ t : (H_j - \pi_j(\varepsilon, t)) - 0 \wedge \left( \inf_{0 \leq s \leq t} \{ H_j - \pi_j(\varepsilon, s) \} = 0 \right) \}.
\]
Then for $t \in \tilde{B}_j^\varepsilon$
\[
\sum_{\ell = -n_0 + 1}^{j-1} \pi_{\ell,j}(\varepsilon, t) \geq z_j.
\]
For $\ell = -n_0 + 1, \ldots, m$, choose
\[
\tilde{u}_{\ell,j}(\varepsilon, t) \leq \pi_{\ell,j}(\varepsilon, t)
\]
such that
\[
\sum_{\ell = -n_0 + 1}^{m} \tilde{u}_{\ell,j}(\varepsilon, t) = z_j.
\]
Define
\[
u_{\ell,j}(\varepsilon, t) = \begin{cases} 
\pi_{\ell,j}(\varepsilon, t), & t \notin \tilde{B}_j^\varepsilon, \\
\tilde{u}_{\ell,j}(\varepsilon, t), & t \in \tilde{B}_j^\varepsilon.
\end{cases}
\]

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Then
\[ x_j(\varepsilon, t) = x_j(\delta) + \int_0^t \left[ \sum_{l=-n_0+1}^m u_{l,j}(\varepsilon, s) - z_j \right] ds \leq H_j. \]

Furthermore, for \( j = m + 1, \ldots, n \), and \( \ell = -n_0 + 1, \ldots, m \),
\[
E \int_0^\infty e^{-\rho t} |u_{l,j}(\varepsilon, t) - \bar{u}_{l,j}(\varepsilon, t)| dt \\
\leq \max_{1 \leq \ell \leq p} \{ k_r^\ell \} E \int_0^\infty e^{-\rho t} \Pr (\bar{x}_j(\varepsilon, t) > H_j) dt \\
\leq \max_{1 \leq \ell \leq m_c} \{ k_r^\ell \} E \int_0^\infty e^{-\rho t} \Pr \left( \bar{x}_j(\varepsilon, t) \geq H_j - \left( \frac{1}{2} - \sum_{i=1}^m \frac{1}{2i+1}\varepsilon^\delta \right) \right) dt.
\]

Using (45), we get that for \( j = m + 1, \ldots, n \),
\[
E \int_0^\infty e^{-\rho t} |u_{l,j}(\varepsilon, t) - \bar{u}_{l,j}(\varepsilon, t)| dt \leq C_1 \exp \left\{ -\kappa_3 \varepsilon^{-\frac{1}{4}} \right\}, \quad (93)
\]
for some \( C_1 > 0 \). For \( j = 1, \ldots, m \),
\[
\hat{x}_j^{(m-1)}(\varepsilon, t) = x_j(\delta) + \int_0^t \left( \sum_{l=-n_0+1}^{j-1} \bar{u}_{l,j}(\varepsilon, s) - \sum_{l=j+1}^m \bar{u}_{j,l}(\varepsilon, s) - \sum_{l=m+1}^n u_{j,l}(\varepsilon, s) \right) ds.
\]

We know that \( \hat{x}_j^{(m-1)}(t) \geq 0 \). Furthermore, for \( j = 1, \ldots, m \),
\[
E \int_0^\infty e^{-\rho t} \Pr \left( \hat{x}_j^{(m-1)}(\varepsilon, t) \geq H_j - \left( \frac{1}{2} - \sum_{i=1}^{m+1} \frac{1}{2i+1}\varepsilon^\delta \right) \right) dt \\
\leq E \int_0^\infty e^{-\rho t} \Pr \left( \bar{x}_j(\varepsilon, t) \geq H_j - \left( \frac{1}{2} - \sum_{i=1}^{m+1} \frac{1}{2i+1}\varepsilon^\delta \right) \right) dt \\
+ E \int_0^\infty e^{-\rho t} \Pr \left( \hat{x}_j(\varepsilon, t) \geq H_j - \left( \frac{1}{2} - \sum_{i=1}^{m+1} \frac{1}{2i+1}\varepsilon^\delta \right) \right) dt \\
\leq E \int_0^\infty e^{-\rho t} \Pr \left( \bar{x}_j(\varepsilon, t) \geq H_j - \left( \frac{1}{2} - \sum_{i=1}^{m+1} \frac{1}{2i+1}\varepsilon^\delta \right) \right) dt \\
+ E \int_0^\infty e^{-\rho t} \Pr \left( |\hat{x}_j(\varepsilon, t) - \bar{x}_j(\varepsilon, t)| \geq \frac{\varepsilon^\delta}{2^{m+1}} \right) dt.
\]

Similar to (79), we get
\[
E \int_0^\infty e^{-\rho t} \Pr \left( \hat{x}_j^{(m-1)}(\varepsilon, t) \geq H_j - \left( \frac{1}{2} - \sum_{i=1}^{m+1} \frac{1}{2i+1}\varepsilon^\delta \right) \right) dt \\
\leq C_2 \exp \left\{ -\kappa_{41} \varepsilon^{-\frac{1}{4}} \right\}, \quad (94)
\]
for some $C_2 > 0$ and $\kappa_{41} > 0$.

Now we consider the system

$$
\begin{align*}
\hat{x}_{j}^{m-1}(\varepsilon, t) &= x_j(\delta) + \int_0^t \left[ \sum_{\ell=-n_0+1}^{m-1} \bar{u}_{\ell,j}(\varepsilon, s) - \sum_{\ell=m+1}^n u_{j,\ell}(\varepsilon, s) \right] ds, \\
\hat{x}_{m}^{m-1}(\varepsilon, t) &= x_m(\delta) + \int_0^t \left[ \sum_{\ell=-n_0+1}^{m-1} \bar{u}_{\ell,m}(\varepsilon, s) - \sum_{\ell=m+1}^n u_{m,\ell}(\varepsilon, s) \right] ds.
\end{align*}
$$

by repeating the above procedure, we get Lemma 4.5.

\[ \square \]

References


