EXERCISES 9.4

9.4.1. Consider

$$L(u) = f(x) \quad \text{with} \quad L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$$

subject to two homogeneous boundary conditions. *All* homogeneous solutions ϕ_h (if they exist) satisfy $L(\phi_h) = 0$ and the same two homogeneous boundary conditions. Apply Green's formula to prove that there are no solutions u if $f(x)$ is not orthogonal (weight 1) to all $\phi_h(x)$.

9.4.2. Modify Exercise 9.4.1 if

$$L(u) = f(x)$$

$$u(0) = \alpha \quad \text{and} \quad u(L) = \beta$$

*(a) Determine the condition for a solution to exist.

(b) If this condition is satisfied, show that there is an infinite number of solutions using the method of eigenfunction expansion.

9.4.3. Without determining $u(x)$, how many solutions are there of

$$\frac{d^2u}{dx^2} + \gamma u = \sin x$$

(a) If $\gamma = 1$ and $u(0) = u(\pi) = 0$?

*(b) If $\gamma = 1$ and $\frac{du}{dx}(0) = \frac{du}{dx}(\pi) = 0$?

(c) If $\gamma = -1$ and $u(0) = u(\pi) = 0$?

(d) If $\gamma = 2$ and $u(0) = u(\pi) = 0$?

9.4.4. For the following examples, obtain the general solution of the differential equation using the method of undetermined coefficients. Attempt to solve the boundary conditions, and show that the result is consistent with the Fredholm alternative:

(a) Equation (9.4.7)

(b) Equation (9.4.11)

(c) Example after (9.4.11)

(d) Second example after (9.4.11)

9.4.5. Are there any values of β for which there are solutions of

$$\frac{d^2u}{dx^2} + u = \beta + x$$

$$u(-\pi) = u(\pi) \quad \text{and} \quad \frac{du}{dx}(-\pi) = \frac{du}{dx}(\pi)?$$

*9.4.6. Consider

$$\frac{d^2u}{dx^2} + u = 1.$$

(a) Find the general solution of this differential equation. Determine all solutions with $u(0) = u(\pi) = 0$. Is the Fredholm alternative consistent with your result?

(b) Redo part (a) if $\frac{du}{dx}(0) = \frac{du}{dx}(\pi) = 0$.

(c) Redo part (a) if $\frac{du}{dx}(-\pi) = \frac{du}{dx}(\pi)$ and $u(-\pi) = u(\pi)$.

9.4.7. Consider

$$\begin{aligned} \frac{d^2u}{dx^2} + 4u &= \cos x \\ \frac{du}{dx}(0) &= \frac{du}{dx}(\pi) = 0. \end{aligned}$$

(a) Determine all solutions using the hint that a particular solution of the differential equation is in the form, $u_p = A\cos x$.

(b) Determine all solutions using the eigenfunction expansion method.

(c) Apply the Fredholm alternative. Is it consistent with parts (a) and (b)?

9.4.8. Consider

$$\frac{d^2u}{dx^2} + u = \cos x,$$

which has a particular solution of the form, $u_p = Ax\sin x$.

*(a) Suppose that $u(0) = u(\pi) = 0$. Explicitly attempt to obtain all solutions. Is your result consistent with the Fredholm alternative?

(b) Answer the same questions as in part (a) if $u(-\pi) = u(\pi)$ and $\frac{du}{dx}(-\pi) = \frac{du}{dx}(\pi)$.

9.4.9. (a) Since (9.4.15) (with homogeneous boundary conditions) is solvable, there is an infinite number of solutions. Suppose that $g_m(x, x_0)$ is one such solution that is not orthogonal to $\phi_h(x)$. Show that there is a unique generalized Green's function $G_m(x, x_0)$ that is orthogonal to $\phi_h(x)$.

(b) Assume that $G_m(x, x_0)$ is the generalized Green's function that is orthogonal to $\phi_h(x)$. Prove that $G_m(x, x_0)$ is symmetric. [*Hint*: Apply Green's formula with $G_m(x, x_1)$ and $G_m(x, x_2)$.]

*9.4.10. Determine the generalized Green's function that is needed to solve

$$\frac{d^2u}{dx^2} + u = f(x)$$
$$u(0) = \alpha \quad \text{and} \quad u(\pi) = \beta.$$

Assume that $f(x)$ satisfies the solvability condition (see Exercise 9.4.2). Obtain a representation of the solution $u(x)$ in terms of the generalized Green's function.