3. *The Fourier sine coefficients of* $f(x) = 1 - x$ *are*

$$4\int_0^{1/2} (1-x)\sin(2n\pi x)\,dx = \frac{2-(-1)^n}{n\pi},\quad n = 1, 2, 3, \ldots,$$

and so

$$1 - x = \sum_{n=1}^{\infty} \frac{2-(-1)^n}{n\pi}\sin(2n\pi x).$$

4. *The Fourier sine coefficients of* $-d^2v/dx^2$ *are*

$$\lambda_n a_n = 4n^2\pi^2 a_n,\ n = 1, 2, 3, \ldots,$$

and so

$$-\frac{d^2v}{dx^2}(x) = \sum_{n=1}^{\infty} 4n^2\pi^2 a_n \sin(2n\pi x).$$

5. *We thus obtain*

$$\sum_{n=1}^{\infty} 4n^2\pi^2 a_n \sin(2n\pi x) = \sum_{n=1}^{\infty} \frac{2-(-1)^n}{n\pi}\sin(2n\pi x),$$

which implies that

$$4n^2\pi^2 a_n = \frac{2-(-1)^n}{n\pi},\quad n = 1, 2, 3, \ldots,$$

or

$$a_n = \frac{2-(-1)^n}{4n^3\pi^3},\quad n = 1, 2, 3, \ldots.$$

The solution v *is then*

$$v(x) = \sum_{n=1}^{\infty} \frac{2-(-1)^n}{4n^3\pi^3}\sin(2n\pi x).$$

This yields

$$u(x) = 3 - 8x + \sum_{n=1}^{\infty} \frac{2-(-1)^n}{4n^3\pi^3}\sin(2n\pi x).$$

Exercises

In Exercises 1–4, solve the BVPs using the method of Fourier series, shifting the data if necessary. If possible,[30] produce a graph of the computed solution by plotting a partial Fourier series with enough terms to give a qualitatively correct graph. (The number of terms can be determined by trial and error; if the plot no longer changes, qualitatively, when more terms are included, it can be assumed that the partial series contains enough terms.)

[30]That is, if you have the necessary technology.

1. (a) $-\frac{d^2u}{dx^2}=1,\ u(0)=u(1)=0$
   (b) $-\frac{d^2u}{dx^2}=1,\ u(0)=0,\ u(1)=1$
2. (a) $-\frac{d^2u}{dx^2}=e^x,\ u(0)=u(1)=0$
   (b) $-\frac{d^2u}{dx^2}=e^x,\ u(0)=u(1)=1$
3. The results of Exercises 5.2.2 and 5.2.3 will be useful for these problems:
   (a) $-\frac{d^2u}{dx^2}=x,\ u(0)=0,\ \frac{du}{dx}(1)=0$
   (b) $-\frac{d^2u}{dx^2}=x,\ \frac{du}{dx}(0)=0,\ u(1)=1$
   (c) $-\frac{d^2u}{dx^2}+2u=1,\ u(0)=u(1)=0$
   (d) $-\frac{d^2u}{dx^2}+u=0,\ u(0)=0,\ u(1)=1$
4. The results of Exercises 5.2.2 and 5.2.3 will be useful for these problems:
   (a) $-\frac{d^2u}{dx^2}=x^2,\ u(0)=u(1)=0$
   (b) $-\frac{d^2u}{dx^2}+2u=\frac{1}{2}-x,\ u(0)=u(2)=0$
   (c) $-\frac{d^2u}{dx^2}=x+\sin(\pi x),\ u(0)=\frac{du}{dx}(1)=0$
   (d) $-\frac{d^2u}{dx^2}=f(x),\ \frac{du}{dx}(0)=u(1)=0$, where
   $$f(x)=\begin{cases}1, & 0<x<\frac{1}{2},\\ 0, & \frac{1}{2}<x<1.\end{cases}$$
5. Solve (5.21) to get (5.22).
6. Suppose u satisfies
   $$u(0)=0,\ \frac{du}{dx}(\ell)=0,$$
   and the Fourier sine series of u is
   $$u(x)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{\ell}\right).$$
   Show that it is not possible to represent the Fourier sine coefficients of $-d^2u/dx^2$ in terms of $b_1,b_2,b_3,\ldots$. This shows that it is not possible to use the "wrong" eigenfunctions (that is, eigenfunctions corresponding to different boundary conditions) to solve a BVP.
7. Suppose u satisfies
   $$u(0)=0,\ \frac{du}{dx}(\ell)=0,$$
   and has Fourier quarter-wave sine series
   $$u(x)=\sum_{n=1}^{\infty}a_n\sin\left(\frac{(2n-1)\pi x}{2\ell}\right).$$

Show that the Fourier series of $-Td^2u/dx^2$ is

$$-T\frac{d^2u}{dx^2}(x) = \sum_{n=1}^{\infty} \frac{T(2n-1)^2\pi^2}{4\ell^2} a_n \sin\left(\frac{(2n-1)\pi x}{2\ell}\right).$$

(Hint: The computation of the Fourier coefficients of $-Td^2u/dx^2$ is similar to (5.23)).

8. Consider an aluminum bar of length 1 m and radius 1 cm. Suppose that the side of the bar is perfectly insulated, the ends of the bar are placed in ice baths, and heat energy is added throughout the interior of the bar at a constant rate of 0.001 W/cm^3. The thermal conductivity of the aluminum alloy is 1.5 W/(cm K). Find and graph the steady-state temperature of the bar. Use the Fourier series method.

9. Repeat the previous exercise, assuming that the right end of the bar is perfectly insulated and the left is placed in an ice bath.

10. Consider the string of Example 5.12. Suppose that the right end of the string is free to move vertically (along a frictionless pole, for example), and a pressure of 2200 dynes per centimeter, in the upward direction, is applied along the string (recall that a dyne is a unit of force—one dyne equals one gram-centimeter per square second). What is the equilibrium displacement of the string? How does the answer change if the force due to gravity is taken into account?

5.4 Finite element methods for BVPs

As we mentioned near the end of the last section, the primary utility of the Fourier series method is for problems with constant coefficients, in which case the eigenpairs can be found explicitly. (For problems in two or three dimensions, in order for the eigenfunctions to be explicitly computable, it is also necessary that the domain on which the equation is to be solved be geometrically simple. We discuss this further in Chapter 8.) For problems with nonconstant coefficients, it is possible to perform analysis to show that the Fourier series method applies in principle—the eigenfunctions exist, they are orthogonal, and so forth. The reader can consult Chapter 5 of Haberman [22] for an elementary introduction to this kind of analysis, which we also discuss further in Section 9.7. However, without explicit formulas for the eigenfunctions, it is not easy to apply the Fourier series method.

These remarks apply, for example, to the BVP

$$\begin{aligned} -\frac{d}{dx}\left(k(x)\frac{du}{dx}\right) &= f(x),\ 0 < x < \ell, \\ u(0) &= 0, \\ u(\ell) &= 0 \end{aligned} \tag{5.32}$$

(where k is a positive function). The operator $K : C^2_D[0,\ell] \to C[0,\ell]$ defined by

$$Ku = -\frac{d}{dx}\left(k(x)\frac{du}{dx}\right) \tag{5.33}$$