EXERCISES 5.3

*5.3.1. Do Exercise 4.4.2(b). Show that the partial differential equation may be put into Sturm-Liouville form.

5.3.2. Consider

$$\rho\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + \alpha u + \beta\frac{\partial u}{\partial t}.$$

(a) Give a brief physical interpretation. What signs must α and β have to be physical?

(b) Allow ρ, α, β to be functions of x. Show that separation of variables works only if $\beta = c\rho$, where c is a constant.

(c) If $\beta = c\rho$, show that the spatial equation is a Sturm-Liouville differential equation. Solve the time equation.

*5.3.3. Consider the non-Sturm-Liouville differential equation

$$\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + [\lambda\beta(x) + \gamma(x)]\phi = 0.$$

Multiply this equation by $H(x)$. Determine $H(x)$ such that the equation may be reduced to the standard Sturm-Liouville form:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + [\lambda\sigma(x) + q(x)]\phi = 0.$$

Given $\alpha(x), \beta(x)$, and $\gamma(x)$, what are $p(x), \sigma(x)$, and $q(x)$?

5.3.4. Consider heat flow with convection (see Exercise 1.5.2):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - V_0\frac{\partial u}{\partial x}.$$

(a) Show that the spatial ordinary differential equation obtained by separation of variables is not in Sturm-Liouville form.

*(b) Solve the initial boundary value problem

$$\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \\ u(x,0) &= f(x). \end{aligned}$$

(c) Solve the initial boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial x}(0,t) &= 0 \\ \frac{\partial u}{\partial x}(L,t) &= 0 \\ u(x,0) &= f(x). \end{aligned}$$

5.3.5. For the Sturm-Liouville eigenvalue problem,

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad \text{with} \quad \frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(L) = 0,$$

verify the following general properties:

(a) There is an infinite number of eigenvalues with a smallest but no largest.
(b) The nth eigenfunction has $n-1$ zeros.
(c) The eigenfunctions are complete and orthogonal.
(d) What does the Rayleigh quotient say concerning negative and zero eigenvalues?

5.3.6. Redo Exercise 5.3.5 for the Sturm-Liouville eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad \text{with} \quad \frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

5.3.7. Which of statements 1–5 of the theorems of this section are valid f the following eigenvalue problem?

$$\begin{aligned} \tfrac{d^2\phi}{dx^2} + \lambda\phi &= 0 \quad \text{with} \\ \phi(-L) &= \phi(L) \\ \tfrac{d\phi}{dx}(-L) &= \tfrac{d\phi}{dx}(L). \end{aligned}$$

5.3.8. Show that $\lambda \geq 0$ for the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0 \quad \text{with} \quad \frac{d\phi}{dx}(0) = 0, \quad \frac{d\phi}{dx}(1) = 0.$$

Is $\lambda = 0$ an eigenvalue?

5.3.9. Consider the eigenvalue problem

$$x^2\frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} + \lambda\phi = 0 \quad \text{with} \quad \phi(1) = 0, \quad \text{and} \quad \phi(b) = 0. \tag{5.3.10}$$

(a) Show that multiplying by $1/x$ puts this in the Sturm-Liouville form. (This multiplicative factor is derived in Exercise 5.3.3.)
(b) Show that $\lambda \geq 0$.
*(c) Since (5.3.10) is an equidimensional equation, determine all positive eigenvalues. Is $\lambda = 0$ an eigenvalue? Show that there is an infinite number of eigenvalues with a smallest, but no largest.
(d) The eigenfunctions are orthogonal with what weight according to Sturm-Liouville theory? Verify the orthogonality using properties of integrals.
(e) Show that the nth eigenfunction has $n-1$ zeros.

5.3.10. Reconsider Exercise 5.3.9 with the boundary conditions

$$\frac{d\phi}{dx}(1) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(b) = 0.$$

EXERCISES 5.5

5.5.1. A Sturm-Liouville eigenvalue problem is called self-adjoint if

$$p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Bigg|_a^b = 0$$

since then $\int_a^b [uL(v) - vL(u)]\ dx = 0$ for any two functions u and v satisfying the boundary conditions. Show that the following yield self-adjoint problems.

(a) $\phi(0) = 0$ and $\phi(L) = 0$

(b) $\frac{d\phi}{dx}(0) = 0$ and $\phi(L) = 0$

(c) $\frac{d\phi}{dx}(0) - h\phi(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$

(d) $\phi(a) = \phi(b)$ and $p(a)\frac{d\phi}{dx}(a) = p(b)\frac{d\phi}{dx}(b)$

(e) $\phi(a) = \phi(b)$ and $\frac{d\phi}{dx}(a) = \frac{d\phi}{dx}(b)$ [self-adjoint only if $p(a) = p(b)$]

(f) $\phi(L) = 0$ and [in the situation in which $p(0) = 0$] $\phi(0)$ bounded and $\lim_{x\to 0} p(x)\frac{d\phi}{dx} = 0$

*(g) Under what conditions is the following self-adjoint (if p is constant)?

$$\begin{aligned}\phi(L) + \alpha\phi(0) + \beta\tfrac{d\phi}{dx}(0) &= 0\\ \tfrac{d\phi}{dx}(L) + \gamma\phi(0) + \delta\tfrac{d\phi}{dx}(0) &= 0\end{aligned}$$

5.5.2. Prove that the eigenfunctions corresponding to different eigenvalues (of the following eigenvalue problem) are orthogonal:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\phi + \lambda\sigma(x)\phi = 0$$

with the boundary conditions

$$\begin{aligned}\phi(1) &= 0\\ \phi(2) - 2\tfrac{d\phi}{dx}(2) &= 0.\end{aligned}$$

What is the weighting function?

5.5.3. Consider the eigenvalue problem $L(\phi) = -\lambda\sigma(x)\phi$, subject to a given set of homogeneous boundary conditions. Suppose that

$$\int_a^b [uL(v) - vL(u)]\ dx = 0$$

for all functions u and v satisfying the same set of boundary conditions. Prove that eigenfunctions corresponding to different eigenvalues are orthogonal (with what weight?).

5.5.4. Give an example of an eigenvalue problem with more than one eigenfunction corresponding to an eigenvalue.

5.5.5. Consider

$$L = \frac{d^2}{dx^2} + 6\frac{d}{dx} + 9.$$

(a) Show that $L(e^{rx}) = (r+3)^2 e^{rx}$.

(b) Use part (a) to obtain solutions of $L(y) = 0$ (a second-order constant-coefficient differential equation).

(c) If z depends on x and a parameter r, show that

$$\frac{\partial}{\partial r} L(z) = L\left(\frac{\partial z}{\partial r}\right).$$

(d) Using part (c), evaluate $L(\partial z/\partial r)$ if $z = e^{rx}$.

(e) Obtain a second solution of $L(y) = 0$, using part (d).

5.5.6. Prove that if x is a root of a sixth-order polynomial *with real coefficients*, then $\overline{x}$ is also a root.

5.5.7. For

$$L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$$

with p and q real, carefully show that

$$\overline{L(\phi)} = L(\overline{\phi}).$$

5.5.8. Consider a fourth-order linear differential operator,

$$L = \frac{d^4}{dx^4}.$$

(a) Show that $uL(v) - vL(u)$ is an exact differential.

(b) Evaluate $\int_0^1 [uL(v) - vL(u)]\ dx$ in terms of the boundary data for any functions u and v.

(c) Show that $\int_0^1 [uL(v) - vL(u)]\ dx = 0$ if u and v are any two functions satisfying the boundary conditions

$$\begin{aligned} \phi(0) &= 0 & \phi(1) &= 0 \\ \frac{d\phi}{dx}(0) &= 0 & \frac{d^2\phi}{dx^2}(1) &= 0. \end{aligned}$$

(d) Give another example of boundary conditions such that

$$\int_0^1 [uL(v) - vL(u)]\ dx = 0.$$

(e) For the eigenvalue problem [using the boundary conditions in part (c)]

$$\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0,$$

show that the eigenfunctions corresponding to different eigenvalues are orthogonal. What is the weighting function?

*5.5.9. For the eigenvalue problem

$$\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0$$

subject to the boundary conditions

$$\begin{aligned} \phi(0) &= 0 \quad & \phi(1) &= 0 \\ \frac{d\phi}{dx}(0) &= 0 \quad & \frac{d^2\phi}{dx^2}(1) &= 0, \end{aligned}$$

show that the eigenvalues are less than or equal to zero ($\lambda \leq 0$). (Don't worry; in a physical context that is exactly what is expected.) Is $\lambda = 0$ an eigenvalue?

5.5.10. (a) Show that (5.5.22) yields (5.5.23) if at least one of the boundary conditions is of the regular Sturm-Liouville type.

(b) Do part (a) if one boundary condition is of the singular type.

5.5.11. *(a) Suppose that

$$L = p(x)\frac{d^2}{dx^2} + r(x)\frac{d}{dx} + q(x).$$

Consider

$$\int_a^b vL(u)\, dx.$$

By repeated integration by parts, determine the adjoint operator L^* such that

$$\int_a^b [uL^*(v) - vL(u)]\, dx = H(x)\Big|_a^b.$$

What is $H(x)$? Under what conditions does $L = L^*$, the **self-adjoint** case? [*Hint*: Show that

$$L^* = p\frac{d^2}{dx^2} + \left(2\frac{dp}{dx} - r\right)\frac{d}{dx} + \left(\frac{d^2p}{dx^2} - \frac{dr}{dx} + q\right)\Bigg].$$

(b) If

$$u(0) = 0 \quad \text{and} \quad \frac{du}{dx}(L) + u(L) = 0,$$

what boundary conditions should $v(x)$ satisfy for $H(x)|_0^L = 0$, called the adjoint boundary conditions?

5.5.12. Consider nonself-adjoint operators as in Exercise 5.5.11. The eigenvalues λ may be complex as well as their corresponding eigenfunctions ϕ.

(a) Show that if λ is a complex eigenvalue with corresponding eigenfunction ϕ, then the complex conjugate $\overline{\lambda}$ is also an eigenvalue with eigenfunction $\overline{\phi}$.

(b) The eigenvalues of the adjoint L^* may be different from the eigenvalues of L. Using the result of Exercise 5.5.11, show that the eigenfunctions of $L(\phi) + \lambda\sigma\phi = 0$ are orthogonal with weight σ(in a complex sense) to eigenfunctions of $L^*(\psi) + \nu\sigma\psi = 0$ if the eigenvalues are different. Assume that ψ satisfies adjoint boundary conditions. You should also use part (a).

5.5.13. Using the result of Exercise 5.5.11, prove the following part of the **Fredholm alternative** (for operators that are not necessarily self-adjoint): A solution of $L(u) = f(x)$ subject to homogeneous boundary conditions may exist only if $f(x)$ is orthogonal to all solutions of the homogeneous adjoint problem.

5.5.14. If L is the following first-order linear differential operator

$$L = p(x)\frac{d}{dx},$$

then determine the adjoint operator L^* such that

$$\int_a^b [uL^*(v) - vL(u)]\, dx = B(x)\Big|_a^b .$$

What is $B(x)$? [*Hint*: Consider $\int_a^b vL(u)\, dx$ and integrate by parts.]

Appendix to 5.5: Matrix Eigenvalue Problem and Orthogonality of Eigenvectors

The matrix eigenvalue problem

$$\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}, \tag{5.5.26}$$

where $\boldsymbol{A}$ is an $n \times n$ real matrix (with entries a_{ij}) and $\boldsymbol{x}$ is an n-dimensional column vector (with components x_i), has many properties similar to those of the Sturm-Liouville eigenvalue problem.

Eigenvalues and eigenvectors. For all values of λ, $\boldsymbol{x}= 0$ is a "trivial" solution of the homogeneous linear system (5.5.26). We ask, for what values of λ are there nontrivial solutions? In general, (5.5.26) can be rewritten as

$$(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}, \tag{5.5.27}$$

where $\boldsymbol{I}$ is the identity matrix. According to the theory of linear equations (elementary linear algebra), a nontrivial solution exists only if

$$\det[\boldsymbol{A} - \lambda\boldsymbol{I}] = 0. \tag{5.5.28}$$

We thus have a minimization theorem for the lowest eigenvalue λ_1. We can ask if there are corresponding theorems for the higher eigenvalues. Interesting generalizations immediately follow from (5.6.13). If we insist that $a_1 = 0$, then

$$RQ[u] = \frac{\sum_{n=2}^{\infty} a_n^2 \lambda_n \int_a^b \phi_n^2 \sigma \, dx}{\sum_{n=2}^{\infty} a_n^2 \int_a^b \phi_n^2 \sigma \, dx}. \tag{5.6.15}$$

This means that in addition we are restricting our function u to be orthogonal to ϕ_1, since $a_1 = \int_a^b u\phi_1\sigma \, dx / \int_a^b \phi_1^2 \sigma \, dx$. We now proceed in a similar way. Since $\lambda_2 < \lambda_n$ for $n > 2$, it follows that

$$RQ[u] \geq \lambda_2,$$

and furthermore the equality holds only if $a_n = 0$ for $n > 2$ [i.e., $u = a_2\phi_2(x)$] since $a_1 = 0$ already. We have just proved the following theorem: The minimum value for all continuous functions *$u(x)$ that are orthogonal to the lowest eigenfunction* and satisfy the boundary conditions is the next-to-lowest eigenvalue. Further generalizations also follow directly from (5.6.13).

EXERCISES 5.6

5.6.1. Use the Rayleigh quotient to obtain a (reasonably accurate) upper bound for the lowest eigenvalue of

(a) $\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0$ with $\frac{d\phi}{dx}(0) = 0$ and $\phi(1) = 0$

(b) $\frac{d^2\phi}{dx^2} + (\lambda - x)\phi = 0$ with $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(1) + 2\phi(1) = 0$

*(c) $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$ with $\phi(0) = 0$ and $\frac{d\phi}{dx}(1) + \phi(1) = 0$ (See Exercise 5.8.10.)

5.6.2. Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0$$

subject to $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(1) = 0$. Show that $\lambda > 0$ (be sure to show that $\lambda \neq 0$).

5.6.3. Prove that (5.6.10) is valid in the following way. Assume $L(u)/\sigma$ is piecewise smooth so that

$$\frac{L(u)}{\sigma} = \sum_{n=1}^{\infty} b_n \phi_n(x).$$

Determine b_n. [*Hint:* Using Green's formula (5.5.5), show that $b_n = -a_n\lambda_n$ if u and du/dx are continuous and if u satisfies the same homogeneous boundary conditions as the eigenfunctions $\phi_n(x)$.]

Since the coefficient $\sigma(x) = 1$ in (5.8.41), the eigenfunctions $\phi_n(x)$ are orthogonal with weight 1. Thus, we know that the generalized Fourier coefficients of the initial condition $f(x)$ are

$$a_n = \frac{\int_0^L f(x)\phi_n(x)\,dx}{\int_0^L \phi_n^2\,dx} = \begin{cases} \int_0^L f(x)\sinh\sqrt{s_1}x\,dx \Big/ \int_0^L \sinh^2\sqrt{s_1}x\,dx & n=1 \\ \int_0^L f(x)\sin\sqrt{\lambda_n}x\,dx \Big/ \int_0^L \sin^2\sqrt{\lambda_n}x\,dx & n \geq 2. \end{cases}$$

In particular, we could show $\int_0^L \sin^2\sqrt{\lambda_n}x\,dx \neq L/2$. Perhaps we should emphasize one additional point. We have utilized the theorem that states that eigenfunctions corresponding to different eigenvalues are orthogonal; it is guaranteed that $\int_0^L \sin\sqrt{\lambda_n}x \sin\sqrt{\lambda_m}x\,dx = 0 (n \neq m)$ and $\int_0^L \sin\sqrt{\lambda_n}x \sinh\sqrt{s_1}x\,dx = 0$. We do not need to verify these by integration (although it can be done).

Other problems with boundary conditions of the third kind appear in the Exercises.

EXERCISES 5.8

5.8.1. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

subject to $u(0,t) = 0$, $\frac{\partial u}{\partial x}(L,t) = -hu(L,t)$, and $u(x,0) = f(x)$.

(a) Solve if $hL > -1$.

(b) Solve if $hL = -1$.

5.8.2. Consider the eigenvalue problem (5.8.8)–(5.8.10). Show that the nth eigenfunction has $n-1$ zeros in the interior if

(a) $h > 0$

(b) $h = 0$

*(c) $-1 < hL < 0$

(d) $hL = -1$

5.8.3. Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0,$$

subject to $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(L) + h\phi(L) = 0$ with $h > 0$.

(a) Prove that $\lambda > 0$ (without solving the differential equation).

*(b) Determine all eigenvalues graphically. Obtain upper and lower bounds. Estimate the large eigenvalues.

(c) Show that the nth eigenfunction has $n-1$ zeros in the interior.

5.8.4. Redo Exercise 5.8.3 parts (b) and (c) only if $h < 0$.

5.8.5. Consider
$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$
with $\frac{\partial u}{\partial x}(0,t) = 0$, $\frac{\partial u}{\partial x}(L,t) = -hu(L,t)$, and $u(x,0) = f(x)$.

(a) Solve if $h > 0$.

(b) Solve if $h < 0$.

5.8.6. Consider (with $h > 0$)
$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$
$$\frac{\partial u}{\partial x}(0,t) - hu(0,t) = 0 \qquad u(x,0) = f(x)$$
$$\frac{\partial u}{\partial x}(L,t) = 0 \qquad \frac{\partial u}{\partial t}(x,0) = g(x).$$

(a) Show that there are an infinite number of different frequencies of oscillation.

(b) Estimate the large frequencies of oscillation.

(c) Solve the initial value problem.

*5.8.7. Consider the eigenvalue problem
$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \text{ subject to } \phi(0) = 0 \text{ and } \phi(\pi) - 2\frac{d\phi}{dx}(0) = 0.$$

(a) Show that usually
$$\int_0^\pi \left(u\frac{d^2v}{dx^2} - v\frac{d^2u}{dx^2}\right)\,dx \neq 0$$
for any two functions u and v satisfying these homogeneous boundary conditions.

(b) Determine all positive eigenvalues.

(c) Determine all negative eigenvalues.

(d) Is $\lambda = 0$ an eigenvalue?

(e) Is it possible that there are other eigenvalues besides those determined in parts (b) through (d)? *Briefly* explain.

5.8.8. Consider the boundary value problem
$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad \text{with} \quad \begin{aligned} \phi(0) - \frac{d\phi}{dx}(0) &= 0 \\ \phi(1) + \frac{d\phi}{dx}(1) &= 0. \end{aligned}$$

(a) Using the Rayleigh quotient, show that $\lambda \geq 0$. Why is $\lambda > 0$?

(b) Prove that eigenfunctions corresponding to different eigenvalues are orthogonal.

*(c) Show that

$$\tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}.$$

Determine the eigenvalues graphically. Estimate the large eigenvalues.

(d) Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with

$$\begin{aligned} u(0,t) - \frac{\partial u}{\partial x}(0,t) &= 0 \\ u(1,t) + \frac{\partial u}{\partial x}(1,t) &= 0 \\ u(x,0) &= f(x). \end{aligned}$$

You may call the relevant eigenfunctions $\phi_n(x)$ and assume that they are known.

5.8.9. Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \text{ with } \phi(0) = \frac{d\phi}{dx}(0) \text{ and } \phi(1) = \beta\frac{d\phi}{dx}(1).$$

For what values (if any) of β is $\lambda = 0$ an eigenvalue?

5.8.10. Consider the special case of the eigenvalue problem of Sec. 5.8:

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \text{ with } \phi(0) = 0 \text{ and } \frac{d\phi}{dx}(1) + \phi(1) = 0.$$

*(a) Determine the lowest eigenvalue to at least two or three significant figures using tables or a calculator.

*(b) Determine the lowest eigenvalue using a root finding algorithm (e.g.. Newton's method) on a computer.

(c) Compare either part (a) or (b) to the bound obtained using the Rayleigh quotient [see Exercise 5.6.1(c)].

5.8.11. Determine all negative eigenvalues for

$$\frac{d^2\phi}{dx^2} + 5\phi = -\lambda\phi \text{ with } \phi(0) = 0 \text{ and } \phi(\pi) = 0.$$

5.8.12. Consider $\partial^2 u/\partial t^2 = c^2\partial^2 u/\partial x^2$ with the boundary conditions

$$\begin{aligned} u &= 0 && \text{at } x = 0 \\ m\frac{\partial^2 u}{\partial t^2} &= -T_0\frac{\partial u}{\partial x} - ku && \text{at } x = L. \end{aligned}$$