The meaning of the mysterious terms in this theorem is as follows:

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^0 f(x,y)=f(x,y)$$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^1 f(x,y)=\left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right)(x,y)$$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x,y)=\left(h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\,\partial y}+k^2\frac{\partial^2 f}{\partial y^2}\right)(x,y)$$

and so on. Letting $f_x=\partial f/\partial x$, $f_y=\partial f/\partial y$, $f_{xx}=\partial^2 f/\partial x^2$, $f_{xy}=\partial^2 f/\partial x\,\partial y$, $f_{yy}=\partial^2 f/\partial y^2$, we can write the first few terms of (5) as

$$f(x+h,y+k)=f+\left(hf_x+kf_y\right)+\tfrac{1}{2}\left(h^2 f_{xx}+2hkf_{xy}+k^2 f_{yy}\right)+\cdots$$

where on the right-hand side the function f and each of the following partial derivatives are evaluated at (x,y).

EXAMPLE 4 What are the first few terms in the Taylor formula for $f(x,y)=\cos(xy)$?

Solution For the given function, we find that

$$\frac{\partial f}{\partial x}=-y\sin(xy)\qquad \frac{\partial f}{\partial y}=-x\sin(xy)$$

$$\frac{\partial^2 f}{\partial x^2}=-y^2\cos(xy)\qquad \frac{\partial^2 f}{\partial x\,\partial y}=-xy\cos(xy)-\sin(xy)\qquad \frac{\partial^2 f}{\partial y^2}=-x^2\cos(xy)$$

Thus, if we let $n=1$ in Taylor's formula (5), the result is

$$\cos[(x+h)(y+k)]=\cos(xy)-hy\sin(xy)-kx\sin(xy)+E_1(h,k)$$

The remainder E_1 is the sum of three terms—namely,

$$-\frac{1}{2}h^2(y+\theta k)^2\cos[(x+\theta h)(y+\theta k)]$$

$$-hk\{(x+\theta h)(y+\theta k)\cos[(x+\theta h)(y+\theta k)]+\sin[(x+\theta h)(y+\theta k)]\}$$

$$-\frac{1}{2}k^2(x+\theta h)^2\cos[(x+\theta h)(y+\theta k)]$$ ■

PROBLEMS 1.1

1. Show that $|x^2-4|<\varepsilon$ when $0<|x-2|<\varepsilon(5+\varepsilon)^{-1}$ and prove $\lim_{x\to 2}x^2=4$ by using these inequalities.
2. Show that the function $f(x)=x\sin(1/x)$, with $f(0)=0$, is continuous at 0 but not differentiable at 0.
3. Show that $f(x)=x^2\sin(1/x)$, with $f(0)=0$, is once differentiable at 0 but not twice.

4. Let $f(x) = x^{-3}(x - \sin x)$ for $x \neq 0$. How should $f(0)$ be defined in order that f be continuous? Will it also be differentiable?

5. **a.** Derive the Taylor series at 0 for the function $f(x) = \ln(x + 1)$. Write this series in summation notation. Give two expressions for the remainder when the series is truncated.

   b. Determine the smallest number of terms that must be taken in the series to yield ln 1.5 with an error less than 10^{-8}.

   c. Determine the number of terms necessary to compute ln 1.6 with error 10^{-10} at most.

6. Determine whether the following function is continuous, once differentiable, or twice differentiable:

$$f(x) = \begin{cases} x^3 + x - 1 & \text{if } x \leq 0 \\ x^3 - x - 1 & \text{if } x > 0 \end{cases}$$

7. (Continuation) Repeat the preceding problem for the function

$$f(x) = \begin{cases} x & \text{if } x \leq 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

8. Criticize this reasoning: The function f defined by

$$f(x) = \begin{cases} x^3 + x & \text{if } x \leq 0 \\ x^3 - x & \text{if } x \geq 0 \end{cases}$$

has the properties

$$\lim_{x \to 0^+} f''(x) = \lim_{x \to 0^+} 6x = 0$$

$$\lim_{x \to 0^-} f''(x) = \lim_{x \to 0^-} 6x = 0$$

Therefore, f'' is continuous.

9. Prove that if f is differentiable at x, then

$$\lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h} = f'(x)$$

Show that for some functions that are not differentiable at x, the preceding limit exists. (See Eggermont [1988] or the following problem.)

10. Prove or disprove this assertion: If f is differentiable at x, then for $\alpha \neq 1$,

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x + \alpha h)}{h - \alpha h}$$

11. Show that $\lim_{x \to 1}(4x + 2) = 6$ by means of an ε-δ proof.

12. Show that $\lim_{x \to 2}(1/x) = \frac{1}{2}$ by means of an ε-δ proof.

13. For the function $f(x) = 3 - 2x + x^2$ and the interval $[a, b] = [1, 3]$, find the number ξ that occurs in the Mean-Value Theorem.

14. (Continuation) Repeat the preceding problem with the function $f(x) = x^6 + x^4 - 1$ and the interval $[0, 1]$.

15. Find the Taylor series for $f(x) = \cosh x$ about the point $c = 0$.

16. If the series for $\ln x$ is truncated after the term involving $(x - 1)^{1000}$ and is then used to compute ln 2, what bound on the error can be given?

17. Find the Taylor series for $f(x) = e^x$ about the point $c = 3$. Then simplify the series and show how it could have been obtained directly from the series for f about $c = 0$.

18. Let k be a positive integer and let $0 < \alpha < 1$. To what classes $C^n(\mathbb{R})$ does the function $x^{k+\alpha}$ belong?

19. Prove: If $f \in C^n(\mathbb{R})$, then $f' \in C^{n-1}(\mathbb{R})$ and $\int_a^x f(t)\,dt$ belong to $C^{n+1}(\mathbb{R})$.

20. Prove Rolle's Theorem directly (not as a special case of the Mean-Value Theorem).

21. Prove: If $f \in C^n(\mathbb{R})$ and $f(x_0) = f(x_1) = \cdots = f(x_n) = 0$ for $x_0 < x_1 < \cdots < x_n$, then $f^{(n)}(\xi) = 0$ for some $\xi \in (x_0, x_n)$. *Hint:* Use Rolle's Theorem n times.

22. Prove that the function $f(x) = x^2$ is continuous everywhere.

23. For small values of x, the approximation $\sin x \approx x$ is often used. Estimate the error in using this formula with the aid of Taylor's Theorem. For what range of values of x will this approximation give results correct to six decimal places?

24. For small values of x, how good is the approximation $\cos x \approx 1 - \frac{1}{2}x^2$? For what range of values will this approximation give correct results rounded to three decimal places?

25. Use Taylor's Theorem with $n = 2$ to prove that the inequality $1 + x < e^x$ is valid for all real numbers except $x = 0$.

26. Derive the Taylor series with remainder term for $\ln(1 + x)$ about 1. Derive an inequality that gives the number of terms that must be taken to yield $\ln 4$ with error less than 2^{-m}.

27. What is the third term in the Taylor expansion of $x^2 + x - 2$ about the point 3?

28. Using the series for e^x, how many terms are needed to compute e^2 correctly to four decimal places (rounded)?

29. Develop the Taylor series for $f(x) = \ln x$ about e, writing the results in summation notation and giving the remainder term. Suppose $|x - e| < 1$ and accuracy $\frac{1}{2} \times 10^{-1}$ is desired. What is the minimum number of terms in the series required to achieve this accuracy?

30. Determine the first two terms of the Taylor series for x^x about 1 and the remainder term E_1.

31. Determine the Taylor polynomial of degree 2 for $f(x) = e^{(\cos x)}$ expanded about the point π.

32. First develop the function $\sqrt{x}$ in a series of powers of $(x - 1)$ and then use it to approximate $\sqrt{0.99999\,99995}$ to ten decimal places.

33. Assume that $|x| < \frac{1}{2}$ and determine by Taylor's Theorem the best upper bound.
    a. $|\cos x - (1 - x^2/2)|$
    b. $|\sin x - x(1 - x^2/6)|$

34. Determine a function that can be termed the **linearization** of $x^3 - 2x$ at 2.

35. How many terms are required in the series

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

to give e with an error of at most $6/10$ unit in the 20th decimal place?

36. Find the first two terms in the Taylor expansion of $x^{1/5}$ about the point $x = 32$. Approximate the fifth root of 31.999999 using these two terms in the series. How accurate is your answer?

37. Find the Taylor polynomial of degree 2 for the function $f(x) = e^{2x} \sin x$ expanded about the point $\pi/2$.

38. Determine the Lagrange form of the remainder when Taylor's Theorem is applied to the function $f(x) = \cos x$, with $n = 2$ and $c = \pi/2$. How small must we make $|x - \pi/2|$ if this remainder term is not to exceed $\frac{1}{2} \times 10^{-4}$ in absolute value?

39. An error term of the form $(-1)^n(n+1)^{-1}\xi^{-n-1}(x-1)^{n+1}$ was obtained in the example illustrating Taylor's formula. Compare this to the error term that arises from the integral form of the remainder.

40. Use Taylor's Theorem with Integral Remainder and the Mean-Value Theorem for Integrals to deduce Taylor's Theorem with Lagrange Remainder.

1.2 Orders of Convergence and Additional Basic Concepts

In numerical calculations, especially on high-performance computers, it often happens that the answer to a problem is not produced all at once. Rather, a sequence of approximate answers is produced, usually exhibiting progressively higher accuracy. Convergence of sequences is an important subject that will be taken up again later, such as in Chapter 3 (p. 73). Here we present just a few introductory concepts.

Convergent Sequences

Let us consider an idealized situation in which a single real number is sought as the answer to a problem. It might be, for example, a zero of a complicated equation or the numerical value of an intractable definite integral. In such a case, a computer program may produce a sequence of real numbers $x_1, x_2, x_3, \ldots$ that are *approaching* the correct answer.

We write

$$\lim_{n\to\infty} x_n = L$$

if there corresponds to each positive ε a real number r such that $|x_n - L| < \varepsilon$ whenever $n > r$. (Here n is an integer.)

For example,

$$\lim_{n\to\infty} \frac{n+1}{n} = 1$$

because

$$\left|\frac{n+1}{n} - 1\right| < \varepsilon$$

whenever $n > \varepsilon^{-1}$.

For another example, recall the equation

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$