$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^1 f(x,y)=\left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right)(x,y)$$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x,y)=\left(h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\,\partial y}+k^2\frac{\partial^2 f}{\partial y^2}\right)(x,y)$$

and so on. Letting $f_x=\partial f/\partial x$, $f_y=\partial f/\partial y$, $f_{xx}=\partial^2 f/\partial x^2$, $f_{xy}=\partial^2 f/\partial x\,\partial y$, $f_{yy}=\partial^2 f/\partial y^2$, we can write the first few terms of (5) as

$$f(x+h,y+k)=f+\left(hf_x+kf_y\right)+\tfrac{1}{2}\left(h^2 f_{xx}+2hkf_{xy}+k^2 f_{yy}\right)+\cdots$$

where on the right-hand side the function f and each of the following partial derivatives are evaluated at (x,y).

EXAMPLE 4 What are the first few terms in the Taylor formula for $f(x,y)=\cos(xy)$?

Solution For the given function, we find that

$$\frac{\partial f}{\partial x}=-y\sin(xy) \qquad \frac{\partial f}{\partial y}=-x\sin(xy)$$

$$\frac{\partial^2 f}{\partial x^2}=-y^2\cos(xy) \qquad \frac{\partial^2 f}{\partial x\,\partial y}=-xy\cos(xy)-\sin(xy) \qquad \frac{\partial^2 f}{\partial y^2}=-x^2\cos(xy)$$

Thus, if we let $n=1$ in Taylor's formula (5), the result is

$$\cos[(x+h)(y+k)]=\cos(xy)-hy\sin(xy)-kx\sin(xy)+E_1(h,k)$$

The remainder E_1 is the sum of three terms—namely,

$$-\frac{1}{2}h^2(y+\theta k)^2\cos[(x+\theta h)(y+\theta k)]$$

$$-hk\{(x+\theta h)(y+\theta k)\cos[(x+\theta h)(y+\theta k)]+\sin[(x+\theta h)(y+\theta k)]\}$$

$$-\frac{1}{2}k^2(x+\theta h)^2\cos[(x+\theta h)(y+\theta k)]$$

■

PROBLEMS 1.1

1. Show that $|x^2-4|<\varepsilon$ when $0<|x-2|<\varepsilon(5+\varepsilon)^{-1}$ and prove $\lim_{x\to 2}x^2=4$ by using these inequalities.
2. Show that the function $f(x)=x\sin(1/x)$, with $f(0)=0$, is continuous at 0 but not differentiable at 0.
3. Show that $f(x)=x^2\sin(1/x)$, with $f(0)=0$, is once differentiable at 0 but not twice.

4. Let $f(x) = x^{-3}(x - \sin x)$ for $x \neq 0$. How should $f(0)$ be defined in order that f be continuous? Will it also be differentiable?

5. **a.** Derive the Taylor series at 0 for the function $f(x) = \ln(x + 1)$. Write this series in summation notation. Give two expressions for the remainder when the series is truncated.

b. Determine the smallest number of terms that must be taken in the series to yield ln 1.5 with an error less than 10^{-8}.

c. Determine the number of terms necessary to compute ln 1.6 with error 10^{-10} at most.

6. Determine whether the following function is continuous, once differentiable, or twice differentiable:

$$f(x) = \begin{cases} x^3 + x - 1 & \text{if } x \leq 0 \\ x^3 - x - 1 & \text{if } x > 0 \end{cases}$$

7. (Continuation) Repeat the preceding problem for the function

$$f(x) = \begin{cases} x & \text{if } x \leq 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

8. Criticize this reasoning: The function f defined by

$$f(x) = \begin{cases} x^3 + x & \text{if } x \leq 0 \\ x^3 - x & \text{if } x \geq 0 \end{cases}$$

has the properties

$$\lim_{x \to 0^+} f''(x) = \lim_{x \to 0^+} 6x = 0$$

$$\lim_{x \to 0^-} f''(x) = \lim_{x \to 0^-} 6x = 0$$

Therefore, f'' is continuous.

9. Prove that if f is differentiable at x, then

$$\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h} = f'(x)$$

Show that for some functions that are not differentiable at x, the preceding limit exists. (See Eggermont [1988] or the following problem.)

10. Prove or disprove this assertion: If f is differentiable at x, then for $\alpha \neq 1$,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x+\alpha h)}{h - \alpha h}$$

11. Show that $\lim_{x \to 1}(4x + 2) = 6$ by means of an ε-δ proof.

12. Show that $\lim_{x \to 2}(1/x) = \frac{1}{2}$ by means of an ε-δ proof.

13. For the function $f(x) = 3 - 2x + x^2$ and the interval $[a, b] = [1, 3]$, find the number ξ that occurs in the Mean-Value Theorem.

14. (Continuation) Repeat the preceding problem with the function $f(x) = x^6 + x^4 - 1$ and the interval $[0, 1]$.

15. Find the Taylor series for $f(x) = \cosh x$ about the point $c = 0$.

16. If the series for $\ln x$ is truncated after the term involving $(x - 1)^{1000}$ and is then used to compute ln 2, what bound on the error can be given?

17. Find the Taylor series for $f(x) = e^x$ about the point $c = 3$. Then simplify the series and show how it could have been obtained directly from the series for f about $c = 0$.

18. Let k be a positive integer and let $0 < \alpha < 1$. To what classes $C^n(\mathbb{R})$ does the function $x^{k+\alpha}$ belong?

19. Prove: If $f \in C^n(\mathbb{R})$, then $f' \in C^{n-1}(\mathbb{R})$ and $\int_a^x f(t)\,dt$ belong to $C^{n+1}(\mathbb{R})$.

20. Prove Rolle's Theorem directly (not as a special case of the Mean-Value Theorem).

21. Prove: If $f \in C^n(\mathbb{R})$ and $f(x_0) = f(x_1) = \cdots = f(x_n) = 0$ for $x_0 < x_1 < \cdots < x_n$, then $f^{(n)}(\xi) = 0$ for some $\xi \in (x_0, x_n)$. *Hint:* Use Rolle's Theorem n times.

22. Prove that the function $f(x) = x^2$ is continuous everywhere.

23. For small values of x, the approximation $\sin x \approx x$ is often used. Estimate the error in using this formula with the aid of Taylor's Theorem. For what range of values of x will this approximation give results correct to six decimal places?

24. For small values of x, how good is the approximation $\cos x \approx 1 - \frac{1}{2}x^2$? For what range of values will this approximation give correct results rounded to three decimal places?

25. Use Taylor's Theorem with $n = 2$ to prove that the inequality $1 + x < e^x$ is valid for all real numbers except $x = 0$.

26. Derive the Taylor series with remainder term for $\ln(1 + x)$ about 1. Derive an inequality that gives the number of terms that must be taken to yield $\ln 4$ with error less than 2^{-m}.

27. What is the third term in the Taylor expansion of $x^2 + x - 2$ about the point 3?

28. Using the series for e^x, how many terms are needed to compute e^2 correctly to four decimal places (rounded)?

29. Develop the Taylor series for $f(x) = \ln x$ about e, writing the results in summation notation and giving the remainder term. Suppose $|x - e| < 1$ and accuracy $\frac{1}{2} \times 10^{-1}$ is desired. What is the minimum number of terms in the series required to achieve this accuracy?

30. Determine the first two terms of the Taylor series for x^x about 1 and the remainder term E_1.

31. Determine the Taylor polynomial of degree 2 for $f(x) = e^{(\cos x)}$ expanded about the point π.

32. First develop the function $\sqrt{x}$ in a series of powers of $(x - 1)$ and then use it to approximate $\sqrt{0.99999\,99995}$ to ten decimal places.

33. Assume that $|x| < \frac{1}{2}$ and determine by Taylor's Theorem the best upper bound.

a. $|\cos x - (1 - x^2/2)|$

b. $|\sin x - x(1 - x^2/6)|$

34. Determine a function that can be termed the **linearization** of $x^3 - 2x$ at 2.

35. How many terms are required in the series

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

to give e with an error of at most 6/10 unit in the 20th decimal place?

36. Find the first two terms in the Taylor expansion of $x^{1/5}$ about the point $x = 32$. Approximate the fifth root of 31.999999 using these two terms in the series. How accurate is your answer?

37. Find the Taylor polynomial of degree 2 for the function $f(x) = e^{2x} \sin x$ expanded about the point $\pi/2$.

33. Find a real number x in the range of the Marc-32 such that $\mathrm{fl}(x) = x(1+\delta)$ with $|\delta|$ as large as possible. Can the bound 2^{-24} be attained by $|\delta|$?
34. Show that $\mathrm{fl}(x) = x/(1+\delta)$ with $|\delta| \le 2^{-24}$ under the assumptions made about the Marc-32.
35. Show that $\mathrm{fl}(x^k) = x^k(1+\delta)^{k-1}$, where $|\delta| \le \varepsilon$, if x is a floating-point machine number in a computer that has unit roundoff ε.
36. Show by examples that often $\mathrm{fl}[\mathrm{fl}(xy)z] \ne \mathrm{fl}[x\,\mathrm{fl}(yz)]$ for machine numbers x, y, and z. This phenomenon is often described informally by saying *machine multiplication is not associative.*
37. Prove that if x and y are machine numbers in the Marc-32, and if $|y| \le |x|2^{-25}$, then $\mathrm{fl}(x+y) = x$.
38. Suppose that x is a machine number in the range $-\infty < x < 0$. In IEEE standard arithmetic, what values are returned by the computer for the computations $-\infty + x$, $-\infty * x$, $x/-\infty$, and $-\infty/x$.
39. Evaluate the repeating decimals.
    a. 0.181818 ...
    b. 2.70270 27027 ...
    c. 98.19819 81981 ...
40. Fix an integer N, and call a real number x *representable* if $x = q2^n$, where $1/2 \le q < 1$ and $|n| \le N$. Prove that if $x_1, x_2, \ldots, x_k$ are representable and if their product is representable, then so is uv, where $u = \max(x_i)$ and $v = \min(x_i)$.
41. Consider machine epsilon, defined in the text to be $\varepsilon = \frac{1}{2}\beta^{1-n}$, for correct rounding in a machine operating with base β and carrying n places in its mantissa. Prove that ε is the smallest machine number that satisfies the inequality $\mathrm{fl}(1+\varepsilon) > 1$.

COMPUTER PROBLEMS 2.1

1. Without using the Fortran 90 intrinsic procedures, write a program to *compute* the value of the machine precision ε on your computer in both single and double precision. Is this the exact value or an approximation to it? *Hint:* Determine the smallest positive machine number ε of the form 2^{-k} such that $1.0 + \varepsilon \ne 1.0$.
2. (Continuation) Repeat for the largest and smallest machine numbers.
3. Students are sometimes confused about the difference between the numbers `tiny` and `epsilon`. Explain the difference. Then design and execute a numerical computer experiment to demonstrate the difference.
4. By repeated division by 2 and printing the results, you may observe that it seems that real numbers smaller than the smallest machine number `tiny` are obtained. Explain why this is not so and what is happening.
5. Obviously, one can determine the sign of a real number by examining a single bit in the computer representation of it. Similarly, one can determine whether an integer is even or odd from a single bit. Explain why this is so. Design a computer experiment to illustrate this situation.
6. Set `X = 1.0/3.0` and print out both the internal machine representation and the decimal number stored in the computer that corresponds to it. Use a large format field for the decimal number. Explain and discuss the results.