approach $g(x_1)$. The condition (6) is weaker than having a bounded derivative. Indeed, if $g'(x)$ exists everywhere and does not exceed L in modulus, then by the Mean-Value Theorem,

$$|g(x_1) - g(x_2)| = |g'(\xi)|\,|x_1 - x_2| \le L|x_1 - x_2|$$

EXAMPLE 2 Show that the function $g(x) = \sum_{i=1}^{n} a_i|x - w_i|$ satisfies a Lipschitz condition with the constant $L = \sum_{i=1}^{n} |a_i|$.

Solution

$$\begin{aligned}|g(x_1) - g(x_2)| &= \left|\sum_{i=1}^{n} a_i|x_1 - w_i| - \sum_{i=1}^{n} a_i|x_2 - w_i|\right| \\ &= \left|\sum_{i=1}^{n} a_i\left\{|x_1 - w_i| - |x_2 - w_i|\right\}\right| \\ &\le \sum_{i=1}^{n} |a_i|\,\big||x_1 - w_i| - |x_2 - w_i|\big| \\ &\le \sum_{i=1}^{n} |a_i||x_1 - x_2| = L|x_1 - x_2|\end{aligned}$$

■

PROBLEMS 8.1

1. Find two solutions of the initial-value problem

$$\begin{cases} x' = x^{1/3} \\ x(0) = 0 \end{cases}$$

Hint: Try $x = ct^{\lambda}$, or observe that the equation is separable.

2. **a.** Use Theorem 1, on initial-value problem existence, to predict in what interval a solution of the initial-value problem (3) exists. Find the largest interval.

   b. Repeat Part **a** for the initial-value problem (2).

3. Show that $x = -t^2/4$ and $x = 1 - t$ are solutions of the initial-value problem

$$\begin{cases} 2x' = \sqrt{t^2 + 4x} - t \\ x(2) = -1 \end{cases}$$

Why does this not contradict Theorem 2, on initial-value problem uniqueness?

4. Solve the initial-value problem $x' = f(t, x)$, $x(0) = 0$ in these special cases:

   a. $f(t, x) = t^3$

   b. $f(t, x) = (1 - t^2)^{-1/2}$

c. $f(t, x) = (1 + t^2)^{-1}$

d. $f(t, x) = (t + 1)^{-1}$

5. Solve the initial-value problem $x' = f(t, x)$, $x(0) = 0$ in the following cases. Use the fact that $dt/dx = (dx/dt)^{-1}$ when $dx/dt \neq 0$.

a. $f(t, x) = x^{-2}$

b. $f(t, x) = 1 + x^2$

c. $f(t, x) = (\sin x + \cos x)^{-1}$

6. Use Theorem 1, on initial-value problem existence, to show that the initial-value problem

$$\begin{cases} x' = \sqrt{|x|} \\ x(0) = 0 \end{cases}$$

has a solution on the entire real line.

7. Show by using Theorem 1, on initial-value problem existence, that the initial-value problem

$$\begin{cases} x' = \tan x \\ x(0) = 0 \end{cases}$$

has a solution in the interval $|t| < \pi/4$.

8. Let f be a continuous function of one variable, defined on all of $\mathbb{R}$. Let $M(r)$ denote the maximum of $|f(x)|$ for $|x| \leq r$. If $M(r) = o(r)$ as $r \to \infty$, then the initial-value problem

$$\begin{cases} x' = f(x) \\ x(0) = 0 \end{cases}$$

has a solution on all of $\mathbb{R}$. Prove this assertion.

9. Prove that the initial-value problem

$$\begin{cases} x' = t^2 + e^x \\ x(0) = 0 \end{cases}$$

has a unique solution in the interval $|t| \leq 0.351$.

10. Prove that if $f(t, x)$ is continuous and bounded in the domain $a \leq t \leq b$, $-\infty < x < \infty$, then the initial-value problem

$$\begin{cases} x' = f(t, x) \\ x(a) = \alpha \end{cases}$$

has a solution in the interval $a \leq t \leq b$.

11. Let R denote the rectangle in the tx-plane defined by $|t - t_0| \leq \alpha$, $|x - x_0| \leq \beta$. Let f be a continuous function defined on this rectangle and satisfying $\beta \geq \alpha|f(t, x)|$. Prove that the initial-value problem $x' = f(t, x)$, $x(t_0) = x_0$ has a solution on the interval $|t - t_0| \leq \alpha$.

12. Establish that the initial-value problem

$$\begin{cases} x' = 1 + x + x^2 \cos t \\ x(0) = 0 \end{cases}$$

has a solution in the interval $-1/3 \leq t \leq 1/3$.

COMPUTER PROBLEM 8