Show that, for example,

$$P_4(t) = 12 - 48t^2 + 16t^4$$
$$P_5(t) = -120t + 160t^3 - 32t^5$$

2. Verify that the function $x(t) = t^2/4$ solves the initial-value problem

$$\begin{cases} x' = \sqrt{x} \\ x(0) = 0 \end{cases}$$

Apply the Taylor-series method of order 1, and explain why the numerical solution differs from the solution $t^2/4$.

3. Compute $x(0.1)$ by solving the differential equation

$$\begin{cases} x' = -tx^2 \\ x(0) = 2 \end{cases}$$

with one step of the Taylor-series method of order 2. (Use a calculator.)

4. Using the ordinary differential equation

$$\begin{cases} x' = x^2 + xe^t \\ x(0) = 1 \end{cases}$$

and one step of the Taylor-series method of order 3, calculate $x(0.01)$.

5. Consider the ordinary differential equation

$$\begin{cases} 5tx' + x^2 = 2 \\ x(4) = 1 \end{cases}$$

Calculate $x(4.1)$ using one step of the Taylor-series method of order 2.

6. An **integral equation** is an equation involving an unknown function within an integral. For example, here is a typical integral equation (of a type known by the name Volterra)

$$x(t) = \int_0^t \cos(s + x(s))\, ds + e^t$$

By differentiating this integral equation, obtain an equivalent initial-value problem for the unknown function.

7. If the Taylor-series method is used to solve an initial-value problem involving the differential equation

$$x' = \cos(tx)$$

what are the formulas for x'', x''', and $x^{(4)}$?

8. Let $x' = f(t, x)$. Determine x'', x''', and $x^{(4)}$ from this equation.

COMPUTER PROBLEMS 8.2

1. Write and test a computer program to solve the following differential equation with ... condition

$$\begin{cases} x' = x + e^t + tx \\ x(1) = 2 \end{cases}$$

on the interval $[1, 3]$. Use the Taylor series of order 5 and $h = 0.01$.

2. Write and test a computer program for solving the following initial-value problem:

$$\begin{cases} x' = 1 + x^2 - t^3 \\ x(0) = -1 \end{cases}$$

Use the Taylor-series method of order 4, with h a binary machine number near 0.01. Find the solution in the interval $[0, 2]$. Account for any peculiar phenomenon in the solution.

3. Methods for solving initial-value problems also can be used to compute definite or indefinite integrals. For example, we can compute

$$\int_0^2 e^{-s^2}\,ds$$

by solving the initial-value problem

$$\begin{cases} x' = e^{-t^2} \\ x(0) = 0 \end{cases}$$

on the t-interval $[0, 2]$. Do this, using the Taylor-series method of order 4. From a table of the **error function**

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2}\,ds$$

we obtain $x(2) \approx 0.88208\,13907$. (See Abramowitz and Stegun [1964, p. 311].)

4. (Continuation) Use Problem **8.2.1** (p. 535) to write a computer program for the Taylor-series method of order 6 applicable to the integral in the preceding computer problem. Test the code with $h = 0.01$ to see whether the same value of $x(2)$ is obtained. Print a table of the function $\frac{1}{2}\sqrt{\pi}\operatorname{erf}(t)$ in steps of 0.01 from $t = 0$ to $t = 2$.

5. Solve the initial-value problem

$$\begin{cases} x' = 1 + x^2 \\ x(0) = 0 \end{cases}$$

on the interval $[0, 1.56]$ using the Taylor-series method of order 4 with $h = 0.01$. Then use the computed value of $x(1.56)$ as the initial value to integrate back to $t = 0$. Compare the results and explain what happened.

6. The equation $\arctan(x/t) = \ln\sqrt{x^2 + t^2}$ defines x implicitly as a function of t. Verify that this implicit function is a solution of the initial-value problem

$$\begin{cases} x' = (t + x)/(t - x) \\ x(1) = 0 \end{cases}$$

Prepare a table of the function $x(t)$ on the interval $[0, 2]$ with steps of ± 0.01. Use the Taylor-series method of order 4.

7. The function

$$\varphi(t) = \int_0^t \sin s^2\,ds$$