c. $(2E^6 - 9E^5 + 12E^4 - 4E^3)x = 0$

d. $(\pi E^2 - \sqrt{2}E + \log 2 \cdot E^0)x = 0$

12. Prove that if p is a polynomial with real coefficients, and if $z \equiv [z_1, z_2, \dots]$ is a (complex) solution of $p(E)z = 0$, then the conjugate of z, the real part of z, and the imaginary part of z are also solutions.

13. Solve.

a. $x_{n+1} - nx_n = 0$

b. $x_{n+1} - x_n = n$

c. $x_{n+1} - x_n = 2$

14. Define an operator Δ by putting

$$\Delta x = [x_2 - x_1, x_3 - x_2, x_4 - x_3, \dots]$$

Show that $E = I + \Delta$. Show that if p is a polynomial, then

$$p(E) = p(I) + p'(I)\Delta + \frac{1}{2}p''(I)\Delta^2 + \frac{1}{3!}p'''(I)\Delta^3 + \cdots + \frac{1}{m!}p^{(m)}(I)\Delta^m$$

15. (Continuation) Prove that if $x = [\lambda, \lambda^2, \lambda^3, \dots]$ and p is a polynomial, then $p(\Delta)x = p(\lambda - 1)x$. Describe how to solve a difference equation written in the form $p(\Delta)x = 0$.

16. (Continuation) Show that

$$\Delta^n = (-1)^n\left[E^0 - nE + \tfrac{1}{2}n(n-1)E^2 - \tfrac{1}{3!}n(n-1)(n-2)E^3 + \cdots + (-1)^n E^n\right]$$

17. Give a complete proof of Theorem 2.

18. Let p be a polynomial such that $p(0) = 0$. Describe the null space of $p(E)$.

19. For $\lambda \in \mathbb{C}$, define $x(\lambda) = [\lambda, \lambda^2, \lambda^2, \dots]$. Prove that if $\lambda_1, \lambda_2, \dots, \lambda_m$ are distinct nonzero complex numbers, then $\{x(\lambda_1), x(\lambda_2), \dots, x(\lambda_m)\}$ is a linearly independent set in V.

20. Prove that if λ is a nonzero root of p having multiplicity k, then the equation $p(E)x = 0$ has solutions $u^{(1)}, u^{(2)}, \dots, u^{(k)}$ in which $u_n^{(j)} = n^{j-1}\lambda^n$.

21. Prove that if $\mu \in (0, \infty)$ and $|\lambda| < 1$, then $\lim_{n\to\infty} n^{\mu}\lambda^n = 0$.

22. Prove in detail that a convergent sequence is bounded.

23. Prove Theorem 3 without the hypothesis that $p(0) \neq 0$.

24. Define a sequence inductively by the equation $x_{n+1} = x_n + x_n^{-1}$, where $x_0 > 0$. Determine the behavior of x_n as $n \to \infty$.

25. Determine whether the difference equation $x_n = x_{n-1} + x_{n-2}$ is stable.

26. Prove that if x is a solution of the difference equation $p(E)x = 0$, then so is Ex.

27. Consider the recurrence relation $x_n = 2(x_{n-1} + x_{n-2})$. Show that the general solution is $z_n = \alpha(1+\sqrt{3})^n + \beta(1-\sqrt{3})^n$. Show that the solution with starting values $x_1 = 1$ and $x_2 = 1 - \sqrt{3}$ corresponds to $\alpha = 0$ and $\beta = (1-\sqrt{3})^{-1}$.

COMPUTER PROBLEMS 1.3

1. Consider the difference equation $x_{n+2} - 2x_{n+1} - 2x_n = 0$. One of its solutions is $x_n = (1-\sqrt{3})^{n-1}$. This solution oscillates in sign and converges to 0. Compute and print out the first 100 terms of this sequence by use of the equation $x_{n+2} = 2(x_{n+1} + x_n)$ starting with $x_1 = 1$ and $x_2 = 1 - \sqrt{3}$. Explain the curious phenomenon that occurs.