THEOREM 6 Theorem on Global Truncation Error Approximation

If the local truncation errors in the numerical solution are $\mathcal{O}(h^{m+1})$, then the global truncation error is $\mathcal{O}(h^m)$.

Proof In Theorem 5, on global truncation error bound, let δ be $\mathcal{O}(h^{m+1})$. Since $e^z - 1$ is $\mathcal{O}(z)$ and $nh = t$, we find a decrease of 1 in the order from using the formula in Theorem 5. ■

PROBLEMS 8.5

1. Discuss these multistep methods in light of Theorem 1 (p. 558), on multistep method stability and consistency:

   a. $x_n - x_{n-2} = 2hf_{n-1}$

   b. $x_n - x_{n-2} = h\left[\frac{7}{3}f_{n-1} - \frac{2}{3}f_{n-2} + \frac{1}{3}f_{n-3}\right]$

   c. $x_n - x_{n-1} = h\left[\frac{3}{8}f_n + \frac{19}{24}f_{n-1} - \frac{5}{24}f_{n-2} + \frac{1}{24}f_{n-3}\right]$

2. A method is said to be **weakly unstable** if p has a zero λ such that $\lambda \neq 1$, $|\lambda| = 1$, and $q(\lambda) < \lambda p'(\lambda)$. Show that the Milne method given by Equation (12) is weakly unstable.

3. Show that every multistep method in which $p(z) = z^k - z^{k-1}$ and $\sum_{i=0}^{k} b_i = 1$ is stable, consistent, convergent, and weakly stable.

4. Determine the numerical characteristics of the multistep method whose equation is

$$x_n + 4x_{n-1} - 5x_{n-2} = h\,[4f_{n-1} + 2f_{n-2}]$$

5. Is there any reason for distrusting this numerical scheme for solving $x' = f(t, x)$?

$$x_n - 3x_{n-1} + 2x_{n-2} = h\,[f_n + 2f_{n-1} + f_{n-2} - 2f_{n-3}]$$

   Explain.

6. Which of these multistep methods is *convergent*?

   a. $x_n - x_{n-2} = h(f_n - 3f_{n-1} + 4f_{n-2})$

   b. $x_n - 2x_{n-1} + x_{n-2} = h(f_n - f_{n-1})$

   c. $x_n - x_{n-1} - x_{n-2} = h(f_n - f_{n-1})$

   d. $x_n - 3x_{n-1} + 2x_{n-2} = h(f_n + f_{n-1})$

   e. $x_n - x_{n-2} = h(f_n - 3f_{n-1} + 2f_{n-2})$

7. A multistep method is said to be **strongly stable** if $p(1) = 0$, $p'(1) \neq 0$, and all other roots of p satisfy the inequality $|z| < 1$. Prove that a strongly stable method is convergent, using Theorem 1, on multistep stability and consistency. Prove also that for any value of λ, a strongly stable method will solve the problem $x' = \lambda x$, $x(0) = 1$ without introducing any extraneous errors of exponential growth.

8. Prove that

$$\frac{Ah^{m+1} + \mathcal{O}(h^{m+2})}{B - Ch} = \frac{A}{B}h^{m+1} + \mathcal{O}(h^{m+2})$$