or

$$(1 + 4a^2\lambda^2 \sin^2 \tfrac{1}{2}\theta \cos^2 \tfrac{1}{2}\theta) \cos^4 \tfrac{1}{2}\theta \leq 1,$$

which is equivalent to

$$\begin{aligned} 4a^2\lambda^2 \sin^2 \tfrac{1}{2}\theta \cos^2 \tfrac{1}{2}\theta \cos^4 \tfrac{1}{2}\theta &\leq 1 - \cos^4 \tfrac{1}{2}\theta \\ &= (1 - \cos^2 \tfrac{1}{2}\theta)(1 + \cos^2 \tfrac{1}{2}\theta) \\ &= \sin^2 \tfrac{1}{2}\theta (1 + \cos^2 \tfrac{1}{2}\theta). \end{aligned}$$

Canceling the common nonnegative factor of $\sin^2 \frac{1}{2}\theta$, we obtain the condition

$$4a^2\lambda^2 \cos^6 \tfrac{1}{2}\theta \leq 1 + \cos^2 \tfrac{1}{2}\theta,$$

which must hold for all values of θ. We first consider the particular case of θ equal to 0, obtaining the necessary condition that

$$a^2\lambda^2 \leq \tfrac{1}{2}. \tag{2.2.14}$$

We now show that this condition is also sufficient, i.e., that θ equal to 0 is the "worst case." Assuming that (2.2.14) holds, and using the fact that $\cos^2 \frac{1}{2}\theta$ is at most 1, we have

$$4a^2\lambda^2 \cos^6 \tfrac{1}{2}\theta \leq 2\cos^2 \tfrac{1}{2}\theta \leq 1 + \cos^2 \tfrac{1}{2}\theta .$$

Thus the forward-time central-space scheme with the smoother (2.2.13) is stable if and only if

$$|a\lambda| \leq \frac{1}{\sqrt{2}}.$$

This scheme is not recommended for use in actual computation. For example, it requires more work per time step than does the Lax–Friedrichs scheme, and the time-step limitation is more severe. The forward-time central-space scheme (1.3.3), without the smoother, is unstable; see Example 2.2.2. □

Example 2.2.6. An interesting example of the relation between consistency and stability is a scheme for the equation

$$u_t + au_{xxx} = f \tag{2.2.15}$$

obtained by applying the ideas of the Lax–Friedrichs scheme (1.3.5). The scheme is

$$v_m^{n+1} = \tfrac{1}{2}(v_{m+1}^n + v_{m-1}^n) - \tfrac{1}{2}akh^{-3}(v_{m+2}^n - 2v_{m+1}^n + 2v_{m-1}^n - v_{m-2}^n) + kf_m^n. \tag{2.2.16}$$

This scheme is consistent with equation (2.2.15) if $k^{-1}h^2$ tends to zero as h and k tend to zero; see Exercise 2.2.3. This is similar to the result for the Lax–Friedrichs scheme as discussed in Example 1.4.2.

The amplification factor for the scheme (2.2.16) is

$$g(\theta) = \cos\theta + 4akh^{-3} i \sin\theta \sin^2 \tfrac{1}{2}\theta,$$

and it is easily shown (see Exercise 2.2.3) that the scheme is stable only if

$$4|a|kh^{-3}$$

is bounded.

The consistency condition, that $k^{-1}h^2$ tend to zero, and the stability condition, that $4akh^{-3}$ be bounded as k and h tend to zero, cannot both be satisfied. Thus, this scheme is not a convergent scheme, since it cannot be both consistent and stable. □

Exercises

2.2.1. Show that the backward-time central-space scheme (1.6.1) is consistent with equation (1.1.1) and is unconditionally stable.

2.2.2. Show that if one takes $\lambda = k^{1/2}$, i.e., $k = h^2$, then the forward-time central-space scheme (1.3.3) is stable and consistent with equation (1.1.1). (See Example 2.2.2.)

2.2.3. Verify the consistency and stability conditions of scheme (2.2.16) as given in Example 2.2.6.

2.2.4. Show that the box scheme

$$\frac{1}{2k}\left[(v_m^{n+1} + v_{m+1}^{n+1}) - (v_m^n + v_{m+1}^n)\right] + \frac{a}{2h}\left[(v_{m+1}^{n+1} - v_m^{n+1}) + (v_{m+1}^n - v_m^n)\right] = f_m^n$$

is consistent with the one-way wave equation $u_t + au_x = f$ and is stable for all values of λ.

2.2.5. Show that the scheme

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+2}^n - 3v_{m+1}^n + 3v_m^n - v_{m-1}^n}{h^3} = f_m^n$$

is consistent with the equation (2.2.15) and, if $\nu = kh^{-3}$ is constant, then it is stable when $0 \le a\nu \le 1/4$.

2.2.6. Determine the stability of the following scheme, sometimes called the Euler backward scheme, for $u_t + au_x = f$:

$$v_m^{n+1/2} = v_m^n - \frac{a\lambda}{2}(v_{m+1}^n - v_{m-1}^n) + kf_m^n,$$
$$v_m^{n+1} = v_m^n - \frac{a\lambda}{2}(v_{m+1}^{n+1/2} - v_{m-1}^{n+1/2}) + kf_m^{n+1}.$$

The variable $v^{n+1/2}$ is a temporary variable, as is $\tilde{v}$ in Example 2.2.5.