For the Crank-Nicolson scheme we obtain

$$g(\theta) = \frac{1 - i\frac{1}{2}a\lambda\sin\theta}{1 + i\frac{1}{2}a\lambda\sin\theta}$$

so

$$|g(\theta)|^2 = \frac{1 + (\frac{1}{2}a\lambda\sin\theta)^2}{1 + (\frac{1}{2}a\lambda\sin\theta)^2} = 1.$$

This scheme is stable for any value of λ; it is said to be *unconditionally stable.*

EXERCISES 3.1

1. Show that the (forward-backward) MacCormack scheme

$$\tilde{v}_m^{n+1} = v_m^n - a\lambda(v_{m+1}^n - v_m^n) + kf_m^n$$

$$v_m^{n+1} = \tfrac{1}{2}(v_m^n + \tilde{v}_m^{n+1} - a\lambda(\tilde{v}_m^{n+1} - \tilde{v}_{m-1}^{n+1}) + kf_m^{n+1})$$

is a second-order accurate scheme for the one-way wave equation (1.1.1). Show that for $f = 0$ it is identical with the Lax-Wendroff scheme (3.1.2).

2. Solve $u_t + u_x = 0$, $-1 \le x \le 1$, $0 \le t \le 1.2$ with $u(0,x) = \sin 2\pi x$ and periodic boundary conditions, i.e., $u(t,1) = u(t,-1)$. Use two methods:

(a) Forward-time backward-space with $\lambda = 0.8$
(b) Lax-Wendroff with $\lambda = 0.8$

Demonstrate the first-order accuracy of the solution of (a) and the second-order accuracy of the solution of (b) using $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}$, and $\frac{1}{80}$. Measure the error in the L^2 norm and maximum norm. The L^2 norm is

$$\left(h\sum_m |v_m^n - u(t_n, x_m)|^2\right)^{1/2},$$

where the sum is over all the grid points. (Do not sum both the grid point at $x = -1$ and $x = 1$ as separate points.)

3. Show that the box scheme

$$\begin{aligned}
&\frac{1}{2k}\left[(v_m^{n+1} + v_{m+1}^{n+1}) - (v_m^n + v_{m+1}^n)\right] \\
&+ \frac{a}{2h}\left[(v_{m+1}^{n+1} - v_m^{n+1}) + (v_{m+1}^n - v_m^n)\right] \qquad (3.1.23)\\
&= \frac{1}{4}\left(f_{m+1}^{n+1} + f_m^{n+1} + f_{m+1}^n + f_m^n\right)
\end{aligned}$$

an approximation to the one-way wave equation $u_t + au_x = f$ that is accurate order $(2,2)$ and is stable for all values of λ.

4. Using the box scheme (3.1.23), solve the one-way wave equation $u_t + u_x = \sin(x - t)$ on the interval $[0, 1]$ for $0 \le t \le 1.2$ with $u(0, x) = \sin x$ and with $u(t, 0) = -(1 + t)\sin t$ as the boundary condition.

Demonstrate the second-order accuracy of the solution using $\lambda = 1.2$ and $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}$, and $\frac{1}{80}$. Measure the error in the L^2 norm and maximum norm. The L^2 norm is

$$\left(h \sum_m |v_m^n - u(t_n, x_m)|^2 \right)^{1/2},$$

where the sum is over all the grid points.

Demonstrate that the box scheme of Exercise 2.2.4 is only first-order accurate. To implement the box scheme note that v_0^{n+1} is given by the boundary data, and then each value of v_{m+1}^{n+1} can be determined from v_m^{n+1} and the other values.

5. Solve the equation

$$u_t + u_x = \cos^2 u$$

with the scheme (3.1.7), treating the $\cos^2 u$ term as $f(t, x)$. Show that the scheme is first-order accurate. The exact solution is given by

$$\tan\left[u(t, x)\right] = \tan\left[u_0(x - t)\right] + t.$$

Use a smooth function, such as $\sin(x - t)$, as initial data and boundary data.

6. Modify the scheme of Exercise 5 to be second-order accurate and explicit. There are several ways to do this. One way uses $\cos^2 v_m^{n+1} = \cos^2 v_m^n - \sin 2v_m^n \left(v_m^{n+1} - v_m^n\right) + O\left(k^2\right)$. Another way is to evaluate explicitly the f_t term in the derivation of the Lax-Wendroff scheme and eliminate all derivatives with respect to t using the equation.

7. Determine the order of accuracy of the Euler backward scheme in Exercise 2.2.6.

8. Determine the accuracy and stability of the following scheme for $u_t + u_{tx} + au_x = f$.

$$v_m^{n+1} + \left(1 + \frac{ka}{2}\right) \frac{v_{m+1}^{n+1} - v_{m-1}^{n+1}}{2h} = v_m^n + \left(1 - \frac{ka}{2}\right) \frac{v_{m+1}^n - v_{m-1}^n}{2h} + \frac{k}{2}(f_m^{n+1} + f_m^n)$$

Hint: This scheme is somewhat symmetric about time level $n + \frac{1}{2}$, much like the Crank-Nicolson scheme.

9. Using equations (3.1.18), (3.1.19), and (3.1.20), show that the Lax-Wendroff scheme is the only explicit one-step second-order accurate scheme that uses only the grid points x_{m-1}, x_m, and x_{m+1} to compute the solution at x_m for the next time step.