```
        compute e        [Equation (13)]
            t ← t + h
            output k, t, x, e
        if iflag = 0 then stop
        if |e| ≥ δ then
                t ← s
                h ← h/2
                x ← y
                k ← k − 1
        else
            if |e| < δ/128 then h ← 2h
        end if
end do
```

Embedded Runge-Kutta procedures are composed of a pair of formulas of orders p and q (usually $q = p + 1$) that share the same function evaluation points. Generally they provide an efficient technique for solving nonstiff initial-value problems. Many higher-order embedded Runge-Kutta formulas have been derived in recent years. Examples of some of these are given in the problems. Although higher-order Runge-Kutta methods with complicated coefficients have been derived, the classic fourth-order formula remains the most popular except for adaptive schemes. Additional information on Runge-Kutta methods can be found in numerous sources—for example, Butcher [1987], Fehlberg [1969], Gear [1971], Jackson, Enright, and Hull [1978], Prince and Dormand [1981], Shampine and Gordon [1975], Thomas [1986], and Verner [1978].

PROBLEMS 8.3

1. Write out the second-order Runge-Kutta formulas when $\alpha = 2/3$.
2. If an initial-value problem involving the differential equation $x + 2tx' + xx' = 3$ is to be solved using a Runge-Kutta method, what function must be programmed?
3. Prove that the Runge-Kutta formula (8) is of order 4 in the special case that $f(t, x)$ is independent of x. Show that in this case the Runge-Kutta formula is equivalent to Simpson's rule. (See Equation (6) in Section 7.2, p. 483.)
4. Derive the **modified Euler's method**,

$$x(t + h) = x(t) + hf\big(t + \tfrac{1}{2}h, x(t) + \tfrac{1}{2}hf\big(t, x(t)\big)\big)$$

   by performing Richardson's extrapolation on Euler's method using step sizes h and $h/2$. *Hint:* Assume the error term is Ch^2.
5. Derive the **third-order Runge-Kutta formula**

$$x(t + h) = x(t) + \tfrac{1}{9}(2F_1 + 3F_2 + 4F_3)$$

   where

$$\begin{cases} F_1 = hf(t, x) \\ F_2 = hf(t + \frac{1}{2}h, x + \frac{1}{2}F_1) \\ F_3 = hf(t + \frac{3}{4}h, x + \frac{3}{4}F_2) \end{cases}$$

Show that it agrees with the Taylor-series method of order 3 for the differential equation $x' = x + t$.

6. Prove that when the fourth-order Runge-Kutta method is applied to the problem $x' = \lambda x$, the formula for advancing this solution will be

$$x(t+h) = \left[1 + h\lambda + \tfrac{1}{2}h^2\lambda^2 + \tfrac{1}{6}h^3\lambda^3 + \tfrac{1}{24}h^4\lambda^4\right]x(t)$$

7. (Continuation) Prove that the local truncation error in the preceding problem is $\mathcal{O}(h^5)$.

COMPUTER PROBLEMS 8.3

1. Write a computer program to solve an initial-value problem $x' = f(t, x)$ with $x(t_0) = x_0$ on an interval $t_0 \le t \le t_m$ or $t_m \le t \le t_0$. Use the fourth-order Runge-Kutta method. Test it on this example:

$$\begin{cases} (e^t + 1)x' + xe^t - x = 0 \\ x(0) = 3 \end{cases}$$

Determine the *analytic* solution and compare it to the computed solution on the interval $-2 \le t \le 0$. Use $h = -0.01$.

2. Using various values of λ, such as 5, -5, or -10, numerically solve the following initial-value problem using the fourth-order Runge-Kutta method:

$$\begin{cases} x' = \lambda x + \cos t - \lambda \sin t \\ x(0) = 0 \end{cases}$$

Compare the numerical solution to the analytic solution on the interval $[0, 5]$. Use step size $h = 0.01$. What effect does λ have on the numerical accuracy?

3. Program and test the Runge-Kutta-Fehlberg algorithm described in the text. For test cases, use the examples in Computer Problems **8.2.7**, **11**, **15** (pp. 537–539).

4. Write and execute a program to solve this initial-value problem:

$$\begin{cases} x' = e^{xt} + \cos(x - t) \\ x(1) = 3 \end{cases}$$

Use the fourth-order Runge-Kutta formulas with $h = 0.01$. Stop the computation just *before* the solution overflows.

5. Solve $x' = x^2$, $x(0) = 1$ on the interval $[0, 2]$ using the adaptive Runge-Kutta-Fehlberg procedure. Show that the true solution is $x(t) = 1/(1-t)$. What happens in the algorithm near the discontinuity at $t = 1$?

6. Numerically compare the following **fourth-order Runge-Kutta-Gill method** with the classical Runge-Kutta method:

$$x(t+h) = x(t) + \tfrac{1}{6}\left[F_1 - \left(2 + \sqrt{2}\right)F_2 + \left(2 + \sqrt{2}\right)F_3 + F_4\right]$$

where

$$\begin{cases} F_1 = hf(t, x) \\ F_2 = hf\left(t + \tfrac{1}{2}h, x + \tfrac{1}{2}F_1\right) \\ F_3 = hf\left(t + \tfrac{1}{2}h, x + \tfrac{1}{2}\left(\sqrt{2} - 1\right)F_1 + \tfrac{1}{2}\left(2 - \sqrt{2}\right)F_2\right) \\ F_4 = hf\left(t + h, x - \tfrac{1}{2}\sqrt{2}F_2 + \tfrac{1}{2}\left(2 + \sqrt{2}\right)F_3\right) \end{cases}$$