PROBLEMS 8.4

1. Derive the first-order (one-step) Adams-Moulton formula and verify that it is equivalent to the trapezoid rule. (See Section 7.2, p. 481.)
2. Verify the correctness of the system of equations in (6).
3. Use the method of undetermined coefficients to derive Equation (8).
4. Use the method of undetermined coefficients to derive the **fourth-order Adams-Bashforth formula**

$$x_{n+1} = x_n + \frac{h}{24}[55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}]$$

5. Derive the **fourth-order Adams-Moulton formula**

$$x_{n+1} = x_n + \frac{h}{24}[9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2}]$$

6. Prove that if every element of Π_m is correctly integrated by the formula

$$\int_0^1 f(x)\,dx \approx \sum_{i=-n}^{n} A_i f(i)$$

then the same is true of the formula

$$\int_{t_0}^{t_0+h} f(x)\,dx \approx h \sum_{i=-n}^{n} A_i f(t_0 + ih)$$

7. Derive the **second-order Adams-Bashforth formula**

$$x_{n+1} = x_n + h\left[\tfrac{3}{2} f_n - \tfrac{1}{2} f_{n-1}\right]$$

8. **a.** Using the method of undetermined coefficients, determine A and B in the following Adams-Bashforth formula. (Do not change to a corresponding numerical integration formula.)

$$x_{n+1} = x_n + h[A f_n + B f_{n-1}]$$

   b. Repeat Part **a** by changing it to a corresponding numerical integration formula.

   c. Repeat the problem by integrating the Newton interpolation polynomial based on the nodes x_n and x_{n-1}.

9. **a.** Use the method of undetermined coefficients to find A and B in the implicit Adams-Moulton formula of second order

$$x_{n+1} = x_n + h[A f_{n+1} + B f_n]$$

   b. Use a numerical integration formula to derive the formula.

   c. Use an interpolation formula to derive the formula.

10. **a.** Use the method of undetermined coefficients to derive a multistep formula of the form

$$x_{n+1} = x_n + h[A f_{n+1} + B f_n + C f_{n-1}]$$

   b. Repeat Part **a** for

$$x_{n+1} = x_n + h[A f_n + B f_{n-1} + C f_{n-2}]$$

11. Derive an implicit multistep formula based on Simpson's rule (involving uniformly spaced points x_{n-1}, x_n, x_{n+1}) for numerically solving the ordinary differential equation $x' = f$.

12. The formula

$$x_{n+1} = (1 - A)x_n + Ax_{n-1} + \frac{h}{12}\left[(5 - A)x'_{n+1} + 8(1 + A)x'_n + (5A - 1)x'_{n-1}\right]$$

is known to be exact for all polynomials of degree m or less for all A. Determine A so that it will be exact for all polynomials of degree $m + 1$. Find A and m.

13. Compute the coefficients in a multistep method of the form

$$x_{n+1} = x_n + h[Af_n + Bf_{n-2} + Cf_{n-4}]$$

The formula should correctly integrate an equation $x' = f(t, x)$ when the right-hand side is of the form $f(t, x) = a + bt + ct^2$.

14. Find a method of order 4 in the form of Equation (9), taking $k = 2$.

15. Prove that the multistep method of Equation (9) is of order m if and only if $Lp_i = 0$ for $0 \le i \le m$ and $Lp_{m+1} \ne 0$, where $p_i(t) = t^i$ and L is as given in Equation (10).

16. (Continuation) Determine the order of this method by the procedure in the preceding problem:

$$x_n = x_{n-2} + 2hf_{n-1}$$

17. Determine the order of this method by computing $d_0, d_1, \ldots$:

$$x_n = x_{n-3} + \tfrac{3}{8}h[f_n + 3f_{n-1} + 3f_{n-2} + f_{n-3}]$$

18. Let $a_0, a_1, \ldots, a_m$ be given, and assume that $\sum_{i=0}^{k} a_i = 0$. Do there necessarily exist $b_0, b_1, \ldots, b_m$ so that the multistep method having these coefficients will be of order at least m? (A theorem or an example is required.)

19. Prove that for Euler's method $d_0 = d_1 = 0$ and $d_j = 1/j!$ for $j \ge 2$.

COMPUTER PROBLEMS 8.4

1. Write and test a subprogram or procedure for the fifth-order Adams-Bashforth-Moulton method coupled with a fifth-order Runge-Kutta method (see Computer Problems **8.3.7–9**, p. 548). Keep only the five most recent values of (t_i, x_i). Print statements should be included in the routine.

2. Write a computer program that calls the routine in the preceding computer problem and solves the initial-value problem

$$\begin{cases} x' = (t - e^{-t})/(x + e^x) \\ x(0) = 0 \end{cases}$$

on the interval $[-1, 1]$. Use $h = 1/238$. Verify (analytically) that the true solution is given implicitly by the equation $x^2 - t^2 + 2e^x - 2e^{-t} = 0$. Use this equation in the program to provide a check on the computed solution. Use the Runge-Kutta method to get started.

3. Compute the solution of

$$\begin{cases} y' = -2xy^2 \\ y(0) = 1 \end{cases}$$

at $x = 1.0$ using $h = 0.25$ and the fourth-order Adams-Bashforth-Moulton method (Problems **8.4.4–5**, p. 555) together with the fourth-order Runge-Kutta method. Give the computed solution to five significant digits at 0.25, 0.5, 0.75, and 1.0. Compare your results to the exact solution $y = 1/(1 + x^2)$.