4. Show that the $(2, \infty)$ leapfrog scheme for the one-way wave equation

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + \left[1 + \left(\frac{h\delta}{2}\right)^2\right]^{-1/2} \frac{\sinh^{-1} \frac{1}{2}h\delta}{\frac{1}{2}h\delta} \delta_0 v_m^n = f_m^n$$

is stable, if λ is constant, if and only if $|a\lambda| < 1/\pi$. See equation (3.2.15).

4.2 Stability for General Multistep Schemes

We now discuss the stability conditions for a general multistep scheme. As for one-step schemes, we assume that the differential equation is of first order in the differentiation with respect to t.

The stability of the multistep scheme

$$P_{k,h}v = R_{k,h}f \tag{4.2.1}$$

is determined by considering the roots of the amplification polynomial $\Phi(g, \theta)$ given by

$$\Phi(g, \theta) = k\, p_{k,h}\left(\frac{\ln g}{k}, \theta h^{-1}\right)$$

or, equivalently,

$$\Phi\left(e^{sk}, h\xi\right) = k\, p_{k,h}\,(s, \xi)\,.$$

Alternatively, Φ can be obtained by requiring that

$$v_m^n = g^n e^{im\theta} \tag{4.2.2}$$

is a solution to the equation (4.2.1) with $f = 0$. $\Phi(g, \theta)$ is the polynomial of which g must be a root so that (4.2.2) can be a solution of (4.2.1). We assume that the scheme involves $\sigma + 1$ time levels, so that Φ is a polynomial of order σ. Note that J in Definition 1.5.1 will be taken to be σ.

Since we are primarily concerned with the roots of this polynomial, there is no difficulty in dealing with a scalar multiple of $\Phi(g, \theta)$ rather than $\Phi(g, \theta)$ itself. However, the relationship between $\Phi(g, \theta)$ and the symbol $p(s, \xi)$ is important in proving convergence results for multistep schemes in Chapter 10.

Example 4.2.1 Consider the multistep scheme for the one-way wave equation given by

$$\frac{3v_m^{n+1} - 4v_m^n + v_m^{n-1}}{2k} + a\frac{v_{m+1}^{n+1} - v_{m-1}^{n+1}}{2h} = f_m^{n+1}. \tag{4.2.3}$$

For this scheme the amplification polynomial is

$$\Phi(g, \theta) = \tfrac{1}{2}(3 + 2ia\lambda \sin\theta)g^2 - 2g + \tfrac{1}{2}.$$

The analysis of the stability of this scheme is not as easy as that of the leapfrog scheme, and therefore we postpone the analysis until the next section, in which we present a general method for analyzing the stability of multistep schemes. This scheme is accurate of order $(2, 2)$ and unconditionally stable; see Exercise 4.3.3. □

This scheme is second-order accurate and dissipative of order 4 for small values of ε.

To show that any scheme can have dissipation added to it, we consider the amplification polynomial and modify it as in formula (4.3.11). To be more precise the scheme corresponding to

$$\Phi^{\varepsilon}(g,\theta) = \Phi(g,\theta) + \varepsilon \sin^{2r} \tfrac{1}{2}\theta \, g\Phi'(g,\theta) \tag{5.1.7}$$

will have all roots inside the unit circle except at θ equal to 0. ($\Phi'(g,\theta)$ is the derivative of Φ with respect to g.) Another choice for a dissipative scheme is

$$\Phi^{\varepsilon}(g,\theta) = \Phi(g,\theta) + \varepsilon \sin^{2r} \tfrac{1}{2}\theta \left[g\Phi'(g,\theta) - n\Phi(g,\theta) \right]. \tag{5.1.8}$$

The preceding general procedures are not always advisable to use, but it does give one guidance in adding dissipation to a scheme (see Exercises 5.1.3 and 5.1.4).

If we use the methods of Section 4.3, then we can determine if the scheme is dissipative by checking if the amplification polynomial is a Schur polynomial for values of θ other than θ equal to 0. For a dissipative scheme the amplification polynomial is a Schur polynomial for θ not equal to zero.

EXERCISES 5.1

1. Show that the scheme (5.1.6) is dissipative of order 4 and stable if $0 < \varepsilon < 2$.

2. Show that the modified leapfrog scheme (5.1.5) is stable for ε satisfying

$$0 < \varepsilon \le 1 \quad \text{if} \quad 0 < a^2\lambda^2 \le \tfrac{1}{2}$$

and

$$0 < \varepsilon \le 4a^2\lambda^2 \left(1 - a^2\lambda^2\right) \quad \text{if} \quad \tfrac{1}{2} \le a^2\lambda^2 < 1.$$

3. Construct the modified scheme corresponding to formula (5.1.7) using the multistep scheme (4.2.3). Compare this scheme with

$$\frac{3v_m^{n+1} - 4v_m^n + v_m^{n-1}}{2k} + a\delta_0 v_m^{n+1} = \frac{\varepsilon}{2k}\left(\frac{i}{2}h\delta\right)^{2r} v_m^{n-1}.$$

4. Construct the leapfrog scheme with added dissipation using the method given by formula (5.1.8). Compare this scheme with the scheme (5.1.5).

5. Construct the Crank-Nicolson scheme with added dissipation using the method given by formulas (5.1.7) and (5.1.8). Compare these schemes with each other and with the scheme (5.1.6).

6. Show that the scheme of Exercise 3.2.6 is dissipative of order 6 for

$$0 < |a\lambda| < \left(\frac{\sqrt{17}-1}{6}\right)^{1/2}.$$