# Chapter 6

# Parabolic Partial Differential Equations

## 6.1 Overview of Parabolic Partial Differential Equations

The simplest parabolic equation is the one-dimensional heat equation,

$$u_t = bu_{xx} \tag{6.1.1}$$

where $b$ is a positive number. This equation arises in the study of heat transfer, in which case the function $u(t,x)$ gives the temperature at time $t$ and location $x$ resulting from the initial temperature distribution. Equations similar to (6.1.1) arise in many other applications, including viscous fluid flow and diffusion processes. As for the one-way wave equation (1.1.1), we are interested in the initial value problem for the heat equation (6.1.1); that is, we wish to determine the solution $u(t,x)$ for $t$ positive, given the initial condition that $u(0,x) = u_0(x)$ for some function $u_0$.

We can obtain a formula for the solution to (6.1.1) by using the Fourier transform of (6.1.1) in space to obtain the equation

$$\hat{u}_t = -b\omega^2 \hat{u}.$$

Using the initial values, this equation has the solution

$$\hat{u}(t,\omega) = e^{-b\omega^2 t}\hat{u}_0(\omega),$$

and thus by the Fourier inversion formula

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} e^{-b\omega^2 t} \hat{u}_0(\omega)\, d\omega. \tag{6.1.2}$$

Formula (6.1.2) shows that $u$ at time $t$ is obtained from $u_0$ by damping the high-frequency modes of $u_0$. It also shows why the solution operator for a parabolic equation is called a dissipative operator, since all high-frequencies are dissipated.

which is true for all $\theta$ if and only if

$$b\mu \leq \tfrac{1}{2}. \qquad (6.3.2)$$

Scheme (6.3.1) is dissipative of order 2 as long as $b\mu$ is strictly less than $\frac{1}{2}$ and positive. Therefore, we usually take $b\mu < \frac{1}{2}$ so that the scheme will be dissipative. Dissipativity is a desirable property for schemes for parabolic equations to have, since then the finite difference solution will become smoother in time, as does the solution of the differential equation. As we will show later (e.g., Theorem 6.3.2) dissipative schemes for (6.1.1) satisfy estimates analogous to (6.2.2) and are often more accurate for nonsmooth initial data. See Section 10.4 and Exercise 6.3.10.

The stability condition (6.3.2) means the time-step $k$ is at most $(2b)^{-1}h^2$, which means that when the spatial accuracy is increased by reducing $h$ in half, then $k$, the time-step, must be reduced by one-fourth. This restriction on $k$ can be quite severe for practical computation, and other schemes are usually more efficient. Notice that even though the scheme is accurate of order $(1,2)$, because of the stability condition the scheme (6.3.1) is second-order accurate if $\mu$ is constant.

We now list some other schemes and their properties. We will give the schemes for the inhomogeneous heat equation

$$u_t = bu_{xx} + f(t,x), \qquad (6.3.3)$$

and we will assume that $b$ is positive.

## The Backward-Time Central-Space Scheme

The backward-time central-space scheme is

$$\frac{v_m^{n+1} - v_m^n}{k} = b\frac{v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}}{h^2} + f_m^n. \qquad (6.3.4)$$

This scheme is implicit and unconditionally stable. It is accurate of order $(1,2)$ and is dissipative when $\mu$ is bounded away from 0.

## The Crank-Nicolson Scheme

The Crank-Nicolson scheme (see [10]) is given by

$$\begin{aligned}\frac{v_m^{n+1} - v_m^n}{k} =& \frac{1}{2}b\frac{v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}}{h^2} \\ &+ \frac{1}{2}b\frac{v_{m+1}^{n} - 2v_m^{n} + v_{m-1}^{n}}{h^2} + \frac{1}{2}(f_m^{n+1} + f_m^n).\end{aligned} \qquad (6.3.5)$$

The Crank-Nicolson scheme is implicit, unconditionally stable, and second-order accurate, i.e., accurate of order $(2,2)$. It is dissipative of order 2 if $\mu$

constant, but not dissipative if $\lambda$ is constant. Even though the Crank-Nicolson scheme (6.3.5) is second-order accurate, whereas the scheme (6.3.4) is only first-order accurate, with nonsmooth initial data and with $\lambda$ held constant, the dissipative scheme (6.3.4) may be more accurate than the Crank-Nicolson scheme, which is not dissipative when $\lambda$ is constant (also see Exercise 6.3.10). This is discussed further and illustrated in Section 10.4.

## The Leapfrog Scheme

The leapfrog scheme is

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} = b\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2} + f_m^n, \qquad (6.3.6)$$

and this scheme is unstable for all values of $\mu$.

## The Du Fort–Frankel Scheme

The Du Fort–Frankel scheme may be viewed as a modification of the leapfrog scheme. It is

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} = b\frac{v_{m+1}^n - \left(v_m^{n+1} + v_m^{n-1}\right) + v_{m-1}^n}{h^2} + f_m^n. \qquad (6.3.7)$$

This scheme is explicit and yet unconditionally stable. The order of accuracy is given by $O\left(h^2\right) + O\left(k^2\right) + O\left(k^{-2}h^2\right)$. The scheme is nondissipative, and this limits its usefulness.

The Du Fort–Frankel scheme is distinctive in that it is both explicit and unconditionally stable. It can be rewritten as

$$(1 + 2b\mu)\, v_m^{n+1} = 2b\mu\left(v_{m+1}^n + v_{m-1}^n\right) + (1 - 2b\mu)\, v_m^{n-1}.$$

To determine the stability we must solve for the roots of the amplification polynomial equation (see Section 4.2):

$$(1 + 2b\mu)\, g^2 - 4b\mu\cos\theta\; g - (1 - 2b\mu) = 0.$$

The two solutions of this equation are

$$g_\pm = \frac{2b\mu\cos\theta \pm \sqrt{1 - 4b^2\mu^2\sin^2\theta}}{1 + 2b\mu}.$$

If $1 - 4b^2\mu^2\sin^2\theta$ is nonnegative, then we have

$$|g_\pm| \le \frac{2b\mu|\cos\theta| + \sqrt{1 - 4b^2\mu^2\sin^2\theta}}{1 + 2b\mu} \le \frac{2b\mu + 1}{1 + 2b\mu} = 1$$

tend to zero as $h$ and $k$ tend to zero with $nk$ fixed, and therefore $n(k - k'$ must tend to zero for $nk = t$ fixed. We then have

$$\begin{aligned} n(k - k') &= t\left(1 - \frac{k'}{k}\right) \\ &= t\left(1 - \frac{\mu'}{\mu}\right). \end{aligned}$$

Thus $1 - \mu'/\mu$ must tend to zero for the scheme to be convergent. But

$$1 - \frac{\mu'}{\mu} = \frac{e^{-2b\mu} - (1 - 2b\mu)}{2b\mu} = O(b\mu)$$

as $\mu$ tends to zero. This shows that scheme (6.3.12) is convergent only if tends to zero with $h$ and $k$. This makes this scheme less efficient than the star dard forward-central scheme (6.3.1). In fact, for explicit second-order accurat schemes, the forward-central scheme is essentially optimal.

## EXERCISES 6.3

**1.** Justify the claims about the stability and accuracy of schemes (6.3.4 (6.3.5), and (6.3.6).

**2.** Show that if $\lambda = k/h$ is a constant, then the Du Fort–Frankel scheme i consistent with

$$b\lambda^2 u_{tt} + u_t = bu_{xx} + f(t, x).$$

**3.** Prove Theorem 6.3.1 for the equation (6.1.1). *Hint:* If $u_0$ is nonnegative an not identically zero, then $u(t, x)$ will be positive for all $x$ when $t$ is positive.

**4.** Show that scheme (6.3.12) with $\mu$ held constant as $h$ and $k$ tends to zer is consistent with

$$u_t = b' u_{xx}$$

where $b'$ is defined by

$$e^{-2b\mu} = 1 - 2b'\mu.$$

**5.** Show that a scheme for (6.1.1) of the form

$$v_m^{n+1} = \alpha v_m^n + \frac{1 - \alpha}{2}\left(v_{m+1}^n + v_{m-1}^n\right),$$

with $\alpha$ constant as $h$ and $k$ tend to zero, is consistent with the heat equatio (6.1.1) only if

$$\alpha = 1 - 2b\mu.$$

**10.** Solve the initial-boundary value problem for (6.1.1) on $-1 \le x \le 1$ with initial data given by

$$u_0(x) = \begin{cases} 1 & \text{if } |x| < \frac{1}{2}, \\ \frac{1}{2} & \text{if } |x| = \frac{1}{2}, \\ 0 & \text{if } |x| > \frac{1}{2}. \end{cases}$$

Solve up to $t = \frac{1}{2}$. The boundary data and the exact solution are given by

$$u(t,x) = \frac{1}{2} + 2\sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{\cos \pi(2\ell+1)x}{\pi(2\ell+1)} e^{-\pi^2(2\ell+1)^2 t}.$$

Use the Crank-Nicolson scheme (6.3.5) with $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}$. Compare the accuracy and efficiency when $\lambda = 1$ and also when $\mu = 10$.

Demonstrate by the computations that when $\lambda$ is constant, the error in the solution does not decrease when measured in the supremum norm, but it does decrease in the $L^2$ norm.

**11.** a. Show that the scheme for (6.1.1) given by

$$\left(1 - \frac{kb}{2}\delta^2\right)\left(\frac{v_m^{n+1} - v_m^n}{k}\right) = b\delta^2 v_m^n$$

is the Crank-Nicolson scheme (6.3.5).

b. Show that the implicit scheme

$$\left(1 - \frac{kb}{2}\delta^2\right)\left(\frac{v_m^{n+1} - v_m^n}{k}\right) = b\left(1 - \frac{h^2}{12}\delta^2\right)\delta^2 v_m^n$$

is accurate of order $(2,4)$ and stable if $b\mu \le \frac{3}{2}$.

**12. Maximum Norm Stability** Show that the forward-time central-space scheme satisfies the estimate

$$\|v^{n+1}\|_\infty \le \|v^n\|_\infty$$

for all solutions if and only if $2b\mu \le 1$.

**13. Maximum Norm Stability** Show that the Crank-Nicolson scheme satisfies the estimate

$$\|v^{n+1}\|_\infty \le \|v^n\|_\infty$$

for all solutions if $b\mu \le 1$. *Hint:* Show that if $v_{m'}^{n+1}$ is the largest value of $v_m^{n+1}$, then

$$\begin{aligned} v_{m'}^{n+1} &\le -\frac{b\mu}{2} v_{m'-1}^{n+1} + (1 + b\mu) v_{m'}^{n+1} - \frac{b\mu}{2} v_{m'+1}^{n+1} \\ &\le \|v^n\|_\infty. \end{aligned}$$