1.1.3. Solve the initial value problem for

$$u_t + \frac{1}{1+\frac{1}{2}\cos x}\, u_x = 0.$$

Show that the solution is given by $u(t,x) = u_0(\xi)$, where ξ is the unique solution of

$$\xi + \tfrac{1}{2}\sin\xi = x + \tfrac{1}{2}\sin x - t.$$

1.1.4. Show that the initial value problem for (1.1.5) is equivalent to the family of initial value problems for the ordinary differential equations

$$\frac{d\tilde{u}}{d\tau} = f(\tau, \xi + a\tau, \tilde{u})$$

with $\tilde{u}(0,\xi) = u_0(\xi)$. Show that the solution of (1.1.5), $u(t,x)$, is given by $u(t,x) = \tilde{u}(t, x - at)$.

1.1.5. Use the results of Exercise 1.1.4 to show that the solution of the initial value problem for

$$u_t + u_x = -\sin^2 u$$

is given by

$$u(t,x) = \tan^{-1}\left(\frac{\tan[u_0(x-t)]}{1 + t\tan[u_0(x-t)]}\right).$$

An equivalent formula for the solution is

$$u(t,x) = \cot^{-1}\left(\cot[u_0(x-t)] + t\right).$$

1.1.6. Show that all solutions to

$$u_t + a\,u_x = 1 + u^2$$

become unbounded in finite time. That is, $u(t,x)$ tends to infinity for some x as t approaches some value t^*, where t^* is finite.

1.1.7. Show that the initial value problem for the equation

$$u_t + \left(1 + x^2\right) u_x = 0$$

is not well defined. *Hint:* Consider the region covered by the characteristics originating on the x-axis.

1.1.8. Obtain the solution of the system

$$\begin{aligned} u_t + u_x + v_x &= 0, \qquad & u(x,0) &= u_0(x),\\ v_t + u_x - v_x &= 0, \qquad & v(x,0) &= v_0(x). \end{aligned}$$

As before, we use the Taylor series on (3.1.6) evaluated at (t_n, x_m) to obtain

$$P_{k,h}\phi = \phi_t + \frac{k}{2}\phi_{tt} + a\phi_x - \frac{a^2k}{2}\phi_{xx} + O(k^2) + O(h^2). \tag{3.1.8}$$

For a smooth function $f(t,x)$, (3.1.7) becomes

$$R_{k,h}f = f + \frac{k}{2}f_t - \frac{ak}{2}f_x + O(k^2) + O(h^2),$$

and if $f = \phi_t + a\phi_x = P\phi$, this is

$$\begin{aligned} R_{k,h}P\phi &= \phi_t + a\phi_x + \frac{k}{2}\phi_{tt} + \frac{k}{2}a\phi_{xt} - \frac{ak}{2}\phi_{xt} - \frac{a^2k}{2}\phi_{xx} + O(k^2) + O(h^2) \\ &= \phi_t + a\phi_x + \frac{k}{2}\phi_{tt} - \frac{a^2k}{2}\phi_{xx} + O(k^2) + O(h^2), \end{aligned}$$

which agrees with (3.1.8) to $O(k^2) + O(h^2)$. Hence the Lax–Wendroff scheme (3.1.2) is accurate of order $(2, 2)$. □

We also see from this analysis that the Lax–Wendroff scheme with $R_{k,h}f_m^n = f_m^n$, i.e.,

$$v_m^{n+1} = v_m^n - \frac{a\lambda}{2}\left(v_{m+1}^n - v_{m-1}^n\right) + \frac{a^2\lambda^2}{2}\left(v_{m+1}^n - 2v_m^n + v_{m-1}^n\right) + k\,f_m^n, \tag{3.1.9}$$

is accurate of order $(1, 2)$.

Notice that to determine the order of accuracy we use the form (3.1.2) of the Lax–Wendroff scheme rather than (3.1.1), which is derived from (3.1.2) by multiplying by k and rearranging the terms. Without an appropriate normalization, in this case demanding that $P_{k,h}u$ be consistent with Pu, we can get incorrect results by multiplying the scheme by powers of k or h. An equivalent normalization is that $R_{k,h}$ applied to the function that is 1 everywhere gives the result 1, i.e.,

$$R_{k,h}1 = 1. \tag{3.1.10}$$

Definition 3.1.1 is not completely satisfactory. For example, it cannot be applied to the Lax–Friedrichs scheme, which contains the term $k^{-1}h^2\phi_{xx}$ in the Taylor series expansion of $P_{k,h}\phi$. We therefore give the following definition, which is more generally applicable. We assume that the time step is chosen as a function of the space step, i.e., $k = \Lambda(h)$, where Λ is a smooth function of h and $\Lambda(0) = 0$.

Definition 3.1.2. *A scheme $P_{k,h}v = R_{k,h}f$ with $k = \Lambda(h)$ that is consistent with the differential equation $Pu = f$ is accurate of order r if for any smooth function $\phi(t,x)$,*

$$P_{k,h}\phi - R_{k,h}P\phi = O(h^r).$$

accuracy 0.75. Convergence estimates proved in Chapter 10 give the rate of convergence of solutions if the initial data are not smooth.

Exercises

3.1.1. Using equations (3.1.21), (3.1.22), and (3.1.23), show that the Lax–Wendroff scheme is the only explicit one-step second-order accurate scheme that uses only the grid points $x_{m-1}, x_m,$ and x_{m+1} to compute the solution at x_m for the next time step.

3.1.2. Solve $u_t + u_x = 0,\ -1 \le x \le 1,\ 0 \le t \le 1.2$ with $u(0,x) = \sin 2\pi x$ and periodicity, i.e., $u(t,1) = u(t,-1)$. Use two methods:

(a) Forward-time backward-space with $\lambda = 0.8$,
(b) Lax–Wendroff with $\lambda = 0.8$.

Demonstrate the first-order accuracy of the solution of (a) and the second-order accuracy of the solution of (b) using $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{40},$ and $\frac{1}{80}$. Measure the error in the L^2 norm (3.1.24) and the maximum norm. (In the error computation, do not sum both grid points at $x = -1$ and $x = 1$ as separate points.)

3.1.3. Solve the equation of Exercise 1.1.5,

$$u_t + u_x = -\sin^2 u,$$

with the scheme (3.1.9), treating the $-\sin^2 u$ term as $f(t,x)$. Show that the scheme is first-order accurate. The exact solution is given in Exercise 1.1.5. Use a smooth function, such as $\sin(x-t)$, as initial data and boundary data.

3.1.4. Modify the scheme of Exercise 3.1.3 to be second-order accurate and explicit. There are several ways to do this. One way uses

$$\sin^2 v_m^{n+1} = \sin^2 v_m^n + \sin 2v_m^n \left(v_m^{n+1} - v_m^n\right) + O\left(k^2\right).$$

Another way is to evaluate explicitly the f_t term in the derivation of the Lax–Wendroff scheme and eliminate all derivatives with respect to t using the differential equation.

3.1.5. Determine the order of accuracy of the Euler backward scheme in Exercise 2.2.6.

3.1.6. Show that the scheme discussed in Example 2.2.6 has the symbol

$$\frac{e^{sk} - \cos h\xi}{k} + 4ai\frac{\sin^2 \frac{1}{2}h\xi \sin h\xi}{h^3}$$

and discuss the accuracy of the scheme.