Chapter 6

Parabolic Partial Differential Equations

6.1 Overview of Parabolic Partial Differential Equations

The simplest parabolic equation is the one-dimensional heat equation

$$u_t = bu_{xx}, \tag{6.1.1}$$

where b is a positive number. This equation arises in the study of heat transfer, in which case the function $u(t,x)$ gives the temperature at time t and location x resulting from the initial temperature distribution. Equations similar to (6.1.1) arise in many other applications, including viscous fluid flow and diffusion processes. As for the one-way wave equation (1.1.1), we are interested in the initial value problem for the heat equation (6.1.1); i.e., we wish to determine the solution $u(t,x)$ for t positive, given the initial condition that $u(0,x) = u_0(x)$ for some function u_0.

We can obtain a formula for the solution to (6.1.1) by using the Fourier transform of (6.1.1) in space to obtain the equation

$$\hat{u}_t = -b\omega^2\hat{u}.$$

Using the initial values, this equation has the solution

$$\hat{u}(t,\omega) = e^{-b\omega^2 t}\hat{u}_0(\omega),$$

and thus by the Fourier inversion formula

$$u(t,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega x} e^{-b\omega^2 t}\hat{u}_0(\omega)\,d\omega. \tag{6.1.2}$$

Formula (6.1.2) shows that u at time t is obtained from u_0 by damping the high-frequency modes of u_0. It also shows why the solution operator for a parabolic equation is called a dissipative operator, since all high frequencies are dissipated.

The Backward-Time Central-Space Scheme

The backward-time central-space scheme is

$$\frac{v_m^{n+1} - v_m^n}{k} = b\frac{v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}}{h^2} + f_m^{n+1}. \tag{6.3.3}$$

The amplification factor is

$$g(\theta) = \frac{1}{1 + 4b\mu \sin^2 \frac{1}{2}\theta}.$$

This scheme is implicit and unconditionally stable. It is accurate of order $(1, 2)$ and is dissipative when μ is bounded away from 0.

The Crank–Nicolson Scheme

The Crank–Nicolson scheme (see [12]) is given by

$$\begin{aligned}\frac{v_m^{n+1} - v_m^n}{k} &= \frac{1}{2}b\frac{v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}}{h^2} \\ &+ \frac{1}{2}b\frac{v_{m+1}^{n} - 2v_m^{n} + v_{m-1}^{n}}{h^2} + \frac{1}{2}(f_m^{n+1} + f_m^n).\end{aligned} \tag{6.3.4}$$

The amplification factor is

$$g(\theta) = \frac{1 - 2b\mu \sin^2 \frac{1}{2}\theta}{1 + 2b\mu \sin^2 \frac{1}{2}\theta}.$$

The Crank–Nicolson scheme is implicit, unconditionally stable, and second-order accurate, i.e., accurate of order $(2, 2)$. It is dissipative of order 2 if μ is constant, but not dissipative if λ is constant. Even though the Crank–Nicolson scheme (6.3.4) is second-order accurate, whereas the scheme (6.3.3) is only first-order accurate, with nonsmooth initial data and with λ held constant, the dissipative scheme (6.3.3) may be more accurate than the Crank–Nicolson scheme, which is not dissipative when λ is constant (also see Exercises 6.3.10 and 6.3.11). This is discussed further and illustrated in Section 10.4.

The Leapfrog Scheme

The leapfrog scheme is

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} = b\frac{v_{m+1}^{n} - 2v_m^{n} + v_{m-1}^{n}}{h^2} + f_m^n, \tag{6.3.5}$$

and this scheme is unstable for all values of μ. The amplification polynomial is

$$g^2 + 8\, g\, b\mu \sin^2 \tfrac{1}{2}\theta - 1 = 0$$

(see Section 4.2), so the amplification factors are

$$g_\pm(\theta) = -4b\mu \sin^2 \tfrac{1}{2}\theta \pm \sqrt{(4b\mu \sin^2 \tfrac{1}{2}\theta)^2 + 1}\,.$$

Because the quantity inside the square root is greater than 1 for most values of θ, the scheme is unstable.

The Du Fort–Frankel Scheme

The Du Fort–Frankel scheme may be viewed as a modification of the leapfrog scheme. It is

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} = b\frac{v_{m+1}^n - \left(v_m^{n+1} + v_m^{n-1}\right) + v_{m-1}^n}{h^2} + f_m^n. \tag{6.3.6}$$

This scheme is explicit and yet unconditionally stable. The order of accuracy is given by $O\left(h^2\right) + O\left(k^2\right) + O\left(k^2 h^{-2}\right)$. The scheme is nondissipative, and this limits its usefulness.

The Du Fort–Frankel scheme is distinctive in that it is both explicit and unconditionally stable. It can be rewritten as

$$(1 + 2b\mu)\, v_m^{n+1} = 2b\mu\left(v_{m+1}^n + v_{m-1}^n\right) + (1 - 2b\mu)\, v_m^{n-1}.$$

To determine the stability we must solve for the roots of the amplification polynomial equation (see Section 4.2):

$$(1 + 2b\mu)\, g^2 - 4b\mu \cos\theta\; g - (1 - 2b\mu) = 0.$$

The two solutions of this equation are

$$g_\pm(\theta) = \frac{2b\mu\cos\theta \pm \sqrt{1 - 4b^2\mu^2\sin^2\theta}}{1 + 2b\mu}.$$

If $1 - 4b^2\mu^2\sin^2\theta$ is nonnegative, then we have

$$|g_\pm(\theta)| \le \frac{2b\mu|\cos\theta| + \sqrt{1 - 4b^2\mu^2\sin^2\theta}}{1 + 2b\mu} \le \frac{2b\mu + 1}{1 + 2b\mu} = 1,$$

and if $1 - 4b^2\mu^2\sin^2\theta$ is negative, then

$$|g_\pm(\theta)|^2 = \frac{(2b\mu\cos\theta)^2 + 4b^2\mu^2\sin^2\theta - 1}{(1 + 2b\mu)^2}$$

$$= \frac{4b^2\mu^2 - 1}{4b^2\mu^2 + 4b\mu + 1} < 1.$$

Thus for any value of μ or θ, we have that both g_+ and g_- are bounded by 1 in magnitude. Moreover, when g_+ and g_- are equal, they both have magnitude less than 1, and so this introduces no constraint on the stability (see Section 4.2). Thus the scheme is stable for all values of μ.

Even though the Du Fort–Frankel scheme is both explicit and unconditionally stable, it is consistent only if k/h tends to zero with h and k (see Exercise 6.3.2). Theorem 1.6.2, which states that there are no explicit unconditionally stable schemes for hyperbolic equations, does not extend directly to parabolic equations. However, the proper analogue of the results of Section 1.6 for parabolic equations is the following theorem.

Theorem 6.3.1. *An explicit, consistent scheme for the parabolic system (6.2.1) is convergent only if k/h tends to zero as k and h tend to zero.*

tends to zero as h and k tend to zero. Thus, to be convergent we must have

$$u\left(nk', x_m\right) - u\left(nk, x_m\right)$$

tend to zero as h and k tend to zero with nk fixed, and therefore $n\left(k - k'\right)$ must tend to zero for $nk = t$ fixed. We then have

$$\begin{aligned} n\left(k - k'\right) &= t\left(1 - \frac{k'}{k}\right) \\ &= t\left(1 - \frac{\mu'}{\mu}\right). \end{aligned}$$

Thus $1 - \mu'/\mu$ must tend to zero for the scheme to be convergent. But

$$1 - \frac{\mu'}{\mu} = \frac{e^{-2b\mu} - (1 - 2b\mu)}{2b\mu} = O\left(b\mu\right)$$

as μ tends to zero. This shows that scheme (6.3.9) is convergent only if μ tends to zero with h and k. This makes this scheme less efficient than the standard forward central scheme (6.3.1). In fact, for explicit second-order accurate schemes, the forward central scheme is essentially optimal.

Exercises

6.3.1. Justify the claims about the stability and accuracy of schemes (6.3.3), (6.3.4), and (6.3.5).

6.3.2. Show that if $\lambda = k/h$ is a constant, then the Du Fort–Frankel scheme is consistent with

$$b\lambda^2 u_{tt} + u_t = b u_{xx} + f(t, x).$$

6.3.3. Prove Theorem 6.3.1 for the equation (6.1.1). *Hint:* If u_0 is nonnegative and not identically zero, then $u(t, x)$ will be positive for all x when t is positive.

6.3.4. Show that scheme (6.3.9) with μ held constant as h and k tends to zero is consistent with

$$u_t = b' u_{xx},$$

where b' is defined by

$$e^{-2b\mu} = 1 - 2b'\mu.$$

6.3.5. Show that a scheme for (6.1.1) of the form

$$v_m^{n+1} = \alpha v_m^n + \frac{1 - \alpha}{2}\left(v_{m+1}^n + v_{m-1}^n\right),$$

with α constant as h and k tend to zero, is consistent with the heat equation (6.1.1) only if

$$\alpha = 1 - 2b\mu.$$

6.3.10. Solve the initial-boundary value problem for (6.1.1) on $-1 \le x \le 1$ with initial data given by

$$u_0(x) = \begin{cases} 1 & \text{if } |x| < \frac{1}{2}, \\ \frac{1}{2} & \text{if } |x| = \frac{1}{2}, \\ 0 & \text{if } |x| > \frac{1}{2}. \end{cases}$$

Solve up to $t = 1/2$. The boundary data and the exact solution are given by

$$u(t,x) = \frac{1}{2} + 2\sum_{\ell=0}^{\infty} (-1)^\ell \frac{\cos \pi(2\ell+1)x}{\pi(2\ell+1)} e^{-\pi^2(2\ell+1)^2 t}.$$

Use the Crank–Nicolson scheme (6.3.4) with $h = 1/10,\ 1/20,\ 1/40$. Compare the accuracy and efficiency when $\lambda = 1$ and also when $\mu = 10$.

Demonstrate by the computations that when λ is constant, the error in the solution does not decrease when measured in the supremum norm, but it does decrease in the L^2 norm.

6.3.11. Solve the initial boundary value problem for $u_t = u_{xx}$ on $-1 \le x \le 1$ for $0 \le t \le 0.5$ with initial data given by

$$u_0(x) = \begin{cases} 1 - |x| & \text{for } |x| < \frac{1}{2}, \\ \frac{1}{4} & \text{for } |x| = \frac{1}{2}, \\ 0 & \text{for } |x| > \frac{1}{2}. \end{cases}$$

Use the boundary conditions

$$u(t,-1) = u^*(t,-1) \quad \text{and} \quad u_x(t,1) = 0,$$

where $u^*(t,x)$ is the exact solution given by

$$u^*(t,x) = \frac{3}{8} + \sum_{\ell=0}^{\infty} \left(\frac{(-1)^\ell}{\pi(2\ell+1)} + \frac{2}{\pi^2(2\ell+1)^2} \right) \cos \pi(2\ell+1)x \, e^{-\pi^2(2\ell+1)^2 t}$$
$$+ \sum_{m=0}^{\infty} \frac{\cos 2\pi(2m+1)x}{\pi^2(2m+1)^2} e^{-4\pi^2(2m+1)^2 t}.$$

Consider three schemes:

(a) The explicit forward-time central-space scheme with $\mu = 0.4$.
(b) The Crank–Nicolson scheme with $\lambda = 1$.
(c) The Crank–Nicolson scheme with $\mu = 5$.

For the boundary condition at $x_M = 1$, use the scheme applied at x_M and set $v^n_{M+1} = v^n_{M-1}$ to eliminate the values at x_{M+1} for all values of n.

For each scheme compute solutions for $h = 1/10,\ 1/20,\ 1/40$, and $1/80$.

Compare the accuracy and efficiency for these schemes.