Thermodynamic Calculus manipulations :

① inversion / reciprocal rule : $\left.\frac{\partial x}{\partial y}\right)_z = 1 \Big/ \left.\frac{\partial y}{\partial x}\right)_z$

Same as single variable result since z is fixed here.

example : $\left.\frac{\partial V}{\partial T}\right)_P = 1 \Big/ \left.\frac{\partial T}{\partial V}\right)_P$

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② triple product rule / cyclic relation

$$\left.\frac{\partial x}{\partial y}\right)_z \left.\frac{\partial z}{\partial x}\right)_y \left.\frac{\partial y}{\partial z}\right)_x = -1$$

example / proof : consider $P(T, V)$.

We have $dP = \left.\frac{\partial P}{\partial T}\right)_V dT + \left.\frac{\partial P}{\partial V}\right)_T dV$

Now hold P constant so that $dP = 0$.

Then we can combine the differentials dT and dV at constant P conditions :

$$0 = \left.\frac{\partial P}{\partial T}\right)_V \left.\frac{\partial T}{\partial V}\right)_P + \left.\frac{\partial P}{\partial V}\right)_T$$

rearranging and using the inversion rule gives the triple product rule.

③ addition of variable / chain rule

$$\left.\frac{\partial x}{\partial y}\right)_z = \left.\frac{\partial x}{\partial w}\right)_z \Big/ \left.\frac{\partial y}{\partial w}\right)_z$$

example : $\left.\frac{\partial U}{\partial T}\right)_V = \left.\frac{\partial U}{\partial P}\right)_V \left.\frac{\partial P}{\partial T}\right)_V$

④ non-natural derivative / chain rule

$$A(x,y) \longrightarrow \left.\frac{\partial A}{\partial y}\right)_z = \left.\frac{\partial A}{\partial y}\right)_x + \left.\frac{\partial A}{\partial x}\right)_y \left.\frac{\partial x}{\partial y}\right)_z$$

example/proof:

$$H = H(T,P) \Rightarrow dH = \left.\frac{\partial H}{\partial T}\right)_P dT + \left.\frac{\partial H}{\partial P}\right)_T dP$$

Now divide by dT at constant V:

$$\left.\frac{\partial H}{\partial T}\right)_V = \left.\frac{\partial H}{\partial T}\right)_P \cdot 1 + \left.\frac{\partial H}{\partial P}\right)_T \left.\frac{\partial P}{\partial T}\right)_V$$

This is a special case of the more general chain rule:
Consider F in terms of either (x,y) or (u,v)

Then $$\left.\frac{\partial F}{\partial x}\right)_y = \left.\frac{\partial F}{\partial u}\right)_v \left.\frac{\partial u}{\partial x}\right)_y + \left.\frac{\partial F}{\partial v}\right)_u \left.\frac{\partial v}{\partial x}\right)_y$$

⑤ Exact and inexact differentials

An expression $M(x,y)\,dx + N(x,y)\,dy$ is exact if there exists an f such that $M = \left.\frac{\partial f}{\partial x}\right)_y$ and $N = \left.\frac{\partial f}{\partial y}\right)_x$

How to test: use the fact that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for any analytic function f.

Therefore, exact iff $\left.\frac{\partial M}{\partial y}\right)_x = \left.\frac{\partial N}{\partial x}\right)_y$