a.

$$x(t) = c_3 \cos \omega t + c_4 \sin \omega t \qquad \frac{dx}{dt} = -\omega c_3 \sin \omega t + \omega c_4 \cos \omega t$$

$$0 = c_3(1) + c_4(0) \qquad v_0 = 0 + \omega c_4$$

$$0 = c_3 \qquad \frac{v_0}{\omega} = c_4$$

The solution to the differential equation under these conditions is $x(t) = \frac{v_0}{\omega} \sin \omega t$.

b.

$$x(t) = c_3 \cos \omega t + c_4 \sin \omega t \qquad \frac{dx}{dt} = -\omega c_3 \sin \omega t + \omega c_4 \cos \omega t$$

$$A = c_3(1) + c_4(0) \qquad v_0 = 0 + \omega c_4$$

$$A = c_3 \qquad \frac{v_0}{\omega} = c_4$$

The solution to the differential equation under these conditions is $x(t) = A \cos \omega t + \frac{v_0}{\omega} \sin \omega t$. Both of these solutions can be written in the form $x(t) = A \cos \omega t + B \sin \omega t$, which (as shown in Problem 2–3) oscillates with frequency $\omega/2\pi$.

2–5. The general solution to the differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x(t) = 0$$

is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

For convenience, we often write this solution in the equivalent forms

$$x(t) = A \sin(\omega t + \phi)$$

or

$$x(t) = B \cos(\omega t + \psi)$$

Show that all three of these expressions for $x(t)$ are equivalent. Derive equations for A and ϕ in terms of c_1 and c_2, and for B and ψ in terms of c_1 and c_2. Show that all three forms of $x(t)$ oscillate with frequency $\omega/2\pi$. *Hint*: Use the trigonometric identities

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

and

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\begin{aligned} x(t) &= A \sin(\omega t + \phi) \\ &= A \sin \omega t \cos \phi + A \cos \omega t \sin \phi \\ &= c_1 \cos \omega t + c_2 \sin \omega t \end{aligned}$$

where we define $c_1 = A \sin \phi$ and $c_2 = A \cos \phi$. These equations can be solved for A and ϕ in terms of c_1 and c_2:

$$\begin{aligned} c_1^2 + c_2^2 &= A^2(\sin^2 \phi + \cos^2 \phi) \\ A &= \left(c_1^2 + c_2^2\right)^{1/2} \\ \phi &= \sin^{-1} \frac{c_2}{\left(c_1^2 + c_2^2\right)^{1/2}} = \tan^{-1} \frac{c_2}{c_1} \end{aligned}$$

Likewise,

$$\begin{aligned} x(t) &= B \cos(\omega t + \phi) \\ &= B \cos \omega t \cos \psi - B \sin \omega t \sin \psi \\ &= c_1 \cos \omega t + c_2 \sin \omega t \end{aligned}$$

where we define $c_1 = B \cos \psi$ and $c_2 = -B \sin \psi$. Solving for B and ϕ in terms of c_1 and c_2 gives

$$\begin{aligned} c_1^2 + c_2^2 &= B^2\left(\cos^2 \psi + \sin^2 \psi\right) \\ B &= \left(c_1^2 + c_2^2\right)^{1/2} \\ \psi &= \cos^{-1} \frac{c_1}{\left(c_1^2 + c_2^2\right)^{1/2}} = \tan^{-1} \frac{c_2}{c_1} \end{aligned}$$

Because ϕ and ψ are constants, $x(t) = A \sin(\omega t + \phi)$ and $x(t) = B \cos(\omega t + \phi)$ oscillate with a frequency of $\nu = \omega/2\pi$. We showed in Problem 2–3 that $x(t) = A \cos \omega t + B \sin \omega t$ oscillates with the frequency $\omega/2\pi$.

2–6. In all the differential equations we have discussed so far, the values of the exponents α that we have found have been either real or purely imaginary. Let us consider a case in which α turns out to be complex. Consider the equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 0$$

If we substitute $y(x) = e^{\alpha x}$ into this equation, we find that $\alpha^2 + 2\alpha + 10 = 0$ or that $\alpha = -1 \pm 3i$. The general solution is

$$\begin{aligned} y(x) &= c_1 e^{(-1+3i)x} + c_2 e^{(-1-3i)x} \\ &= c_1 e^{-x} e^{3ix} + c_2 e^{-x} e^{-3ix} \end{aligned}$$

Show that $y(x)$ can be written in the equivalent form

$$y(x) = e^{-x}(c_3 \cos 3x + c_4 \sin 3x)$$

Thus we see that complex values of the α's lead to trigonometric solutions modulated by an exponential factor. Solve the following equations.

a. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 0$

b. $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 25y = 0$

c. $\dfrac{d^2y}{dx^2} + 2\beta\dfrac{dy}{dx} + (\beta^2 + \omega^2)y = 0$

d. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = 0 \qquad y(0) = 1; \; \dfrac{dy}{dx}(\text{at } x = 0) = -3$

---

where we have used the subscripts on E to denote explicitly the dependence of the energy on the values of n_x and n_y.

2–14. Extend Problems 2–9 and 2–13 to three dimensions, where a particle is constrained to move freely throughout a rectangular box of sides a, b, and c. The Schrödinger equation for this system is

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + \left(\frac{8\pi^2 mE}{h^2}\right)\psi(x, y, z) = 0$$

and the boundary conditions are that $\psi(x, y, z)$ vanishes over all the surfaces of the box.

As in Problem 2–13, we can separate the variables to produce three differential equations, one for each dimension:

$$\frac{\partial^2 X}{\partial x^2} + \frac{8\pi^2 mE}{h^2}X = p^2$$
$$\frac{\partial^2 Y}{\partial y^2} + \frac{8\pi^2 mE}{h^2}Y = q^2$$
$$\frac{\partial^2 Z}{\partial Z^2} + \frac{8\pi^2 mE}{h^2}Z = r^2$$

where $p^2 + q^2 + r^2 = 8\pi^2 mE/h^2$. Following the method described in Problem 2–13, we find

$$\psi(x, y, z) = A \sin\frac{n_x \pi x}{a} \sin\frac{n_y \pi y}{b} \sin\frac{n_z \pi z}{c} \qquad \begin{array}{l} n_x = 1, 2, 3, \ldots \\ n_y = 1, 2, 3, \ldots \\ n_z = 1, 2, 3, \ldots \end{array}$$

and

$$E_{n_x n_y n_z} = \frac{n_x^2 h^2}{8ma^2} + \frac{n_y^2 h^2}{8mb^2} + \frac{n_z^2 h^2}{8mc^2} \qquad \begin{array}{l} n_x = 1, 2, 3, \ldots \\ n_y = 1, 2, 3, \ldots \\ n_z = 1, 2, 3, \ldots \end{array}$$

2–15. Show that Equations 2.46 and 2.48 are equivalent. How are G_{nm} and ϕ_{nm} in Equation 2.48 related to the quantities in Equation 2.46?

$$T_{nm}(t) = G_{nm}\cos(\omega_{nm}t + \phi_{nm}) \qquad (2.48)$$

Using the trigonometric identities from Problem 2–5, we write this as

$$\begin{aligned} T_{nm}(t) &= G_{nm}\left[\cos(\omega_{nm}t)\cos\phi_{nm} - \sin(\omega_{nm}t)\sin\phi_{nm}\right] \\ &= G_{nm}\cos(\omega_{nm}t)\cos\phi_{nm} - G_{nm}\sin(\omega_{nm}t)\sin\phi_{nm} \\ &= E_{nm}\cos\omega_{nm}t + F_{nm}\sin\omega_{nm}t \qquad (2.46) \end{aligned}$$

where $G\cos\phi_{nm} = E_{nm}$ and $-G\sin\phi_{nm} = F_{nm}$.

Many problems in classical mechanics can be reduced to the problem of solving a differential equation with constant coefficients (cf. Problem 2–7). The basic starting point is Newton's second law, which says that the rate of change of momentum is equal to the force acting on a body. Momentum p equals mv, and so if the mass is constant, then in one dimension we have

$$\frac{dp}{dt} = m\frac{dv}{dt} = m\frac{d^2x}{dt^2} = f$$

If we are given the force as a function of x, then this equation is a differential equation for x(t), which is called the trajectory of the particle. Going back to the simple harmonic oscillator discussed in Problem 2–7, if we let x be the displacement of the mass from its equilibrium position, then Hooke's law says that f(x) = −kx, and the differential equation corresponding to Newton's second law is

$$\frac{d^2x}{dt^2} + kx(t) = 0$$

a differential equation that we have seen several times.

2–16. Consider a body falling freely from a height x_0 according to Figure 2.9a. If we neglect air resistance or viscous drag, the only force acting upon the body is the gravitational force mg. Using the coordinates in Figure 2.9a, mg acts in the same direction as x and so the differential equation corresponding to Newton's second law is

$$m\frac{d^2x}{dt^2} = mg$$

Show that

$$x(t) = \frac{1}{2}gt^2 + v_0 t + x_0$$

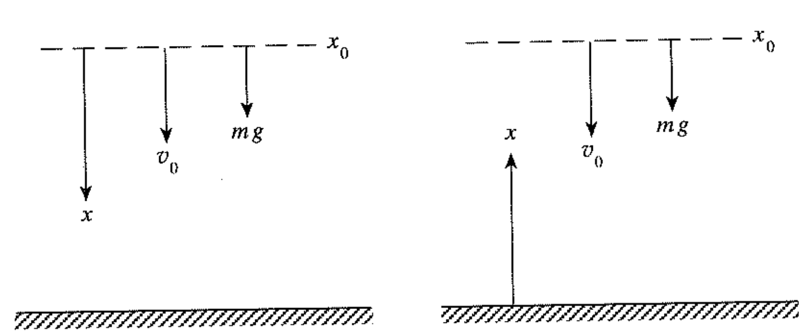
where x_0 and v_0 are the initial values of x and v. According to Figure 2.9a, $x_0 = 0$ and so

$$x(t) = \frac{1}{2}gt^2 + v_0 t$$

If the particle is just dropped, then $v_0 = 0$ and so

$$x(t) = \frac{1}{2}gt^2$$

Discuss this solution. Now do the same problem using Figure 2.9b as the definition of the various quantities involved, and show that although the equations may look different from those above, they say exactly the same thing because the picture we draw to define the direction of x, v_0, and mg does not affect the falling body.



(a) (b)

FIGURE 2.9
(a) A coordinate system for a body falling from a height x_0, and (b) a different coordinate system for a body falling from a height x_0.

We solve the equation (written using the coordinates in Figure 2.9a)

$$m\frac{d^2x}{dt^2} = mg$$

by integrating twice to obtain

$$x = \frac{gt^2}{2} + ct + k$$

At $t = 0$, $x = 0$, and so $k = 0$. Likewise, at $t = 0$, $dx/dt = v_0$, and so $c = v_0$. Thus, we obtain the result

$$x(t) = \frac{1}{2}gt^2 + v_0t \tag{1}$$

When $v_0 = 0$, we have $x(t) = \frac{1}{2}gt^2$, which is a formula for the acceleration of a falling body from rest; the distance $x(t)$ increases quadratically with time. Newton's equation in the coordinate system of Figure 2.9b is

$$m\frac{d^2x}{dt^2} = -mg$$

whose general solution is

$$x(t) = at^2 + bt + c$$

The particle is falling from an initial height of x_0, so $x(0) = x_0$. Also $dx/dt = -v_0$ initially, and so

$$x(t) = -\frac{gt^2}{2} - v_0t + x_0 \tag{2}$$

Both Equations 1 and 2 say the very same thing. For example, to find the time that it takes for the mass to strike the ground, let $x(t) = x_0$ in Equation 1 and $x(t) = 0$ in Equation 2 to obtain

$$\frac{1}{2}gt^2 + v_0t = x_0$$

in each case.

2–17. Derive an equation for the maximum height a body will reach if it is shot straight upward with a velocity v_0. Refer to Figure 2.9b but realize that in this case v_0 points upward. How long will it take for the body to return to its initial position, $x = 0$?

Using the coordinate system of Figure 2.9b and Equation 2 derived in Problem 2–16 (with $v_0 = -v_0$ and $x(0) = 0$), we find that

$$x(t) = -gt^2 + v_0t$$

and

$$\frac{dx}{dt} = -gt + v_0$$

To determine how long it will take for the body to return to earth, we first calculate how long it will take the body to reach its maximum height. At its maximum height, the velocity is zero ($dx/dt = 0$). Therefore the time needed to reach the maximum height, t_{max}, is given by

$$0 = -gt_{max} + v_0$$

or

$$t_{max} = \frac{v_0}{g}$$

The maximum height that the mass will attain is $x(t_{max}) = v_0^2/2g$. From the instant the body is shot, it will take $2t_{max}$ (or $2v_0/g$) to return to earth because it takes the same amount of time for the body to return from its maximum height as it takes the body to reach that height.

2–18. Consider a simple pendulum as shown in Figure 2.10. We let the length of the pendulum be l and assume that all the mass of the pendulum is concentrated at its end as shown in Figure 2.10. A physical example of this case might be a mass suspended by a string. We assume that the motion of the pendulum is set up such that it oscillates within a plane so that we have a problem in plane polar coordinates. Let the distance along the arc in the figure describe the motion of the pendulum, so that its momentum is $mds/dt = mld\theta/dt$ and its rate of change of momentum is $mld^2\theta/dt^2$. Show that the component of force in the direction of motion is $-mg\sin\theta$, where the minus sign occurs because the direction of this force is opposite to that of the angle θ. Show that the equation of motion is

$$ml\frac{d^2\theta}{dt^2} = -mg\sin\theta$$

Now assume that the motion takes place only through very small angles and show that the motion becomes that of a simple harmonic oscillator. What is the natural frequency of this harmonic oscillator? *Hint*: Use the fact that $\sin\theta \approx \theta$ for small values of θ.

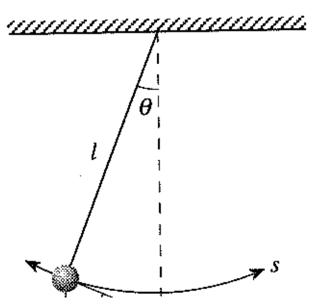


FIGURE 2.10
The coordinate system describing an oscillating pendulum.

The component of the force mg along the arc in Figure 2–10 is $mg\sin\theta$, but in a direction opposite to the motion. Newton's law states that the change in momentum is equal to the forces acting on the body. Therefore

$$m\frac{d^2s}{dt^2} = -mg\sin\theta$$

Since $s = l\theta$,

$$ml\frac{d^2\theta}{dt^2} = -mg\sin\theta \tag{a}$$

For small angles, $\sin\theta \approx \theta$ and (a) becomes

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

The general solution to this equation is (Example 2–4)

$$x(t) = c_1 \cos t\sqrt{\frac{g}{l}} + c_2 \sin t\sqrt{\frac{g}{l}}$$

The natural frequency of the pendulum is $(g/l)^{1/2}$.

2–19. Consider the motion of a pendulum like that in Problem 2–18 but swinging in a viscous medium. Suppose that the viscous force is proportional to but oppositely directed to its velocity; that is,

$$f_{\text{viscous}} = -\lambda\frac{ds}{dt} = -\lambda l\frac{d\theta}{dt}$$

where λ is a viscous drag coefficient. Show that for small angles, Newton's equation is

$$ml\frac{d^2\theta}{dt^2} + \lambda l\frac{d\theta}{dt} + mg\theta = 0$$

Show that there is no harmonic motion if

$$\lambda^2 > \frac{4m^2 g}{l}$$

Does it make physical sense that the medium can be so viscous that the pendulum undergoes no harmonic motion?

Now we have both the force of gravity and the viscous force acting on the system, so, again by Newton's Law,

$$ml\frac{d^2\theta}{dt^2} = -mg\sin\theta - \lambda l\frac{d\theta}{dt}$$

$$ml\frac{d^2\theta}{dt^2} + mg\sin\theta + \lambda l\frac{d\theta}{dt} = 0$$

For small angles $\sin\theta \approx \theta$, so

$$\frac{d^2\theta}{dt^2} + \frac{\lambda}{m}\frac{d\theta}{dt} + \frac{g}{l}\theta = 0 \qquad \text{(a)}$$

Substituting $\theta(t) = e^{\alpha t}$ into (a) and dividing through by $\theta(t)$ gives

$$\alpha^2 + \frac{\lambda}{m}\alpha + \frac{g}{l} = 0$$

Solving for α gives

$$\alpha = -\frac{\lambda}{2m} \pm \frac{\left(\frac{\lambda^2}{m^2} - \frac{4g}{l}\right)^{1/2}}{2} = -\frac{\lambda}{2m} \pm \left(\frac{\lambda^2}{4m^2} - \frac{g}{l}\right)^{1/2}$$

and so the solution to the differential equation is (Problem 2–6)

$$\theta(t) = e^{-\lambda t/2m}\left(c_1 e^{\beta t} + c_2 e^{\beta t}\right)$$

where $\beta = \left(\frac{\lambda^2}{4m^2} - \frac{g}{l}\right)^{1/2}$. If $\lambda^2 < 4m^2 g/l$, then β is imaginary and the motion is harmonic. However, if $\lambda^2 \geq 4m^2 g/l$, then β is real and there is no harmonic motion. The viscosity is so large that the pendulum simply approaches its vertical position without oscillating.

2–20. Consider two pendulums of equal lengths and masses that are connected by a spring that obeys Hooke's law (Problem 2–7). This system is shown in Figure 2.11. Assuming that the motion takes place in a plane and that the angular displacement of each pendulum from the vertical is small, show that the equations of motion for this system are

$$m\frac{d^2x}{dt^2} = -m\omega_0^2 x - k(x-y)$$

$$m\frac{d^2y}{dt^2} = -m\omega_0^2 y - k(y-x)$$

where ω_0 is the natural vibrational frequency of each isolated pendulum, [i.e., $\omega_0 = (g/l)^{1/2}$] and k is the force constant of the connecting spring. In order to solve these two simultaneous differential equations, assume that the two pendulums swing harmonically and so try

$$x(t) = Ae^{i\omega t} \qquad y(t) = Be^{i\omega t}$$

Substitute these expressions into the two differential equations and obtain

$$\left(\omega^2 - \omega_0^2 - \frac{k}{m}\right)A = -\frac{k}{m}B$$

$$\left(\omega^2 - \omega_0^2 - \frac{k}{m}\right)B = -\frac{k}{m}A$$

Now we have two simultaneous linear homogeneous algebraic equations for the two amplitudes A and B. We shall learn in MathChapter E that the determinant of the coefficients must vanish in order for there to be a nontrivial solution. Show that this condition gives

$$\left(\omega^2 - \omega_0^2 - \frac{k}{m}\right)^2 = \left(\frac{k}{m}\right)^2$$

Now show that there are two natural frequencies for this system, namely,

$$\omega_1^2 = \omega_0^2 \quad \text{and} \quad \omega_2^2 = \omega_0^2 + \frac{2k}{m}$$

Interpret the motion associated with these frequencies by substituting ω_1^2 and ω_2^2 back into the two equations for A and B. The motion associated with these values of A and B are called *normal modes* and any complicated, general motion of this system can be written as a linear combination of these normal modes. Notice that there are two coordinates (x and y) in this problem and two

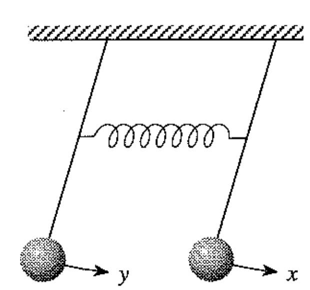


FIGURE 2.11
Two pendulums coupled by a spring that obeys Hooke's law.