Chem 3322 homework #1 solutions

Problem 1, 16 marks – **classical wave equation**

a) Show that $u(x,t) = \sin(kx - \omega t)$ satisfies the classical wave equation by directly using the function $\sin(kx - \omega t)$ in the wave equation.

Solution:

For this problem we need the chain rule, which states, in general, that

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \tag{1}$$

On the left hand side of Eq. (6) we have

$$\frac{\partial^2}{\partial x^2} \sin(kx - \omega t) = \frac{\partial}{\partial x}[\cos(kx - \omega t)\frac{\partial(kx - \omega t)}{\partial x}] = \frac{\partial}{\partial x}[k\cos(kx - \omega t)] \tag{2}$$

Taking one more partial gives

$$= -k^2 \sin(kx - \omega t) = -k^2 u(x,t) \tag{3}$$

On the right hand side of Eq. (6) we have

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2} \sin(kx - \omega t) = \frac{1}{v^2}\frac{\partial}{\partial t}[\cos(kx - \omega t)\frac{\partial(kx - \omega t)}{\partial t}] = \frac{1}{v^2}\frac{\partial}{\partial t}[-\omega\cos(kx - \omega t)] \tag{4}$$

Taking one more partial gives

$$= -\frac{1}{v^2}\omega^2 \sin(kx - \omega t) = -\frac{1}{v^2}\omega^2 u(x,t) \tag{5}$$

Then, using that $v = \omega/k$, we can see that the left hand side and the right hand side are equal, and thus we have shown that this $u(x,t)$ satisfies the wave equation.

b) Show that $u(x,t) = \sin(kx - \omega t)$ satisfies the classical wave equation by using the trigonometric identity $\sin(A - B) = \sin A \cos B - \cos A \sin B$.

Solution:

The wave equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \tag{6}$$

On the left hand side, for the given $u(x,t)$, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2[\sin(kx)\cos(\omega t) - \cos(kx)\sin(\omega t)]}{\partial x^2} = \frac{\partial}{\partial x}[k\cos(kx)\cos(\omega t) + k\sin(kx)\sin(\omega t)] \tag{7}$$

$$= -k^2 \sin(kx)\cos(\omega t) + k^2 \cos(kx)\sin(\omega t) = -k^2 u(x,t) \tag{8}$$

On the right hand side, we have

$$\frac{1}{v^2}\frac{\partial^2[\sin(kx)\cos(\omega t) - \cos(kx)\sin(\omega t)]}{\partial t^2} = \frac{1}{v^2}\frac{\partial}{\partial t}[-\omega \sin(kx)\sin(\omega t) - \omega \cos(kx)\cos(\omega t)] \tag{9}$$

$$= \frac{1}{v^2}[-\omega^2 \sin(kx)\cos(\omega t) + \omega^2 \cos(kx)\sin(\omega t)] = -\frac{1}{v^2}\omega^2 u(x,t) \tag{10}$$

Then, using that $v = \omega/k$, we can see that the left hand side and the right hand side are equal, and thus we have shown that this $u(x,t)$ satisfies the wave equation.

c) Show that $u(x,t) = e^{i(kx-\omega t)}$ satisfies the classical wave equation by using the Euler identity $e^{i\theta} = \cos\theta + i\sin\theta$.

Solution:

On the left hand side we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left[\cos(kx-\omega t) + i\sin(kx-\omega t)\right] \tag{11}$$

$$= \frac{\partial}{\partial x}\left[-k\sin(kx-\omega t) + ik\cos(kx-\omega t)\right] = -k^2\cos(kx-\omega t) - ik^2\sin(kx-\omega t)$$

On the right hand side we have, taking one time derivative already,

$$\frac{1}{v^2}\frac{\partial}{\partial t}\left[\omega\sin(kx-\omega t) - i\omega\cos(kx-\omega t)\right] = \frac{1}{v^2}\left[-\omega^2\cos(kx-\omega t) - i\omega^2\sin(kx-\omega t)\right] \tag{12}$$

Then, using that $v = \omega/k$, we can see that the left hand side and the right hand side are equal, and thus we have shown that this $u(x,t)$ satisfies the wave equation.

d) Show that $u(x,t) = e^{i(kx-\omega t)}$ satisfies the classical wave equation by directly differentiating the function $e^{i(kx-\omega t)}$.

Solution:

Here again we need the chain rule. On the left hand side of Eq. (6) we have

$$\frac{\partial^2}{\partial x^2}e^{i(kx-\omega t)} = \frac{\partial}{\partial x}[e^{i(kx-\omega t)}\frac{\partial(i(kx-\omega t))}{\partial x}] = \frac{\partial}{\partial x}[ike^{i(kx-\omega t)}] = -k^2 e^{i(kx-\omega t)} \tag{13}$$

On the right hand side of Eq. (6) we have

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2}e^{i(kx-\omega t)} = \frac{1}{v^2}\frac{\partial}{\partial t}[-i\omega e^{i(kx-\omega t)}] = -\frac{1}{v^2}\omega^2 e^{i(kx-\omega t)} \tag{14}$$

Then, using that $v = \omega/k$, we can see that the left hand side and the right hand side are equal, and thus we have shown that this $u(x,t)$ satisfies the wave equation.

<u>Problem 2, 10 marks</u> – different wavelength components

a) Show that $u(x,t) = \sin(k_1 x)\cos(\omega_1 t) - \cos(k_2 x)\sin(\omega_2 t)$ is not a classical wave if $k_2 = 2k_1$ and $\omega_1 = \omega_2$.

Solution:

On the left hand side of Eq. (6) we have

$$\frac{\partial^2}{\partial x^2}[\sin(k_1 x)\cos(\omega_1 t) - \cos(k_2 x)\sin(\omega_2 t)] \tag{15}$$

$$= -k_1^2 \sin(k_1 x)\cos(\omega_1 t) + k_2^2 \cos(k_2 x)\sin(\omega_2 t) \tag{16}$$

Using $k_2 = 2k_1$ and $\omega_1 = \omega_2$ gives

$$= k_1^2[-\sin(k_1 x)\cos(\omega_1 t) + 4\cos(k_2 x)\sin(\omega_2 t)] \tag{17}$$

On the right hand side of Eq. (6) we have

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2}[\sin(k_1 x)\cos(\omega_1 t) - \cos(k_2 x)\sin(\omega_2 t)] \tag{18}$$

$$= \frac{1}{v^2}[-\omega_1^2 \sin(k_1 x)\cos(\omega_1 t) + \omega_2^2 \cos(k_2 x)\sin(\omega_2 t)] \tag{19}$$

Using $k_2 = 2k_1$ and $\omega_1 = \omega_2$ gives

$$= \frac{\omega_1^2}{v^2}[-\sin(k_1 x)\cos(\omega_1 t) + \cos(k_2 x)\sin(\omega_2 t)] \tag{20}$$

By comparing the expressions in Equations (17) and (20) you should see that the extra factor of 4 in Eq. (17) prevents us from making them the same, and we have to conclude that this function does not represent a wave.

b) Show that $u(x,t) = \sin(k_1 x)\cos(\omega_1 t) - \cos(k_2 x)\sin(\omega_2 t)$ is a classical wave if $k_2 = 2k_1$ and $\omega_2 = 2\omega_1$. What is the propagation speed of this wave?

Solution:

Now the change is that Equation (20) becomes

$$= \frac{\omega_1^2}{v^2}[-\sin(k_1 x)\cos(\omega_1 t) + 4\cos(k_2 x)\sin(\omega_2 t)] \tag{21}$$

so that the left hand side and right hand side are equal provided that we take the propagation speed to be $v = \omega_1 / k_1$.

Problem 3, 10 marks – Taylor series

For (a), (b), and (c) you can look up the answers using any resource.

a) Write down, up to (and including) 7th powers of x, the Taylor series for $\sin x$.

Solution:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{22}$$

b) Write down, up to 7th powers of x, the Taylor series for $\cos x$.

Solution:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{23}$$

c) Write down, up to 7th powers of x, the Taylor series for e^x.

Solution:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \tag{24}$$

d) Write down, up to 7th powers of x, the Taylor series for e^{ix} by using your answer (c).

Solution:

$$e^{ix} = 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \frac{i^6x^6}{6!} + \frac{i^7x^7}{7!} + \cdots \tag{25}$$

Now, we have the relations $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$, and $i^7 = -i$, giving

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \cdots \tag{26}$$

or, rearranging,

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right) \tag{27}$$

e) By comparing your answer (d) to the Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$ show how you could identify the $\sin x$ and $\cos x$ Taylor series (assuming you didn't know them).

Solution: By looking at Equation (27) and using the Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$, we can immediately identify the Taylor series for $\sin x$ and $\cos x$ in Equations (22) and (23).

Problem 4, 10 marks – **operators**

a) We usually denote an operator by a capital letter with a carat over it, *e.g.*, $\hat{A}$. Thus, we write

$$\hat{A}f(x) = g(x) \tag{28}$$

to indicate that the operator $\hat{A}$ operates on $f(x)$ to give a new function $g(x)$.

Evaluate (see page 75) $\hat{A}f(x)$ where $f(x) = 2x^2$ and where

$$\hat{A} = \frac{d^2}{dx^2} + 2\frac{d}{dx} + 3 \tag{29}$$

Solution:

Following pages 75 and 76, we have

$$\hat{A}f(x) = 4 + 8x + 6x^2 \tag{30}$$

b) Consider the operator (see page 79)

$$\hat{C} = \hat{A}\hat{B} - \hat{B}\hat{A} \tag{31}$$

Specifically, take $\hat{A} = x$ and $\hat{B} = d/dx$. What does this operator $\hat{C}$ do to a function $f(x)$? Based on your answer, express this operator in a simpler form.

Solution:

Following page 79, we have

$$\hat{C}f(x) = \hat{A}\left(\hat{B}f(x)\right) - \hat{B}\left(\hat{A}f(x)\right) \tag{32}$$

$$= x\frac{df(x)}{dx} - \frac{d}{dx}\left(xf(x)\right) \tag{33}$$

$$= x\frac{df(x)}{dx} - x\frac{df(x)}{dx} - f(x) \tag{34}$$

from the product rule

$$= -f(x) \tag{35}$$

Therefore the operator is just multiplication by minus one.