Problem 1 – Schrödinger Equation with a step potential, 10 marks

For the step potential we discussed in class, we decided that we could have different
wavelengths on either side of the step but a single frequency. In this problem will examine
this observation in a little more detail. As we discussed, such a wavefunction must be
continuous, and its first derivative with respect to position must also be continuous, since
the Schrödinger equation includes a second derivative with respect to position (a twice-
differentiable function must be continuous and must have a continuous first derivative.)

With the potential energy defined as

\[ V(x) = \begin{cases} 
0 & \text{for } x < 0 \\
1 & \text{for } x \geq 0 
\end{cases} \]  

we will consider the following wavefunction

\[ \psi(x, t) = \begin{cases} 
Ae^{i(k_1 x - \omega t)} + Be^{-i(k_1 x + \omega t)} & \text{for } x < 0 \\
Ce^{i(k_2 x - \omega t)} & \text{for } x \geq 0 
\end{cases} \]  

The interpretation of this wavefunction is that we are considering a particle, initially in
the $x < 0$ region of space, traveling to the right. As such, the $A$ coefficient represents the
particle traveling to the right in the $x < 0$ region of space. The $B$ coefficient represents the
particle traveling to the left in the $x < 0$ region of space, meaning that it has been reflected
from the step. The $C$ coefficient represents the particle traveling to the right in the $x > 0$
region of space, meaning that it has been transmitted through the step.

Show that this wavefunction solves the Schrödinger equation as long as

\[ \hbar \omega = \frac{\hbar^2 k_1^2}{2m} = \frac{\hbar^2 k_2^2}{2m} + 1 \]  

Also, work out the relationships between the coefficients $A$, $B$, and $C$ such that the
wavefunction is continuous and has a continuous first derivative over all of space (with
$k_1 \neq k_2$).