DRAFT: Generalizing Ham Sandwich Cuts to Equitable Subdivisions

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1 Abstract

We prove a generalization of famous Ham Sandwich Theorem for the plane. Given gn red points and gm blue points in the plane in general position, there exists a subdivision of the plane into g disjoint convex polygons, each of which contains n red points and m blue points. For $g = 2$ this problem is equivalent to the Ham Sandwich Theorem in the plane. We also present an efficient algorithm for constructing an equitable subdivision.

2 Introduction

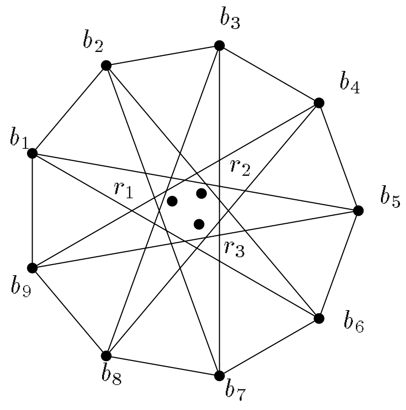
The planar case of the well-known discrete Ham Sandwich Theorem [16] states that, for finite sets of red and blue points in the plane, there exists a line dividing both red and blue points into sets of equal size. The Ham Sandwich problem is well studied from an algorithmic point of view [2, 4, 5, 6, 7, 11, 12, 13, 15, 17]. An optimal algorithm of Lo et al. [12] finds a Ham Sandwich cut in linear time. Very recently Kaneko and Kano [10] considered balanced partitions of two sets in the plane. They gave the following conjecture.

Conjecture 1 *Let $m \geq 2, n \geq 2$ and g be positive integers. Let R and B be two disjoint sets of points in the plane such that no three points of $R \cup B$ are collinear, $|R| = gn$ and $|B| = gm$. Then $R \cup B$ can be partitioned into g subsets $P_1, \ldots, P_g$ satisfying the following two conditions: (i) P_i and P_j are linearly separable for all $1 \leq i < j \leq g$; and (ii) $|P_i \cap R| = n$ and $|P_i \cap B| = m$ for all $1 \leq i \leq g$.*

For $g = 2$ the conjecture is equivalent to Ham Sandwich Theorem [16, p.212]. Kaneko and Kano proved the conjecture in the case of $n = 2$.

In this paper we prove the conjecture of Kaneko and Kano. Actually we prove stronger theorem providing a subdivision of the plane. Note that, for $g = 2^k$, conjecture 1 can be proved by applying the Ham Sandwich Theorem in a divide-and-conquer fashion. For general g, such a strategy is not possible. Specifically, Figure 1 illustrates an example of 3 red and 9 blue points that does not admit an equitable 2-cutting, that is a partition of the plane by a line such that the i-th halfplane contains $g_i n$ red and $g_i m$ blue points for some integers $g_1, g_2 < g$ (in other words, there is no a halfplane with exactly one red and three blue points). However it is sufficient to apply *equitable*

3-cutting. An equitable 3-cutting is a partition of the plane by 3 rays with common apex into 3 convex wedges, each of which contains a proportional number of red and blue points. In other words, the i-th wedge contains $g_i n$ red points and $g_i m$ blue points, for some $0 \le g_1, g_2, g_3 < g$ with $g_1 + g_2 + g_3 = g$.



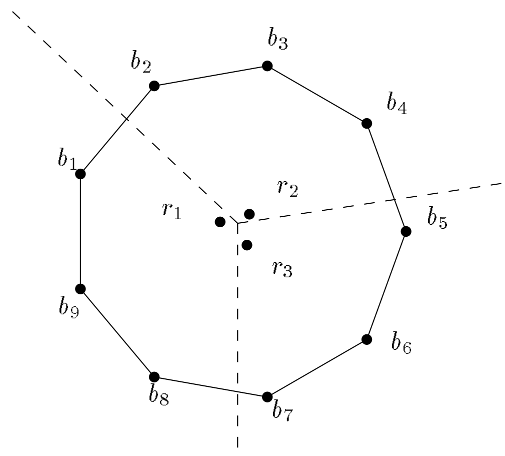


Figure 1: An example of 3 red points r_1, r_2, and r_3 and 9 blue points $b_1, \ldots, b_9$ for $g = 3$. a) There is no equitable 2-cutting. b) An equitable 3-cutting.

Theorem 2 (3-cutting) *For any gn red points ($g \ge 2$) and gm blue points in the plane in general position, there is equitable 3-cutting of red and blue points.*

When $g = 2$, Theorem 2 is equivalent to the Ham Sandwich Theorem. For $g \ge 3$, we can apply Theorem 2 recursively for the wedges with $g_i \ge 2$.

We present an efficient algorithm for partitioning the plane in $O(N^{4/3} \log^3 N \log g)$ time where $N = g(n + m)$ is total number of points.

3 Preliminaries

Let R be a set of gn red points and let B be a set of gm blue points in the plane in general position, i.e. there are no 3 points lying on the same line. The integer g is the number of groups into which the sets R and B are to be divided. A *2-cutting* is a partition of the plane by the line into two halfplanes. A 2-cutting is *equitable* (or more specifically (g_1, g_2)-equitable) if the i-th open halfplane contains exactly $g_i n$ red and $g_i m$ blue points for some integers $g_1, g_2 < g$ with $g_1 + g_2 = g$.

If there is an equitable 2-cutting of the red and blue points, then the problem can be reduced to the subproblems for the number of groups g_1 and g_2. Otherwise, for any $g_1 \in \{1, \ldots, g-1\}$, a halfplane containing exactly $g_1 n$ red points (we call it g_1-*halfplane*) contains either less or greater than $g_1 m$ blue points. We assign to the halfplane the sign -1 or $+1$, respectively.

Lemma 3 *If, for some $g_1 \in \{1, \ldots, g-1\}$, there are two g_1-halfplanes with opposite signs, then there is a (g_1, g_2)-equitable 2-cutting.*

Proof: We apply the geometric duality. The dual of a point $p = (p_1, p_2)$ is the line $y = -p_1x + p_2$, and the dual of a line $y = ax + b$ is the point (a, b).

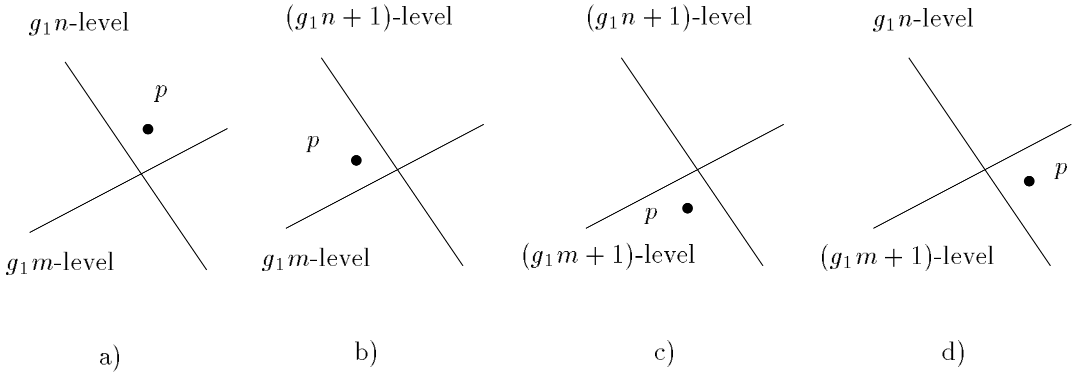


Figure 2: The point p is dual of an equitable 2-cutting.

Let l_1 and l_2 be the lines that define two g_1-halfplanes with signs -1 and $+1$. We can rotate the plane so that the lines become non-vertical and the g_1-halfplanes become lower halfplanes. In dual setting the red and blue points correspond red and blue lines. The *k-level* of an arrangement of lines is a polygonal line formed by all edges of level k, where the level of an edge is the number of lines lying strictly below its interior point. The point dual an equitable 2-cutting lies between the g_1n-level and $(g_1n + 1)$-level of red lines and between the g_1m-level and $(g_1m + 1)$-level of blue lines. An equitable 2-cutting can be found if two levels of different colors are crossed, see Fig. 3. Otherwise $(g_1n + 1)$-level of red lines lies below the g_1m-level of blue lines or g_1n-level of red lines lies above the $(g_1m + 1)$-level of blue lines. In this case the points p_1, p_2 dual to l_1, l_2 lie below the g_1m-level of blue lines or above the $(g_1m + 1)$-level of blue lines (the points p_1, p_2 lie between the g_1n-level and $(g_1n + 1)$-level of red lines). It is impossible because the point p_1 is below g_1m-level and the point p_2 is above $(g_1m + 1)$-level of blue lines. ∎

Lemma 3 allows us to assume that, for any g_1, all g_1-halfplanes have the same sign. However, it seems that the signs of all halfplanes cannot be computed efficiently. In practice, we will use Lemma 3 to find an equitable 2-cutting if, for some g_1, two g_1-halfplanes with opposite signs discovered. To begin we define signs for a canonical set of halfplanes. Draw $g - 1$ vertical lines that divide red points into g sets of n points. We assume that there are no 2 points lying on the same vertical line (otherwise we can rotate the points). We define the *sign* of g_1, $s(g_1)$, as the sign of the left halfplane formed by g_1-th vertical line.

A *3-cutting* is a partition of the plane into 3 wedges W_1, W_2 and W_3 by 3 rays with a common point that is called the *apex* of the 3-cutting. A 3-cutting is *convex* if its wedges are convex. A convex 3-cutting is *equitable* (or more specifically (g_1, g_2, g_3)-equitable) if the open wedge W_i contains exactly g_in red and g_im blue points for some integers $0 \le g_1, g_2, g_3 < g$ with $g_1 + g_2 + g_3 = g$. Note that an equitable 3-cutting is an equitable 2-cutting for some $g_i = 0$.

We will find a 3-cutting with one ray going down, see Fig. 1b), 3. We call the wedges adjacent to this ray *left* and *right*. The remaining wedge is *upper*. The rays are *left*, *right* and *downward*. (Note that at most one of the wedges can be non-convex.)

Though our goal is to find 3-cutting whose rays don't pass through data points, it is convenient to define a *canonical red 3-cutting* for a triple (g_1, g_2, g_3) that violates this condition. Let p be a point in the plane and $p_1, \ldots, p_{gn}$ be a list of red points in clockwise order starting from (but omitting) the downward ray from p. (If p is red, then $p_1 = p$. For a red point q below p, $p_{gn} = q$.) The points $p_{g_1 n}$ and $p_{(g_1+g_2)n}$ define a red 3-cutting with apex p. A *canonical blue 3-cutting* is defined similarly.

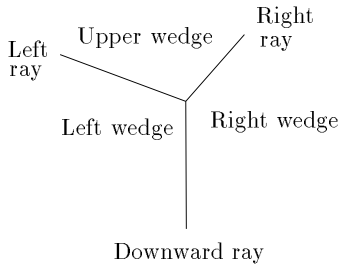


Figure 3: The notation of a 3-cutting.

4 A 3-cutting exists

Note that any (g_1, g_2)-equitable 2-cutting can be viewed as a $(g_1, g_2, 0)$-equitable 3-cutting. Thus the Ham Sandwich Theorem implies that Theorem 2 is true for even g. The case of odd g is more complicated.

4.1 Odd $g \geq 3$

We use the following topological lemma (a variant of one from Goodman, Pach and Yap [9]).

Lemma 4 *Let $\mathcal{R}$ be any closed region in the plane. Let A be a finite arrangement of curves in $\mathcal{R}$, in which each face of A is labeled with a 1, 2, 3, or 4. Suppose that there are points p and q, labeled 1 and 3, on the boundary $\delta\mathcal{R}$ such that one component of $\delta\mathcal{R} - \{p, q\}$ uses labels 1,2,3 and the other component uses 1,3,4. Then some point of $\mathcal{R}$ lies on the boundary between two faces whose labels differ by 2.*

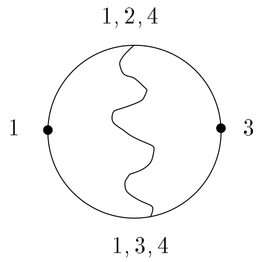


Figure 4: 1-3 or 3-4 transition.

Proof: Subtract the faces labeled 2 and 4 from $\mathcal{R}$, and walk along the boundary of the resulting region starting from p. We either reach q, in which case we have seen a 1-3 transition, or we return to p, in which case we have walked from one component of $\delta R - \{p, q\}$ to the other, and have seen a 2-4 transition. ∎

The following theorem provides an equitable 3-cutting or 2-cutting for given numbers of points in groups.

Theorem 5 *Let R be any set of gn red points and B be any set gm blue points in the plane in general position. Let g_1, g_2 and g_3 be positive integers with $g_1 + g_2 + g_3 = g$. If $s(g_1) = s(g_2) = s(g_3)$ then there exists either*

- *a (g_1, g_2, g_3)-equitable 3-cutting, or*
- *a $(g_i, g - g_i)$-equitable 2-cutting for some i.*

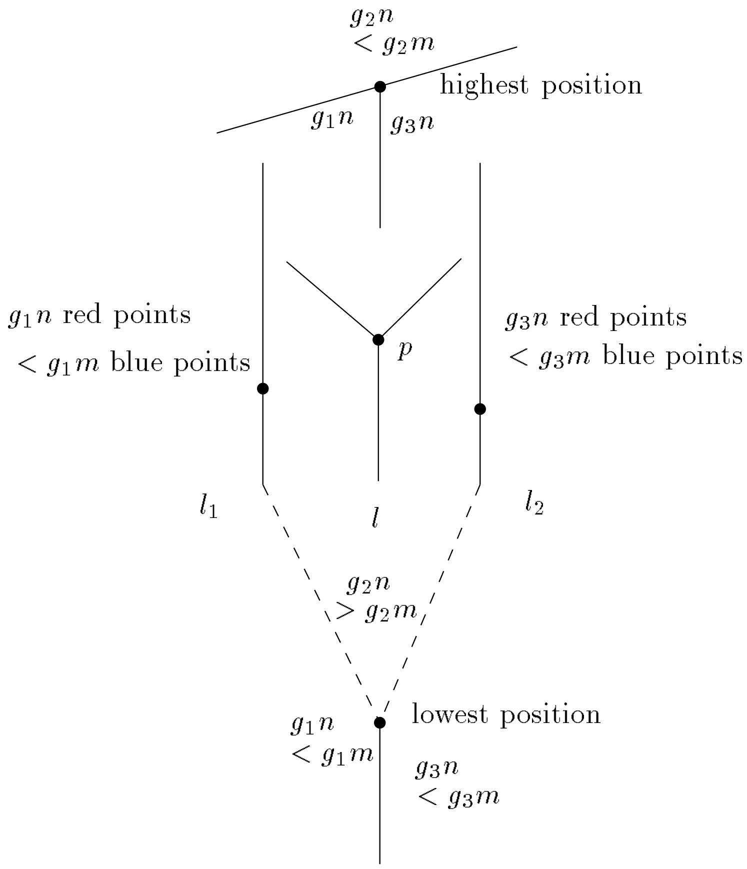


Figure 5: The region $\mathcal{R}$ for triple (g_1, g_2, g_3).

Proof:

We can assume the sign $s(g_1) = -1$ (if $s(g_1) = 1$ we exchange the colors red and blue). We define a region $\mathcal{R}$ and label each $p \in \mathcal{R}$ as follows: Let $\mathcal{R}$ consist of those points in which the canonical red 3-cutting for triple (g_1, g_2, g_3) with apex p are convex. Recall that no vertical line contains more than one data point. Let $x_1, \ldots, x_{gn}$ be a sorted list of x-coordinates of red points. Consider 2 vertical lines l_1 and l_2 passing through $g_1 n$-th and $(g_1 + g_2)n$-th coordinates, i.e. $x = x_{g_1 n}$ and $x = x_{(g_1+g_2)n}$. The closed halfplane left of l_1 contains exactly $g_1 n$ red points and less than $g_1 m$ blue points since $s(g_1) = -1$. Symmetrically, the open halfplane right of l_2 contains exactly $g_3 n$ red points and less than $g_3 m$ blue points since $s(g_3) = -1$. For any point left of l_1, the left wedge of a canonical red 3-cutting is non-convex. Symmetrically, for any point right of l_2, the right wedge of a canonical red 3-cutting is non-convex. Hence the region $\mathcal{R}$ is bounded by the lines l_1 and l_2. Note that an open strip between the lines l_1 and l_2 contains exactly $g_2 n$ red points and greater than $g_2 m$ blue points.

Consider any vertical line l between l_1 and l_2. For a point p on l, the angles of left and right wedges of the canonical red 3-cutting are both less than π. Furthermore each such angle is a monotone function of the y-coordinate of the point p (the angles decrease when p goes up). It follows that the angle of the upper wedge is monotone. If the point p has y-coordinate less than minimum of y-coordinates of data points, then both the left and right rays go up and the upper wedge is convex. For a point p on l above all y-coordinates of red and blue points, the upper wedge of the canonical red 3-cutting has an angle greater than π. Hence there is a point on the line l such that the points below p belong to $\mathcal{R}$.

The lines passing through pairs of points of $R \cup B$ and the vertical lines through the points of $R \cup B$ form an arrangement A. For any point p from the interior of a face $f \in A$, the left (right) rays of the two canonical cuttings are distinct. We are interested in the clockwise order of left rays (RB or BR) and the clockwise order of the right rays of the two canonical cuttings. We assign a label to p according to 1=RB-RB, 2=RB-BR, 3=BR-BR, and 4=BR-RB. All points of the same face of the arrangement A have the same label, which is taken to be the label of the face.

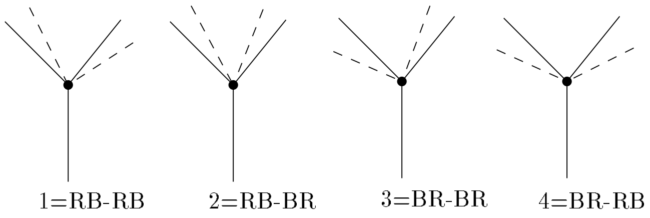


Figure 6: The labels. The dashed rays are blue.

Consider the labels on the boundary of $\mathcal{R}$. At $y = -\infty$ all labels are 2=RB-BR due to the signs of g_1 and g_3. The faces forming the left boundary of $\mathcal{R}$, all of which are incident on and right of the line l_1, have labels RBxx by the sign of g_1, which implies labels 1 or 2. Symmetrically, the faces of the right boundary of $\mathcal{R}$ have labels xxBR, which implies labels 2 or 3. Along the remaining boundary of R, the red partition uses a straight line at top, so the sign of g_2 rules out label 2=RB-BR. The top face along the line l_1 has label 1=RB-RB. The top face along the line l_2 has label 3=BR-BR. By Lemma 4, therefore, we must have a 1-3 or 2-4 transition. This implies the existence of an equitable 3-cutting by Lemma 6. ■

Lemma 6 *Let p be a point of the region $\mathcal{R}$ that lies on the boundary between two faces whose labels differ by 2. One of the faces around p contains the apex of an equitable 3-cutting.*

Proof:

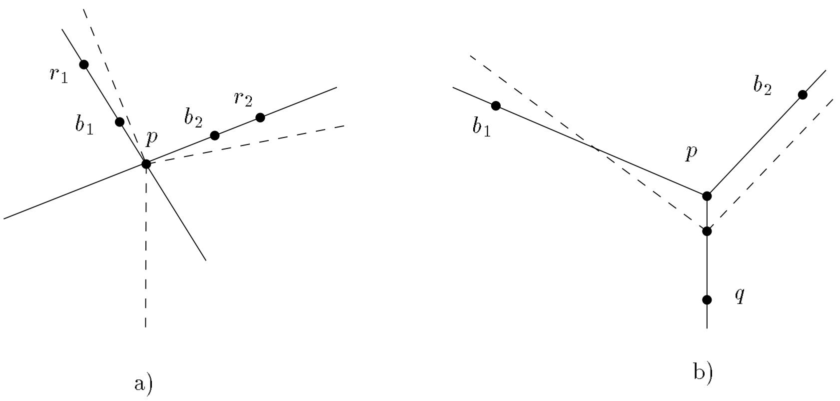


Figure 7: Construction of an equitable 3-cutting (dashed rays) for a point p with 2-transition of labels. a) no point below p. b) the red point q below p.

First consider a special case where no data point lies below the apex p. Let q be a point in a face whose boundary contains the point p. Let r_1 and r_2 be red points that define canonical red 3-cutting at q, with $\overrightarrow{qr_1}$ being the left ray. Similarly let b_1 and b_2 be blue points that define canonical blue 3-cutting at q. Actually these points define canonical red and blue 3-cuttings for all points around p. If the line passing through the points r_1 and b_1 does not contain p, then the order of rays $\overrightarrow{qr_1}$ and $\overrightarrow{qb_1}$ is still the same when the point q is rotated around p. This implies that the faces around p use only two consecutive labels (1=RB-RB, 2=RB-BR or 3=BR-BR, 4=BR-RB) contradicting the assumptions. Hence the points r_1 and b_1 lie on a ray emanating from p. Similarly some ray emanating from p intersects the points r_2 and b_2. The required equitable 3-cutting can be obtained can be obtained by rotating the rays $\overrightarrow{pb_1}$ and $\overrightarrow{pb_2}$ by a sufficiently small angle clockwise.

Now suppose some point q, say red, lies below p or coincides with p. Clearly, all canonical blue 3-cuttings for points in a sufficiently small ball with center at p are defined by the same blue points, say b_1 and b_2. We prove that an equitable 3-cutting C can be obtained by a translation of the blue 3-cutting at p downward by a sufficiently small distance and a rotation the left ray by a sufficiently small angle clockwise such that the data points on the ray $\overrightarrow{pb_1}$ (except p, if $p \in R$) remain in the left wedge. The 3-cutting C divides the blue points into subsets of g_1m, g_2m and g_3m points.

Split the ball containing p by a vertical line passing through p. Let H be the left open halfball. If the orders of left rays of red and blue 3-cuttings for some points in H are changed (in other words there are two points in H with labels RBxx and BRxx), then there is a red point r_1 in the ray pb_1 that forms all left rays of the red 3-cuttings for points in H. Therefore the left wedge of C consist of g_1n red points. Otherwise the change of the order of left rays for points whose labels differ by 2 is caused by inserting the red point p into the left wedge of a red 3-cutting (with an apex in the right halfball). It follows that the left wedge of C again consist of g_1n red points.

Using a similar argument, we can show that the number of red points in the right wedge of of C is g_3n. Note that the right wedge of C is convex because it is covered by a convex right wedge of red 3-cutting for a point with a label xxRB. Hence C is an equitable 3-cutting. ∎

It remains to show how to construct an appropriate triple (g_1, g_2, g_3).

4.2 Triple g_1, g_2 and g_3

Given a sequence of signs $s(1), \ldots, s(g-1) \in \{-1, 1\}$ defined by vertical cuts. We can apply Lemma 3 if the i-th line cuts off less (greater) than im blue points and the $(g-i)$-th line cuts off greater (less) than im blue points. In other words the signs $s(i)$ and $s(g-i)$ are the same. To apply Theorem 5 we need a triple with the same signs. The following theorem provides one of these conditions.

Theorem 7 *For any sequence of signs $s(1), \ldots, s(g-1)$, there is a pair (g_1, g_2) or a triple (g_1, g_2, g_3) with sum g and the same signs.*

Proof: If the pair $(1, g-1)$ has the same signs, we are done. Suppose $s(1) \neq s(g-1)$. Let k be the smallest index with $s(k) \neq s(1)$ and $s(k-1) = s(1)$. Then either $s(k) = s(g-k)$ or the triple $(1, k-1, g-k)$ have the same signs. ∎

In fact one of the numbers in the triple can be specified.

Corollary 8 *For any sequence of signs $s(1),\ldots,s(g-1)$, there is a pair (g_1,g_2) or a triple $(g_1,g_2,1)$ with sum g and the same signs.*

Applying Theorem 7 directly adds extra factor g to the running time of constructing a subdivision the plane into g regions. The following theorem reduces this factor to $\log g$.

Theorem 9 *For any sequence of signs $s(1),\ldots,s(g-1)$, there is a pair (g_1,g_3) or a triple (g_1,g_2,g_3) with sum g and the same signs such that any $g_i \le \lfloor \frac{2g}{3} \rfloor$.*

Proof: For even g the pair $(g/2,g/2)$ has the same signs. Suppose that $g = 2k+1$. We can assume $s(1) = -1$ (otherwise we exchange the colors red and blue). If $s(k) = -1$ then triple $(1,k,k)$ satisfies the theorem, so suppose $s(k) = 1$. If $s(k+1) = 1$ then the pair $(k,k+1)$ satisfies the theorem, so suppose $s(k+1) = -1$. Let $k_1 \le k$ be the smallest number such that $s(k_1) = -1$ and $s(j) = 1$ for all j, $k_1 < j \le k$. Denote $k_2 = \lfloor \frac{g}{3} \rfloor$.

Suppose $k_1 \ge k_2$. Then the triple $(1, k_1, g-k_1-1)$ has the same signs since $s(g-k_1-1) = -s(k_1+1) = -1$. Note that $g-k_1-1 \le g - \lfloor \frac{g}{3} \rfloor - 1$. One can prove $g - \lfloor \frac{g}{3} \rfloor - 1 \le \lfloor \frac{2g}{3} \rfloor$ by considering all residues of g modulo 3.

Alternatively $k_1 < k_2$. In this case $g > 5$ because $k_2 = 1$ for $g = 5$. It follows that $g > 6$ and $k_2 + 1 = \lfloor \frac{g}{3} \rfloor + 1 \le \frac{g}{3} + 1 < \frac{g+1}{2} = k$. Hence $s(k_2+1) = 1$. The final triple with the same signs depends on the residue of g modulo 3.

- If $g = 3k_2$, then the triple (k_2,k_2,k_2) has the same signs.
- If $g = 3k_2+1$, then the triple (k_2,k_2,k_2+1) has the same signs.
- If $g = 3k_2+2$, then the triple (k_2,k_2+1,k_2+1) has the same signs.

■

5 Subdivision Theorems

Applying Theorem 2 in a divide-and-conquer fashion we prove our main result.

Theorem 10 (Equitable Subdivision) *Given gn red and gm blue points in the plane in general position. There exists a subdivision of the plane into g convex regions each of which contains n red and m blue points.*

The 3-cutting Theorem and the Equitable Subdivision Theorem are discrete. In the rest of this Section we prove continuous versions of these Theorems.

Let ρ_1 and ρ_2 be measurable functions $\mathsf{R}^2 \to [0,\infty)$ with $\int_{\mathsf{R}^2} \rho_i dx = 1$. A convex 3-cutting of the plane into wedges W_1, W_2 and W_3 is *equitable* (or more specifically (g_1,g_2,g_3)-equitable) if $\int_{W_i} \rho_1 dx = \int_{W_i} \rho_2 dx = g_i/g$ for some integers $0 \le g_1,g_2,g_3 < g$ with $g_1+g_2+g_3 = g$.

Theorem 11 (3-cutting, continuous version) *Let ρ_1 and ρ_2 be measurable functions $\mathsf{R}^2 \to [0,\infty)$ with $\int_{\mathsf{R}^2} \rho_i dx = 1$. Then there exists an equitable 2-cutting or an equitable 3-cutting.*

Proof: The idea is to approximate functions ρ_i using a continuous version of the Ham Sandwich Theorem. Consider the function ρ_1. For a region R, the *weight* of R is $\int_R \rho_1(x)dx$. The weight of the plane is 1. Draw any line l_1 (for example, vertical) that defines halfplanes H_1 and H_2 of weight 1/2. There is a line l_2 that halves both halfplanes by the continuous version of the Ham Sandwich Theorem, i.e. two lines l_1 and l_2 partition the plane into 4 wedges of the same weight 1/4 (we apply the Ham Sandwich Theorem for functions ϕ_1 and ϕ_2 defined by

$$\phi_i(p) = \begin{cases} 2\rho_1(p) & \text{if } p \in H_i \\ 0 & \text{otherwise} \end{cases}$$

We partition each of 4 wedges into 4 regions in the same fashion. The number of convex regions after partitioning k times is 4^k. This subdivision S satisfies a *line property* that any line properly intersects at most 3^k regions. The line property can be proved using the fact that a line cannot contain 4 interior points of wedges formed by 2 lines.

Pick a red point in each region. We define 4^k blue points in similar way using the function ρ_2. To apply Theorem 2 the number of red and blue points must be an integer multiple of g. We remove any $r = 4^k - g\lfloor 4^k/g \rfloor$ red and blue points. By 3-cutting Theorem 2, for any k, there exists an equitable 3-cutting of $g\lfloor 4^k/g \rfloor$ red and blue points. Let it is (g_1, g_2, g_3)-equitable 3-cutting of the plane into the wedges W_1, W_2 and W_3. By the line property each ray of 3-cutting cross at most 3^k convex regions of S. The wedge W_i contains at least $g_i\lfloor 4^k/g \rfloor - 2\cdot 3^k - r$ whole regions of S and intersects at most $g_i\lfloor 4^k/g \rfloor + 2\cdot 3^k + r$ regions of S. Each region of S has weight $1/4^k$. Hence

$$g_i\lfloor\frac{4^k}{g}\rfloor - 2\cdot 3^k - r \leq 4^k\int_{W_i}\rho_1 dx \leq g_i\lfloor\frac{4^k}{g}\rfloor + 2\cdot 3^k + r \text{ or}$$

$$|4^k\int_{W_i}\rho_1 dx - g_i\lfloor\frac{4^k}{g}\rfloor| \leq 2\cdot 3^k + r.$$

Using $r = 4^k - g\lfloor\frac{4^k}{g}\rfloor$ we obtain

$$|4^k(\int_{W_i}\rho_1 dx - \frac{g_i}{g}) + r\frac{g_i}{g}| \leq 2\cdot 3^k + r \text{ or}$$

$$4^k|\int_{W_i}\rho_1 dx - \frac{g_i}{g}| \leq 2\cdot 3^k + r + r\frac{g_i}{g} \leq 2\cdot 3^k + 2r.$$

Hence

$$|\int_{W_i}\rho_1 dx - \frac{g_i}{g}| \leq 2(\frac{3}{4})^k + 2\frac{g}{4^k}.$$

One can prove the same bound for ρ_2

$$|\int_{W_i}\rho_2 dx - \frac{g_i}{g}| \leq 2(\frac{3}{4})^k + 2\frac{g}{4^k}.$$

The triples (g_1, g_2, g_3) can be different if we change k. We extract an infinite sequence of numbers $k_1, \ldots, k_i, \ldots$ that define the same triples (g_1, g_2, g_3). Let p_i be an apex of 3-cutting constructed for $k = k_i$. It is clear that p_i lies in the vertical slab defining by the lines $x = a$ and $x = b$ such that

$$\int_{x<a}\rho_1 dx = \int_{x>b}\rho_1 dx = 1/g.$$

We can extract an infinite subsequence of apexes $p_{i_1}, p_{i_2}, \ldots$ that goes to some point $p \in \mathsf{R}^2$ or to an infinite point $(x, \pm\infty)$. In the first case the wedges of corresponding 3-cuttings go to some wedges W_1, W_2 and W_3 that define (g_1, g_2, g_3)-equitable 3-cutting. In remaining case some 2 rays of 3-cuttings go to vertical lines that define 3 slabs of weights g_1, g_2 and g_3 (for both functions ρ_1 and ρ_2). Each of these lines defines a 2-cutting. ∎

Applying 3-cutting Theorem 11 recursively for a regions with number of groups more than 1 we can prove the following Theorem.

Theorem 12 (Equitable Subdivision, continuous version) *Let ρ_1 and ρ_2 be measurable functions $\mathsf{R}^2 \to [0, \infty)$ with $\int_{\mathsf{R}^2} \rho_i dx = 1$. For any integer $g > 0$, there exists a subdivision of the plane into g convex regions $R_1, \ldots, R_g$ such that*

$$\int_{R_j} \rho_i dx = \frac{1}{g}, \text{ for } i = 1, 2 \text{ and } j = 1, \ldots, g.$$

6 Algorithm

The algorithm finds 3-cuttings recursively. The recursive procedure has input g, n, m and two sets R and B of points. If $g = 1$ we stop cuttings. If g is even we use a linear time algorithm of Lo et al. for the Ham Sandwich Problem [12]. Otherwise we find the triple g_1, g_2 and g_3 using Theorem 9. For the triple g_1, g_2 and g_3, we can find an equitable 3-cutting by Theorem 5 (in fact the algorithm can find 2-cutting instead of 3-cutting). For each wedge with $g_i n$ red and $g_i m$ blue points, we continue recursively.

6.1 Computing a triple g_1, g_2 and g_3

To compute the signs $s(1), \ldots, s(g-1)$ we construct $g - 1$ vertical lines separating red points into g pieces of equal size. It can be done in linear time using presorting of red points. Presorting takes $O(N \log N)$ time. We can count the number of blue points in the g strips in linear time using presorting of blue points (we can use presorting of all points). For each vertical line, we compute the number of blue points left of the line. Comparing i-th number with im gives sign $s(i)$. If, for some i, i-th number coincides with im, then i-th vertical line gives the equitable 2-cutting.

Now we can find the triple g_1, g_2 and g_3 using Theorem 9. The most difficult part of the algorithm is finding an equitable 3-cutting.

6.2 One more topological lemma

Unfortunately Lemma 4 does not give a way for computing two faces whose labels differ by 2 (exceptly looking through all faces). We give a general topological lemma for existence of such faces that allows to apply a binary search.

Let G be a dual graph of the arrangement A. The vertices of G correspond to the faces and two vertices are adjacent iff corresponding faces are adjacent. The graph G is planar. We associate the points $(1, 0), (0, 1), (-1, 0)$ and $(0, -1)$ with the labels 1, 2, 3 and 4 respectively. Let S be the square with these vertices. We label the vertices of the graph G by the points associated with labels of corresponding faces. Note that the labels of two vertices corresponding to faces whose labels differ by 2 form a diagonal of the square S. In this case, we say, that the vertices form a *diagonal*. If the vertices are adjacent, we call an edge connecting them a *diagonal*.

If the exterior face of the graph G contains a diagonal edge we are done. Consider a non-diagonal directed edge $e = (u, v)$ of G. It corresponds to a directed side of the square S. We assign a *directed length* to the edge e to be 1 (resp. -1) if the corresponding directed side of the square S has clockwise (resp. counterclockwise) direction. A *directed length* of a path P without diagonal edges is the sum of directed lengths of edges of P.

Recall the labels of the boundary of the region $\mathcal{R}$. The top has labels 1,2 or 4, the bottom has 2, the left side has 1,2 and the right side has 2,3. It means that any closed path along the boundary of G clockwise has the directed length 4.

Observation 13 *The length of any closed path without diagonal edges is an integer multiple of 4.*

Proof: The length of a path P does not change if we remove two consecutive edges of opposite directions. After all deletions the path P consists of edges of the same direction. The length of a path P modulo 4 does not change if we remove any 4 consecutive edges. Now the path P contains only one point. ∎

We define a *number of turns* of a closed path P to be the directed length of P divided by 4. So the clockwise boundary of G has 1 turn. Now we can formulate the topological lemma.

Lemma 14 *Let G be a connected graph embedded to the plane and the vertices of G are labeled by the vertices of the square S. If the boundary of G has odd number of turns, then there is a face of G whose vertices form a diagonal.*

Proof: The proof is constructive and can be used in a binary search. If the graph G is not a cycle, we can find a cutting path P with only two vertices in the boundary of G that are ends of P. The path P divides G into two subgraphs G_1 and G_2 (both graphs contain P). The clockwise directed length of G is equal to the sum of clockwise directed lengths of subgraphs because each edge of P is included twice in the sum with opposite signs. It follows that the number of turns of the boundary of G is the sum of the number of turns of the boundaries of the subgraphs. One of the subgraphs has odd turns because its sum is odd. So we reduced the graph.

If the graph G cannot be reduced, then G is a cycle with one bounded face. The vertices of G use all labels (otherwise the directed length of a cycle is 0). Hence there are two vertices that form a diagonal. ∎

6.3 Binary search

Given a triple g_1, g_2 and g_3. The algorithm finds an equitable 3-cutting for this triple. At first we determine the location of an apex of 3-cutting among the vertical lines passing through red and blue points. We apply the binary search on x-coordinates of data points. Actually each data point p gives us two vertical cuts of the region $\mathcal{R}$, left and right to p, say $x = p_x + \epsilon$ and $x = p_x - \epsilon$ for an sufficiently small ϵ (blue points from the lines l_1 and l_2 give only 2 cuts).

A vertical cut of the region $\mathcal{R}$ threads $\Omega(N^2)$ faces of the arrangement A in worst case. Fortunately, the faces form the sequences with the same labels and the number of sequences is at most $O(n^{4/3})$. Consider a set of red points left of the cutting line l. The left ray of any red 3-cutting with an apex on the line l cuts off $g_1 n$ red points.

This reduces to a well-studied *k-level problem* in a dual setting. Given a set of N lines in the plane in general position. A point p from a line has level k if exactly k lines pass below p. The points of level k form a *k-level* which is a curve consisting of segments. The k-level problem is to bound the maximum number of segments of k-level, see [1] for a recent survey. Let $\psi(N)$ be the maximum value of k-level problem over all k. The best upper bound $\psi(N) = O(N^{4/3})$ is obtained by Dey [3]. The *k-level reporting problem* is to report a k-level.

The points of (g_1n-1)-level corresponds to the lines cutting off g_1n red points. A segment of the (g_1n-1)-level corresponds to the lines that cut off g_1n red points and pass the same point. The (g_1n-1)-level corresponds to segments into which the line l is divided. The number of segments is $O(\psi(N))$. The blue points also generate a partition of the line l into $O(\psi(N))$ segments. (If the halfplane left of l contains less than g_1m blue points, all left wedges of blue canonical 3-cuttings are non-convex and the order of left rays is RB.) Combining both partitions we divide the line l into $O(\psi(N))$ segments such that, for all apexes from the same segment, the left rays of red and blue 3-cuttings pass through the same pair of red and blue points p_r and p_b. Hence the order of left rays can change at most one time along a segment, if it is crossed by the line passing through p_r and p_b. Symmetrically, the points right of l generate $O(\psi(N))$ segments with an invariable order of right rays. Combining all segments we obtain $O(\psi(N))$ segments such that the points from the same segment have the same labels. To look through these segments the algorithm uses 4 pointers to the next endpoint of a segment in one of 4 combining sets. For a pointer, we maintain two convex hulls of subsets that are divided by a corresponding ray. We use the dynamic algorithm by Overmars and van Leeuwen [14] that maintain convex hull in $O(\log^2 N)$ time per update (see [8], for example). Thus we obtain an $O(\psi(N)\log^2 N)$ algorithm to compute a directed length d for a cutting path P (in down-up direction) and a top label t.

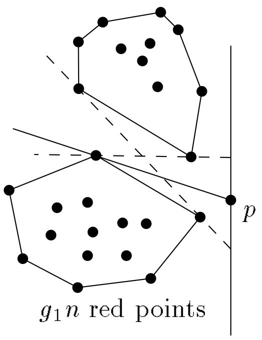


Figure 8: Convex hulls.

The input of the binary search procedure is a strip between two vertical lines, two directed lengths d_1 and d_2 of the left and right boundary of the region in the strip in clockwise order and labels t_1 and t_2 of the top faces. The boundary of a graph corresponding to the region has odd numbers of turns, The cutting line l makes two subgraphs whose boundary lengths can be computed using lengths d_1, d_2 and d and top labels t_1, t_2 and t (note that all top paths do not use label 2). One of the subgraphs has odd number of turns. The binary search proceeds to this subgraph. So we prove the following Theorem.

Theorem 15 *Let $T(N)$ be the running time of an algorithm for reporting a k-level. The binary search can be implement in $O(T(N)\log N)$ time.*

Corollary 16 *The binary search can be implement in $O(N^{4/3}\log^3 N)$ using the algorithm of Overmars and van Leeuwen [14].*

6.4 Searching in the strip

The binary search can be interrupted if one of the vertical cuts contains a diagonal. We find an equitable 3-cutting in linear time using Lemma 6. Otherwise the binary search outputs a vertical strip s with either one data point or without both the red and blue points in the interior of s. In the first case there are two neighbor segments whose labels differ by 2. We again apply the construction of Lemma 6.

Consider the remaining case. Of course our technique allows to continue the binary search among all crossing points of the arrangement A. But the finding of the vertical cuts seems difficult enough. Actually the final strip s defines the set of left and right rays and we can find two rays with crossing point inside s. The strip s contains an apex of the solution but no data points. We look through the sides of the strip but we change their roles. We use the partition of the line l_1 generated by the points right of the strip. Similarly the line l_2 is partitioned by the points left of the strip. The difference is that we use two pointers for each color. For the line l_2, one colored pointer indicates $g_1 n$-th point of the same color, second pointer indicates $g_1 n+1$-th point. So, for a point of the line l_2, we know the range of left red rays. The total number of the segments into which the line l_2 is divided is still $O(\psi(N)$. We also define the blue range and two ranges for the line l_1. Note that the ranges of each segment are defined by 4 points (2 red and 2 blue). We exclude the intersection of the lines passing through these points and the segment by dividing the segment.

Now we are ready to start a climb. We pass segments from a side of the strip if the red and blue ranges don't intersect. Otherwise all rays emanating from a point of a segment lie between two extreme rays emanating from endpoints of the segment and form a trapezoid in the strip s, see Fig. 9. Suppose both segments from l_1 and l_2 form trapezoids. If two trapezoids intersect, then any point of the intersection gives an equitable 3-cutting, see Fig. 9. Otherwise one of the trapezoids lies below other. The lower trapezoid does not contain an apex of an equitable 3-cutting. We pass the segment corresponding this trapezoid. The climbing will stop because the strip contains a solution.

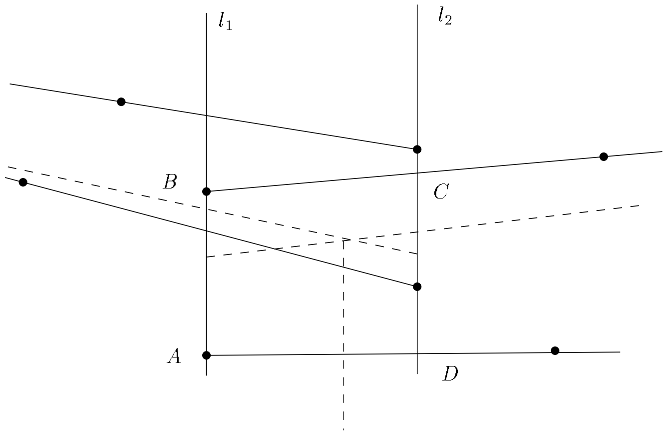


Figure 9: Searching in the strip. Two extreme rays AD and BC form a trapezoid ABCD

6.5 Running time analysis

The running time for the searching in the strip is $O(N^{4/3}\log^2 N)$. The total running time for the 3-cutting algorithm is $O(N^{4/3}\log^3 N)$. To obtain a planar subdivision we use Theorem 9. It gives extra $\log g$. So we proved the following theorem.

Theorem 17 *Given gn red points and gm blue points in the plane in general position. A subdivision of the plane into g convex polygonal regions each of which contains n red and m blue points can be computed in $O(N^{4/3}\log^3 N\log g)$ time where $N = g(n+m)$.*

7 Discussion

Recall that the conjecture of Kaneko and Kano asserts the existence of disjoint convex polygons. Such polygons do not guarantee a subdivision of the plane into convex polygonal regions. Fig. 10 a) shows an example of such polygons. There is no convex subdivision of the plane into 3 polygonal regions such that each region contains one of these polygons. Actually the 3-cutting theorem provided a more restricted class of subdivisions. Fig. 10 b) gives an example of a subdivision of the plane that cannot be constructed by 3-cuttings.

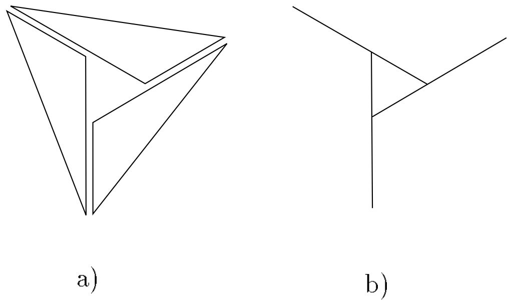


Figure 10: a) 3 polygons that cannot be extended to a subdivision of the plane into 3 convex regions b) Polygonal subdivision that cannot be obtained by 3-cuttings

8 Conclusion

The presented method can be used to solve the problem for arbitrary number of red and blue points in arbitrary position.

Theorem 18 *Let R and B be finite sets of red and blue points in the plane. For any $g > 0$, there exists a subdivision of the plane into g convex polygonal regions so that the interior of each region contains at most $\lfloor \frac{|R|}{g} \rfloor$ red points and at most $\lfloor \frac{|B|}{g} \rfloor$ blue points.*

Actually the proof of Theorem 5 allows us to restrict the 3-cuttings to be found. If gn red and gm blue points in general position don't admit 2-cutting, then there exists an equitable 3-cutting with prescribed direction of one of the rays and the order of groups around apex. It is interesting that the sequence of signs $s(1), \ldots, s(g-1)$ is the source of various triples for the equitable 3-cuttings, for example Theorems 7, 9.

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