Conformal Geometry, Analysis, and Physics

Abstracts

- **Maciej Zworski**

  Title: Outgoing property via Gevrey regularity

  Abstract: My small overlap with Robin Graham's research came from the use of scattering matrices in conformal geometry. We also had to use Borel's lemma to construct certain Poisson operators. My elementary talk will touch upon those concepts in the setting 1D scattering and will explain a recent approach of Gajic–Warnick to distinguishing outgoing solutions (hence resonances) using Gevrey regularity. It is based on joint work with J Galkowski.

- **Kengo Hirachi**

  Title: Normal scale for pseudo-Einstein contact forms and intrinsic CR normal coordinates

  Abstract: In conformal geometry, Robin Graham gave a choice of good scale in the conformal class to simplfies the computation involving the jets of the conformal structure. We will give an analogeous result in CR geometry based on an earlier work by Jerison-Lee. Our choice of the contact form is simpler as we can stay in the class of pseud-Einstein cotact forms, which was not the case for the choice of Jerison-Lee. We also plan to explain its application to the Szego kernel.

- **Yi Wang**

  Title: Yamabe flow of asymptotically flat metrics

  Abstract: In this talk, we will discuss the behavior of the Yamabe flow on an asymptotically flat (AF) manifold. We will first show the long-time existence of the Yamabe flow starting from an AF manifold and discuss the uniform estimates on manifolds with positive Yamabe constant. This would allow us to prove convergence along the Yamabe flow on such manifolds. We also prove that the flow will diverge if the Yamabe constant is nonpositive. It turns out that the behavior of the rescaled flow on manifolds with negative Yamabe constant is explicit. If time permits, we will discuss some further work when the Yamabe constant is zero. This is joint work with Eric Chen and Gilles Carron.

- **Kostas Skenderis**

  Title: Ambient metric and the embedding formalism

  Abstract: In their 1985 paper on Conformal Invariants Fefferman and Graham developed two constructions that associate a n-dimensional conformal manifold to a higher dimensional metric. In one of them the n-dimensional conformal manifold is the conformal boundary of the $n+1$-dimensional (Euclidean) AdS spacetime. This construction is instrumental in setting up the holographic dictionary in the AdS/CFT correspondence via the formalism of holographic renormalization. The second construction associates the n-dimensional conformal manifold with a Lorentzian Ricci-flat metric in $n+2$ dimensions, the ambient metric. This second construction have not had a direct physics application to date. In this talk I will argue that the ambient metric may be used to generalise the embedding formalism for conformal field theories, originally introduced by Dirac in 1936, to curved spacetimes. I will review how the embedding formalism works in flat space and then illustrate the generalisation using an example involving a holographic thermal CFT.

- **Jie Qing**

  Title: Asymptotically hyperbolic Einstein manifolds and conformal geometry

  Abstract: In this talk we will present a survey on our works on asymptotically hyperbolic Einstein manifolds and conformal geometry. We will start with Fefferman-Graham ambient spaces and the definition of asymptotically hyperbolic Einstein manifolds. We will then introduce Mobius coordinate charts and some basic analysis on asymptotically hyperbolic manifolds.

  In the spirit of AdS/CFT correspondence, we will introduce scattering operators and its relations to renormalized volumes on asymptotically hyperbolic Einstein manifolds. We will introduce the geometric correspondences: one is on the scattering poles vs the positivity of the Yamabe constant; the second is on the volume growth vs the Yamabe constant.

  At the end, we will also report some recent work on the uniqueness and compactness of asymptotically hyperbolic Einstein manifolds. Many of the reported work are joint work with Alice Chang and Paul Yang.

- **Nikolaos Eptaminitakis**

  Title: Stability for the X-ray Transform on Asymptotically Hyperbolic Manifolds

  Abstract: Given a Riemannian manifold and a function f defined on it, the geodesic X-ray transform of f is a function on the space of geodesics given by $\gamma \mapsto \int_\gamma f\,ds$, where ds is the arc-length measure on the geodesic γ. Typically the function f is unknown and one is interested in obtaining information about it from its geodesic X-ray transform; for example, one would like to understand whether f is uniquely and stably determined by the transform or whether an inverse can be computed. Such problems have been studied extensively in various settings, starting with J. Radons 1917 seminal paper, where inversion formulas for the transform were derived on the Euclidean plane. The topic of the talk will be the stability of the geodesic X-ray transform on simple asymptotically hyperbolic manifolds, a class of non-compact, simply connected, complete manifolds whose behavior near infinity resembles in many ways that of hyperbolic space. The goal is to establish that small perturbations in the X-ray transform cannot originate from large perturbations of the unknown function f, and find appropriate spaces to measure such perturbations. In a similar spirit to the work of Stefanov-Uhlmann (2004) on simple compact manifolds with boundary, we study microlocally the stability of the normal operator of the X-ray transform, which is given by its composition with a backprojection. We discuss how the normal operator fits within the of the 0-calculus of Mazzeo-Melrose, which is a natural framework for studying pseudodifferential operators on asymptotically hyperbolic manifolds, and construct a parametrix, which eventually leads to the desired stability.

- **Robert Bryant** (Clay Foundation Speaker)

  Title: On solitons for the closed G_2-Laplacian flow.

  Abstract: I will start by recalling the basic geometry of G_2-structures on 7-manifolds and their relation with special holonony in dimension 7. Then, I'll discuss a natural evolution defined for *closed* G_2-structures, known as the closed G_2 Laplacian flow, that is a natural analog of Ricci flow for Riemannian metrics. After discussing the short-time existence and uniqueness question, I will discuss the local 'generality' of solitons for this flow in É. Cartan's sense, introducing a natural exterior differential system whose integral manifolds correspond to these solitons. Finally, I will discuss the special case of *gradient* solitons for this flow, and the remaining question of their generality.

- **Michael Eastwood**

  Title: Conformal versus CR geometry

  Abstract: It is well-known (as seen, for example, in the Fefferman-Graham ambient metric construction) that there are close analogies between conformal differential geometry and hypersurface CR geometry (and the Fefferman circle bundle gives a direct link). What is the CR analogue of an Einstein metric in a conformal class? I shall explain joint work with Lenka Zalabova, which gives an answer to this question and to questions like it.

- **Qing Han**

  Title: Asymptotic analysis for extreme Kerr-type harmonic maps

  Abstract: Motivated by studies of axially symmetric stationary solutions of the Einstein vacuum equations in general relativity, we study singular harmonic maps from domains of the 3-dimensional Euclidean space to the hyperbolic plane, with bounded hyperbolic distance to extreme Kerr harmonic maps. We prove that every such harmonic map has a unique tangent map at the black hole horizon. The harmonic map equation restricted to the unit sphere has a singularity at the north and south poles. The talk is based on joint work with Marcus Khuri, Gilbert Weinstein, and Jingang Xiong.

- **Yueh-Ju Lin**

  Title: Conformally variational Riemannian invariants and polydifferential operators

  Abstract: Conformally variational Riemannian invariants (CVIs) are homogeneous scalar invariants which appear as the conformal gradient of a Riemannian functional. In this talk, I will present several stability and rigidity results for CVIs that generalize similar known results for the scalar curvature and Q-curvature. I will talk about constructing a formally self-adjoint conformally covariant polydifferential operator associated with a given CVI. In addition, I discuss a complete classification of tangential bi-differential operators in terms of ambient Laplacian. This result is a curved analogue of conformally covariant bi-differential operators on spheres classified by Ovsienko-Redou and Clerc. As time permits, I will also present a family of sharp, fully nonlinear Sobolev inequalities involving the Paneitz operator and Ovsienko-Redou operator. This talk is based on several joint works with Jeffrey Case and Wei Yuan.

- **Daniel Fox**

  Title: Minimal algebraic hypersurfaces in spheres from quasicomposition algebras

  Abstract: A cubic form on a metrized vector space determines a commutative algebra for which the given metric is invariant. Hsiang asked for the classification of irreducible homogeneous polynomials whose regular level sets are codimension one minimal cones. For cubic forms this is equivalent to the classification of certain commutative algebras called Hsiang algebras by Tkachev. The regular level sets of the cubic forms of the Hsiang algebras studied by Vladimir Tkachev give examples of algebraic minimal hypersurfaces in spheres that are not homogeneous. He showed these come in an infinite family constructed from Clifford systems and finitely many exceptional cases. The talk will describe a tripling construction that associates with an n-dimensional metrized algebra with involution a $3n$-dimensional metrized commutative algebra and will explain that the triple algebra is a Hsiang algebra if and only if the original algebra is a quasicomposition algebra. The quasicomposition algebras include algebras in dimensions $1, 2, 3, 4, 6, 7, 8$: Hurwitz algebras, commutative para-Hurwitz algebras, cross product algebras, six-dimensional color algebras, and probably nothing more. This yields a unified construction of certain of the Hsiang algebras and raises a number of further questions. This is joint work in progress with Vladimir Tkachev.

- **Pawel Nurowski**

  Title: Poincare-Einstein approach to Penrose's Conformal Cyckic Cosmology

  Abstract: I will discuss two consecutive eons in Penroses conformal cyclic cosmology, and study how the matter content of the past eon determines the matter content of the present eon. I will use the Poincare-Einstein type of expansion to determine the metric in the past eon, and by the reciprocity hypothesis will produce the metric in the present eon. Then, using the Einstein equations in the present eon, i will show what has happened to the matter after its transition through the Big Bang. I will show that a single spherical wave from the previous eon, in the new eon, splits into three portions of radiation: the two spherical waves (one which is a damped continuation from the previous eon, the other which is focusing in the new eon as it encountered a mirror at the Big Bang surface), and a scattered radiation described by the statistical physics.

- **Richard Melrose**

  Title: Boundary weights and Hodge cohomology

  Abstract: I will discuss some aspects of the spaces of pseudodifferential operators intended to allow the identification of the Hodge cohomology of various complete Riemann metrics on the interiors of compact manifolds with corners. The boundary weights of the title refer to the appearance of incomplete metrics in the model problems which arise in the construction of a generalized inverse of the Hodge-Dirac operator.

- **Rafe Mazzeo**

  Title: The Hitchin equations and ALG spaces

  Abstract: There is a twelve-parameter family of Hitchin equations on the four-punctured sphere, the parameters representing different ‘boundary conditions at the punctures. For each allowable choice of parameters, the corresponding moduli space of solutions carries the natural structure of a four-dimensional hyperKaehler manifold of ALG type, and with D4 topology. There is a twelve dimensional family of such hyperKaehler metrics. In work with Fredrickson, Swoboda and Weiss, we show that the map from Hitchin parameters to ALG metrics with this topology is surjective.

- **Azahara Dela Torre**

  Title: The fractional Yamabe problem with singularities: from Conformal Geometry to the study of non-local semilinear elliptic PDEs.

Abstract: The so called Yamabe problem in Conformal Geometry consists in finding a metric conformal to a given one and which has constant scalar curvature. From the analytic point of view, this problem becomes a semilinear elliptic PDE with critical (for the Sobolev embedding) power non-linearity. If we study the problem in the Euclidean space, allowing the presence of positive-dimensional singularities can be transformed into reducing the non-linearity to a Sobolev-subcritical power. When, instead of scalar, we work with non-local curvature we need to deal with the pertinent non-local semilinear elliptic PDE.

In this talk, we will focus on the study of the solutions of these non-local (or fractional) semilinear elliptic PDEs which represent metrics which are singular along positive-dimensional singularities. In collaboration with Ao, Chan, Fontelos, Gonzlez and Wei, we covered the construction of solutions which are singular along (zero and positive-dimensional) smooth submanifolds. This was done through the development of new methods coming from conformal geometry and Scattering theory for the study of non-local ODEs. Due to the limitations of the techniques we used, the particular case of “maximal dimension for the singularity was not covered. In a recent work, in collaboration with H. Chan, we covered the construction for this specific dimension using asymptotic analysis and the stability and semi-linearity of the equation. Moreover we developed new methods to study the singular behavior of the solutions.

- **Claude LeBrun**

  Title: Kodaira Dimension and the Yamabe Problem, Revisted

  Abstract: Dimension four provides a surprisingly idiosyncratic setting for the interplay between scalar curvature and differential topology. This peculiarity becomes especially pronounced when discussing the Yamabe invariant or sigma constant of a smooth compact manifold; and Seiberg-Witten theory makes this especially apparent for those 4-manifolds that arise as compact complex surfaces. For compact complex surfaces of Kaehler type, I showed in the late 1990s that the sign of the Yamabe invariant is always determined by the Kodaira dimension, and calculated the Yamabe invariant exactly in all cases where it is non-positive. In this talk, I will describe recent joint work with Michael Albanese that generalizes these results to all complex surfaces

of non-Kaehler type, except those of class VII. However, I will also explain why excluding this exceptional class is absolutely essential for these purposes.

- **Andras Vasy**

  Title: Waves on black hole backgrounds and the stability problem

  Abstract: I will discuss the behavior of waves on black hole backgrounds as well as its relationship to, and implications for, the black hole stability problem.

  This talk is based on joint work with Peter Hintz, Dietrich Haefner and Oliver Lindblad Petersen.

- **Lionel Mason**

  Title: Holomorphic discs and self-dual Einstein metrics from conformal boundary data

  Abstract: A global twistor construction for conformally self-dual split signature metrics on $S^2 \times S^2$ was developed 15 years ago by the speaker and Claude LeBrun. This encodes the conformal metric into the location of a finite deformation of the real slice in $\mathbb{CP}^3$; the construction entails determining a family of holomorphic discs in $\mathbb{CP}^3$ whose boundaries lie on the deformed real slice. This talk adapts the construction to resolve this version of a problem studied by Graham & Lee 30 years ago to construct global Einstein metrics from conformal boundary data. This problem can be resolved straightforwardly in the case of vanishing cosmological constant, but remains conjectural when nonzero.

- **Andreas Karch**

  Title: Submanifolds in conformal Geometry and their Physics Applications

  Abstract: Holography, the equivalence between a conformal field theory and gravity in a higher dimensional bulk, provides a very concrete connection between ideas from conformal geometry and pressing questions in modern physics. In this talk I'll review the crucial role submanifolds have played in this context and present some recently developed applications in the context of the studies of quantum information in both gravity and field theory.

- **Jeffrey Case**

  Title: Some Q-curvature operators in CR geometry

  Abstract: In conformal geometry, Q-curvature operators are differential operators on differential forms which have a "Q-like" transformation law when restricted to closed forms. In CR geometry, the first example of a Q-curvature operator is the P'-operator. In this talk, I introduce some Q-curvature operators on $\overline{\partial}_b$-closed and on d-closed differential forms, and discuss their application to the Q-curvature of closed strictly pseudoconvex CR manifolds in dimension five.

- **Andreas Cap**

  Title: Tractors and the mass of asymptotically hyperbolic manifolds

  Abstract: This talk reports on my joint work arXiv:2108.01373 with Rod Gover that is motivated by the concept of mass for asymptotically hyperbolic metrics. We work in the setting of conformally compact metrics on arbitrary manifolds with boundary (at infinity) and a class of metrics that are asymptotically hyperbolic in a weak sense and asymptotic to each other to appropriate (higher) order. This makes sure that all the metrics in the class induce the same conformal infinity on the boundary. We then associate to two metrics in the class a two-parameter family of ""relative masses"", that are top degree forms on the boundary with values in the standard tractor bundle of the conformal infinity.

  The construction is manifestly independent of coordinates and it is easily seen to be equivariant with respect to a natural family of diffeomorphisms. Much more subtle considerations shows that a one-parameter subfamily satisfies an additional invariance condition with respect to diffeomorphisms asymptotic to the identity. Assuming that the given class contains hyperbolic metrics, this allows us to associate an invariant to a single metric in the class. In the case of hyperbolic space and the class of metrics determined by the Poincaré metric, the forms we construct can be integrated to parallel sections of the tractor bundle and this recovers that mass of asymptotically hyperbolic metrics introduced by Wang and Chrusciel-Herzlich.

- **Edward Witten**

  Title: Volumes and Ensembles in 3d Gravity

  Abstract: There has been a longtime puzzle concerning the role of connected manifolds with disconnected boundaries in the duality between quantum gravity on a D-manifold X and an ordinary quantum theory on the boundary of X. Some of these studies have suggested that the boundary theory in this duality is really an ensemble average. Here we will explore this question in the case of D=3 using facts from classical hyperbolic geometry. The upshot is to argue that carefully selected observables are not affected by "wormholes," consistent with the idea that the boundary theory is a well-defined quantum theory. (Based on https://arxiv.org/pdf/2202.01372.pdf with J.-M. Schlenker.)

- **Rod Gover**

  Title: Conformal submanifolds, distinguished submanifolds, and integrability

  Abstract: For conformal geometries of Riemannian signature, we provide a comprehensive treatment of the basic local theory of embedded submanifolds of arbitrary codimension. We provide three distinct, but equivalent, fundamental invariants of submanifolds. For any one of these normal objects its derivative by the pullback of the conformal tractor connection recovers the data of the tractor second fundamental form. The vanishing of this, so the normal object being parallel, determines a notion of distinguished submanifold. For the case of curves this exactly characterises conformal circles, while for hypersurfaces it is the totally umbilic condition. So, for other codimensions, this unifying notion interpolates between these extremes, and we prove that this coincides with a property of ambient conformal circles remaining in the submanifold, a property that we term weakly conformally circular. If this holds and the conformal circles coincide with the submanifold distinguished curves then the submanifolds is said to be strongly conformally circular; this holds if and only also a second tractor invariant vanishes, this invariant is a Fialkow type tensor invariant. We prove that the property of being distinguished is also captured by a type of moving incidence relation. This second characterisation is used to provide a very general theory and construction of quantities that are necessarily conserved along the submanifold. The formalism thus leads to a conformal submanifold first integral theory that generalises the ideas available for curves. We prove that any normal solution to an equation from the class of first BGG equations can yield such a conserved quantity, and we show that it is easy to provide explicit formulae for these. We also show that for normal solutions of conformal Killing-Yano equations, a certain zero locus of the solution is necessarily a distinguished submanifold.

This is joint work with Daniel Snell

- **Andrew Waldron**

  Title: Quantization and Contact Structures

  Abstract: A contact form on an odd dimensional manifold determines a generic classical mechanics. This can quantized by generalizing Fedosov's approach to symplectic structures. This gives quantum mechanics, generally covariant with respect to choices of time, generalized position and momentum. Just as many structures in general relativity (GR) are effectively studied by considering causal structures–i.e. conformal classes of metrics–it is also useful to consider conformal classes of contact forms aka contact structures. In the case of GR, key machinery for the study of conformal manifolds is tractor calculus. This theory is a parabolic geometry, and thus extends nicely to contact structures and, as will be explained, their quantization. Key notions are parallel sections with respect to distinguished connections; just as these provide the link between conformal and Einstein structures in GR, here they relate contact structures to quantum mechanics. The general theory will be illustrated for the example of S^3 with its standard contact structure.

- **Eric Bahauad**

  Title: Analytic semigroups, bounded geometry and geometric flows

  Abstract: Semigroup methods provide an elegant abstract framework for studying parabolic evolution equations. In this talk I will describe joint work with Guenther, Isenberg and Mazzeo providing a criterion for a geometric elliptic operator to generate an analytic semigroup on weighted Hlder spaces on manifolds with bounded geometry. I will then outline a few applications to geometric flows.

- **Travis Willse**

  Title: Chains of path geometries on surfaces

  Abstract: A path geometry on a surface Σ can be viewed as (1) a kind of nondegenerate 2-parameter family of unparameterized curves in Σ, or, slightly more generally, (2) a pair of line distributions on a 3-manifold spanning a contact distribution, or (3) an equivalence class of second-order o.d.e.s modulo point transformations. Path geometries are a classic, low-dimensional example of parabolic geometries: They are formally analogous to 3-dimensional CR structures, and they share with CR-structures a notion of chains, a preferred class of curves in $\mathbb{P}(T\Sigma)$. In this project, joint with Gil Bor (C.I.M.A.T.), we investigate the geometry of chains of path geometries. We give a simple but apparently new characterization of projective path geometries (those whose paths are the geodesics of some projective structure) in terms of chains, we describe the chains of four geometrically natural homogeneous path geometries, and we raise some open questions. https://arxiv.org/pdf/2201.09141.pdf

- **Jack Lee**

  Title: Some Remarks on the 5-Dimensional CR Embedding Problem

  Abstract: CR manifolds are the abstract models of real hypersurfaces in $\mathbb{C}^n$. But abstract CR manifolds cannot always be realized as such hypersurfaces, even locally. Louis Boutet de Monvel proved in 1975 that every *compact* strictly convex CR manifold of dimension at least 5 can be embedded locally as a hypersurface, and globally in higher codimension. On the other hand, generic 3-dimensional CR structures cannot be locally embedded.

  The noncompact case is more subtle. Masatake Kuranishi proved in 1982 that every strictly pseudoconvex CR manifold of dimension at least 9 is locally embeddable, and shortly thereafter Takao Akahori was able to push the dimension down to 7. That left the 5-dimensional noncompact case open, and it remains open to this day.

  I'll review the history of the problem, explain some of the reasons why it is harder than the other dimensions, and suggest some strategies for going forward.