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Optimal Control Problems with Symmetry Breaking Cost Functions*

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Abstract. We investigate symmetry reduction of optimal control problems for left-invariant control affine systems on Lie groups, with partial symmetry breaking cost functions. Our approach emphasizes the role of variational principles and considers a discrete-time setting as well as the standard continuous-time formulation. Specifically, we recast the optimal control problem as a constrained variational problem with a partial symmetry breaking Lagrangian and obtain the Euler–Poincaré equations from a variational principle. By using a Legendre transformation, we recover the Lie–Poisson equations obtained by Borum and Bretl [*IEEE Trans. Automat. Control*, 62 (2017), pp. 3209–3224] in the same context. We also discretize the variational principle in time and obtain the discrete-time Lie–Poisson equations. We illustrate the theory with some practical examples including a motion planning problem in the presence of an obstacle.

Key words. Euler–Poincaré equations, Lie–Poisson equations, optimal control, symmetry reduction

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1. Introduction.

Symmetry reduction of optimal control problems (OCPs) for left-invariant control systems on Lie groups has been studied extensively over the past couple of decades. Such symmetries are usually described as an invariance under an action of a Lie group, and the system can be reduced to a lower-dimensional one or decoupled into subsystems by exploiting the symmetry (see, e.g., [1, 3, 14, 16, 25, 26, 31, 33, 34]). Symmetry reduction of OCPs is also desirable from a computational point of view. Given that solving OCPs usually involves iterative/numerical methods such as the shooting method (as opposed to solving a single initial value problem), reducing the system to a lower-dimensional one results in a considerable reduction of the computational cost as well.

The goal of this work is to study symmetry reduction of OCPs for left-invariant control affine systems on Lie groups with partially broken symmetries, more specifically, cost functions that break some, but not all, of the symmetries. Symmetry breaking is common in several physical contexts, from classical mechanics to particle physics. The simplest example is the

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heavy top dynamics (the motion of a rigid body with a fixed point in a gravitational field), where due to the presence of gravity, we get a Lagrangian that is SO(2)-invariant but not SO(3)-invariant, contrary to what happens for the free rigid body. Reduction theory for Lagrangian/Hamiltonian systems on Lie groups with symmetry breaking was developed in [19, 29, 30]. In the context of motion planning, the symmetry breaking appears naturally in the form of a barrier function, as we shall see in section 4.2.

The results in this paper are the Lagrangian/variational counterpart of those in [7, 8, 9]; we also develop a discrete-time version of the results. From the Lagrangian point of view, we obtain the Euler–Poincaré equations from a variational principle, and by using a Legendre transformation, we obtain the Lie–Poisson equations. By discretizing the variational principle in time, we obtain the discrete-time Lie–Poisson equations. We only study the reduction of necessary conditions for optimality for OCPs, which are obtained from a variational perspective. Reduction of necessary and sufficient conditions for optimality has been recently studied in [9] from the Hamiltonian point of view using Pontryagin's maximum principle, and the connection between both formalisms could be stated using a Legendre transformation.

We also work out some practical examples to illustrate the theory, including an optimal control description of the heavy rigid top equations as well as a motion planning problem with an obstacle; the latter provides an example of the symmetry breaking by a barrier function mentioned above.

1.1. Outline.

The paper is organized as follows. In section 2, we introduce some preliminaries about geometric mechanics on Lie groups. In section 3, we study the Euler–Poincaré reduction of OCPs for left-invariant control affine systems on Lie groups, with partial symmetry breaking cost functions, and we also obtain the Lie–Poisson equations by using a Legendre transformation. Furthermore, we consider the discrete-time case and obtain the discrete-time Lie–Poisson equations. In section 4, some practical examples are considered to illustrate the theory. Finally, in section 5, we make some concluding remarks.

2. Preliminaries.

Let $\mathcal{Q}$ be an n-dimensional differentiable manifold with local coordinates $(q^1,\dots,q^n)$, the configuration space of a mechanical system. Denote by $T\mathcal{Q}$ its tangent bundle with induced local coordinates $(q^1,\dots,q^n,\dot{q}^1,\dots,\dot{q}^n)$. Given a Lagrangian $L\colon T\mathcal{Q}\to\mathbb{R}$, the corresponding Euler–Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)-\frac{\partial L}{\partial q^i}=0,\ \ i=1,\dots,n,$$

which determine a system of implicit second-order differential equations.

An indispensable tool in the study of mechanical systems is symmetry reduction. In particular, when the configuration space is a Lie group, one can reduce the Euler–Lagrange equations, which generally give a system of second-order equations, to a system of first-order equations on its Lie algebra.

Let G be a Lie group, and let $\mathfrak{g}$ be its Lie algebra. Let $L_g : G\to G$ be the left translation by an element $g\in G$; i.e., $L_g(h)=gh$ for any $h\in G$. Note that the map L_g is a diffeomorphism on G and is also a left action of G on itself. The tangent map of L_g at $h\in G$ is denoted by $T_hL_g\colon T_hG\to T_{gh}G$. Similarly, the cotangent map of L_g at $h\in G$ is denoted by $T_h^*L_g\colon T_{gh}^*G\to T_h^*G$. A vector field $X\in\mathfrak{X}(G)$ is called left-invariant if $X(L_g(h))=T_hL_g(X(h))$ for any g,

$h \in G$. Let $\Phi\colon G \times \mathcal{Q} \to \mathcal{Q}$ be a left action of G on $\mathcal{Q}$. A function $f\colon \mathcal{Q} \to \mathbb{R}$ is said to be G-invariant if $f \circ \Phi_g = f$ for any $g \in G$.

If we assume that the Lagrangian $L\colon TG \to \mathbb{R}$ is G-invariant under the left action of G on TG, then we can obtain a reduced Lagrangian $\ell\colon \mathfrak{g} \to \mathbb{R}$, where $\ell(\xi) = L(g^{-1}g, T_gL_{g^{-1}}(\dot{g})) = L(e, \xi)$. We can now obtain the reduced Euler–Lagrange equations, commonly known as the Euler–Poincaré equations (see, e.g., [2, 20, 28]), which are given by

$$\frac{d}{dt}\frac{\partial \ell}{\partial \xi} = \mathrm{ad}^*_\xi \frac{\partial \ell}{\partial \xi}. \tag{1}$$

The Euler–Poincaré equations together with the reconstruction equation $\xi = T_gL_{g^{-1}}(\dot{g})$ are equivalent to the Euler–Lagrange equations on G. If we assume that the reduced Lagrangian ℓ is hyperregular, then we can obtain the reduced Hamiltonian $h\colon \mathfrak{g}^* \to \mathbb{R}$ (by using a Legendre transformation) given by

$$h(\mu) = \langle \mu, \xi \rangle - \ell(\xi),$$

where $\mu = \frac{\partial \ell}{\partial \xi} \in \mathfrak{g}^*$. The Euler–Poincaré equations (1) can now be written as the Lie–Poisson equations (see, e.g., [2, 20, 28]), which are given by

$$\dot{\mu} = \mathrm{ad}^*_{\frac{\partial h}{\partial \mu}} \mu. \tag{2}$$

3. Optimal control problems on Lie groups.

We will first define a left-invariant control affine system on G, which is assumed to be n-dimensional.

Definition 1. *A left-invariant control affine system on G is given by*

$$\dot{g} = T_eL_g(u),$$

where $g(\cdot) \in C^1([0,T], G)$ and u is a curve in the vector space $\mathfrak{g}$. More precisely, if $\mathfrak{g} = \mathrm{span}\{e_1, \ldots, e_m, e_{m+1}, \ldots, e_n\}$, then u is given by

$$u(t) = e_0 + \sum_{i=1}^{m} u^i(t) e_i,$$

where $e_0 \in \mathfrak{g}$ and the m-tuple of control inputs $[u^1 \ldots u^m]^T$ takes values in $\mathbb{R}^m$.

Remark 2. If $m < n$, then the left-invariant control affine system is underactuated; otherwise it is fully actuated.

In what follows, we will fix $\mathfrak{k} = \mathrm{span}\{e_1, \ldots, e_m\}$, $\mathfrak{p} = \mathrm{span}\{e_{m+1}, \ldots, e_n\}$, with $e_0 \in \mathfrak{p}$ and $E := G \times \mathfrak{k}$. Consider the OCP

$$\min_{u(\cdot)} J := \min_{u(\cdot)} \int_0^T [C(g(t), u(t) - e_0) + V(g(t))] dt$$

subject to **(P1)**

$$\dot{g}(t) = T_eL_{g(t)}(u(t)), \;\; g(0) = g_0, \;\; g(T) = g_T,$$

where the following assumptions hold:

(i) $C : E \to \mathbb{R}$ is a G-invariant function (under a suitable left action of G on E, which will be defined shortly) and is also sufficiently regular.

(ii) $V : G \to \mathbb{R}$ (potential function) is not a G-invariant function and is also sufficiently regular.

(iii) The potential function also depends on a parameter α_0 that may be considered to be an element of the dual space X^* of a finite-dimensional vector space X. Hence, we define the extended potential function as $V_{\text{ext}} : G \times X^* \to \mathbb{R}$, with $V_{\text{ext}}(\cdot, \alpha_0) = V$.

(iv) There is a left representation ρ of G on X; i.e., $\rho_\cdot : G \to \mathrm{GL}(X)$ is a homomorphism. Hence, there is a left representation $\rho^*_\cdot : G \to GL(X^*)$ defined as the adjoint of ρ: for any $g \in G$, we define $\rho^*_{g^{-1}} \in \mathrm{GL}(X^*)$ as the adjoint of $\rho_{g^{-1}} \in \mathrm{GL}(X)$; i.e., for any $x \in X$ and $\alpha \in X^*$,

$$\langle \rho^*_{g^{-1}}(\alpha), x \rangle = \langle \alpha, \rho_{g^{-1}}(x) \rangle.$$

As a result, there is a left action of G on $G \times X^*$ defined as

$$\begin{aligned} \Phi : G \times (G \times X^*) &\longrightarrow G \times X^*, \\ (g, (h, \alpha)) &\longmapsto (L_g(h), \rho^*_{g^{-1}}(\alpha)). \end{aligned} \tag{3}$$

(v) The extended potential function is G-invariant under (3); i.e., $V_{\text{ext}} \circ \Phi_g = V_{\text{ext}}$ for any $g \in G$ or, more concretely, $V_{\text{ext}}(L_g(h), \rho^*_{g^{-1}}(\alpha)) = V_{\text{ext}}(h, \alpha)$ for any $h \in G$ and $\alpha \in X^*$.

Remark 3. Under assumptions (iii)–(v), the potential function is invariant under the left action of the isotropy group

$$G_{\alpha_0} := \{ g \in G \mid \rho^*_g(\alpha_0) = \alpha_0 \}.$$

Remark 4. Note that E is a trivial vector bundle over G, and define the left action of G on E as follows:

$$\begin{aligned} \Psi : G \times E &\longrightarrow E, \\ (g, (h, u)) &\longmapsto (L_g(h), u). \end{aligned} \tag{4}$$

Throughout the paper, we assume that $C : E \to \mathbb{R}$ is G-invariant under (4); i.e., $C \circ \Psi_g = C$ for any $g \in G$.

3.1. Euler–Poincaré reduction. We can solve **(P1)** as a constrained variational problem using the method of Lagrange multipliers (see, e.g, [2, 25]). Define the augmented Lagrangian $L_a : E \times \mathfrak{p}^* \times X^* \to \mathbb{R}$ as follows:

$$L_a(g, u, \lambda, \alpha) = C(g, u - e_0) + V_{\text{ext}}(g, \alpha) + \langle \lambda, u - e_0 \rangle,$$

where $\lambda(\cdot) \in C^1([0, T], \mathfrak{p}^*)$. Define the left action of G on $E \times \mathfrak{p}^* \times X^*$ as follows:

$$\begin{aligned} \Upsilon : G \times (E \times \mathfrak{p}^* \times X^*) &\longrightarrow E \times \mathfrak{p}^* \times X^*, \\ (g, (h, u, \lambda, \alpha)) &\longmapsto (L_g(h), u, \lambda, \rho^*_{g^{-1}}(\alpha)). \end{aligned} \tag{5}$$

Under assumption (v), it follows that $L_a : E \times \mathfrak{p}^* \times X^* \to \mathbb{R}$ is G-invariant under (5); i.e., $L_a \circ \Upsilon_g = L_a$ for any $g \in G$. In particular, under assumptions (iii)–(v), it follows that the augmented Lagrangian $L_a(\cdot,\cdot,\cdot,\alpha_0) =: L_{a,\alpha_0} : E \times \mathfrak{p}^* \to \mathbb{R}$ is G_{α_0}-invariant under (5); i.e., $L_{a,\alpha_0} \circ \Upsilon_g = L_{a,\alpha_0}$ for any $g \in G_{\alpha_0}$. We can now obtain the reduced augmented Lagrangian $\ell_a : \mathfrak{k} \times \mathfrak{p}^* \times X^* \to \mathbb{R}$, which is given by

$$\begin{aligned} \ell_a(u,\lambda,\alpha) &:= L_a(e,u,\lambda,\alpha) \\ &= C(u-e_0) + V_{\text{ext}}(\alpha) + \langle \lambda, u - e_0 \rangle, \end{aligned}$$

where $\alpha = \rho_g^*(\alpha_0)$ and with a slight abuse of notation, we write $C(e, u - e_0) = C(u - e_0)$ and $V_{\text{ext}}(\alpha) = V_{\text{ext}}(e,\alpha)$. A normal extremal for **(P1)** satisfies the following Euler–Poincaré equations (see Theorem 5 below for a proof):

$$\frac{d}{dt}\left(\frac{\partial C}{\partial u} + \lambda\right) = \operatorname{ad}_u^*\left(\frac{\partial C}{\partial u} + \lambda\right) + \mathbf{J}_X\left(\frac{\partial V_{\text{ext}}}{\partial \alpha}, \alpha\right), \tag{6}$$

$$\dot\alpha = \rho_u'^*(\alpha), \quad \alpha(0) = \rho_{g_0}^*(\alpha_0), \tag{7}$$

where $\mathbf{J}_X : T^*X \cong X \times X^* \to \mathfrak{g}^*$ is the momentum map corresponding to the left action of G on X defined using the left representation ρ of G on X. That is, for any x, $\xi \in X$ and $\alpha \in X^*$,

$$\langle \mathbf{J}_X(x,\alpha), \xi \rangle = \langle \alpha, \xi_X(x) \rangle, \tag{8}$$

with ξ_X being the infinitesimal generator of the left action of G on X, and $\rho_\cdot'^* : \mathfrak{g} \to \mathfrak{gl}(X^*)$ is defined as the adjoint of ρ', which is the representation induced by the left representation ρ of G on X. Note that the solution to (7) is given by $\alpha(\cdot) = \rho_{g(\cdot)}^*(\alpha_0)$. Also, note that using the notation of [19], we have $\mathbf{J}_X(x,\alpha) = x \diamond \alpha$. For more details, see [12, 19, 20]. To summarize, we have the following theorem.

Theorem 5. *A normal extremal for* **(P1)***, under assumptions* (i)–(v)*, satisfies the following Euler–Poincaré equations:*

$$\frac{d}{dt}\left(\frac{\partial C}{\partial u} + \lambda\right) = \operatorname{ad}_u^*\left(\frac{\partial C}{\partial u} + \lambda\right) + \mathbf{J}_X\left(\frac{\partial V_{\text{ext}}}{\partial \alpha}, \alpha\right), \tag{9}$$

$$\dot\alpha = \rho_u'^*(\alpha), \quad \alpha(0) = \rho_{g_0}^*(\alpha_0). \tag{10}$$

Proof. The proof follows arguments similar to those given in [19, 25]. Consider the following variational principles:

(a) The variational principle

$$\delta \int_0^T L_{a,\alpha_0}(g(t), u(t), \lambda(t))dt = 0$$

holds for all variations of g (vanishing at the endpoints) and u.

(b) The constrained variational principle

$$\delta \int_0^T \ell_a(u(t), \lambda(t), \alpha(t))dt = 0$$

holds using variations of u and α of the form

$$\delta u = \dot{\eta} + \operatorname{ad}_u \eta,$$
$$\delta\alpha = \rho_\eta'^*(\alpha),$$

where $\eta(\cdot) \in C^1([0,T],\mathfrak{g})$ vanishes at the endpoints.

We will first show that the variational principle (a) implies the constrained variational principle (b). We begin by noting that $L_a : E \times \mathfrak{p}^* \times X^* \to \mathbb{R}$ is G-invariant under (5), i.e., $L_a \circ \Upsilon_g = L_a$ for any $g \in G$ and $\alpha = \rho_g^*(\alpha_0)$, so the integrand in the variational principle (a) is equal to the integrand in the constrained variational principle (b). However, all variations of g vanishing at the endpoints induce and are induced by variations of u of the form $\delta u = \dot{\eta} + \operatorname{ad}_u \eta$, with $\eta(0) = \eta(T) = 0$. The relation between δg and η is given by $\eta = T_g L_{g^{-1}}(\delta g)$ (see Proposition 5.1 in [4], which is Lemma 3.2 in [19]). Thus, if the variational principle (a) holds and if we define $\eta = T_g L_{g^{-1}}(\delta g)$ and $\delta u = T_g L_{g^{-1}}(\dot{g})$, then by Proposition 5.1 in [4], we have $\delta u = \dot{\eta} + \operatorname{ad}_u \eta$. In addition, we have $\delta\alpha = \rho_\eta'^*(\alpha)$. Hence, the variational principle (a) implies the constrained variational principle (b).

A normal extremal for **(P1)** satisfies the variational principle (a) and hence, the constrained variational principle (b), by the above discussion. So, we have

$$\begin{aligned}
0 &= \delta \int_0^T \ell_a(u(t),\lambda(t),\alpha(t))dt \\
&= \int_0^T \left[\left\langle \frac{\partial C}{\partial u}, \delta u\right\rangle + \langle \lambda, \delta u\rangle + \left\langle \delta\alpha, \frac{\partial V_{\text{ext}}}{\partial \alpha}\right\rangle\right] dt \\
&= \int_0^T \left[\left\langle \frac{\partial C}{\partial u} + \lambda, \delta u\right\rangle + \left\langle \delta\alpha, \frac{\partial V_{\text{ext}}}{\partial \alpha}\right\rangle\right] dt \\
&= \int_0^T \left[\left\langle \frac{\partial C}{\partial u} + \lambda, \dot{\eta} + \operatorname{ad}_u \eta\right\rangle + \left\langle \delta\alpha, \frac{\partial V_{\text{ext}}}{\partial \alpha}\right\rangle\right] dt \\
&= \int_0^T \left[\left\langle -\frac{d}{dt}\left(\frac{\partial C}{\partial u} + \lambda\right) + \operatorname{ad}_u^*\left(\frac{\partial C}{\partial u} + \lambda\right), \eta\right\rangle + \left\langle \rho_\eta'^*(\alpha), \frac{\partial V_{\text{ext}}}{\partial \alpha}\right\rangle\right] dt \\
&= \int_0^T \left[\left\langle -\frac{d}{dt}\left(\frac{\partial C}{\partial u} + \lambda\right) + \operatorname{ad}_u^*\left(\frac{\partial C}{\partial u} + \lambda\right) + \mathbf{J}_X\left(\frac{\partial V_{\text{ext}}}{\partial \alpha}, \alpha\right), \eta\right\rangle\right] dt,
\end{aligned}$$

where we have used integration by parts along with the fact that $\eta(0) = \eta(T) = 0$, and so we have

$$\frac{d}{dt}\left(\frac{\partial C}{\partial u} + \lambda\right) = \operatorname{ad}_u^*\left(\frac{\partial C}{\partial u} + \lambda\right) + \mathbf{J}_X\left(\frac{\partial V_{\text{ext}}}{\partial \alpha}, \alpha\right).$$

Finally, by taking the time derivative of α, we have

$$\dot{\alpha} = \rho_u'^*(\alpha), \quad \alpha(0) = \rho_{g_0}^*(\alpha_0). \qquad \blacksquare$$

There are two special cases of Theorem 5 that are of particular interest, and we state them as corollaries.

Corollary 6. *Let $X = \mathfrak{g}$, and let ρ be the adjoint representation of G on X, i.e., $\rho_g = \operatorname{Ad}_g$ for any $g \in G$. Then, the Euler–Poincaré equations (9)–(10) give the following equations:*

$$\frac{d}{dt}\left(\frac{\partial C}{\partial u} + \lambda\right) = \operatorname{ad}_u^*\left(\frac{\partial C}{\partial u} + \lambda\right) - \operatorname{ad}^*_{\frac{\partial V_{\text{ext}}}{\partial \alpha}} \alpha,$$
$$\dot{\alpha} = \operatorname{ad}_u^* \alpha, \quad \alpha(0) = \operatorname{Ad}_{g_0}^* \alpha_0.$$

Proof. We first begin by noting that $\alpha \in X^* = \mathfrak{g}^*$ and ρ^* is the coadjoint representation of G on X^*, i.e., $\rho_g^* = \operatorname{Ad}_g^*$ for any $g \in G$. We also have $\rho'_\xi = \operatorname{ad}_\xi$ for any $\xi \in \mathfrak{g}$, and it follows that $\xi_X(x) = \operatorname{ad}_\xi x$ for any $x \in X$. From (8), we have

$$\begin{aligned}\langle \mathbf{J}_X(x, \alpha), \xi \rangle &= \langle \alpha, \operatorname{ad}_\xi x \rangle \\ &= \langle \alpha, -\operatorname{ad}_x \xi \rangle \\ &= \langle -\operatorname{ad}_x^* \alpha, \xi \rangle,\end{aligned}$$

which gives $\mathbf{J}_X(x, \alpha) = -\operatorname{ad}_x^* \alpha$. ■

Corollary 7. *Let $X = \mathfrak{g}^*$, and let ρ be the coadjoint representation of G on X, i.e., $\rho_g = \operatorname{Ad}_{g^{-1}}^*$ for any $g \in G$. Then, the Euler–Poincaré equations (9)–(10) give the following equations:*

$$\frac{d}{dt}\left(\frac{\partial C}{\partial u} + \lambda\right) = \operatorname{ad}_u^*\left(\frac{\partial C}{\partial u} + \lambda\right) + \operatorname{ad}_\alpha^* \frac{\partial V_{\text{ext}}}{\partial \alpha},$$
$$\dot{\alpha} = -\operatorname{ad}_u \alpha, \quad \alpha(0) = \operatorname{Ad}_{g_0^{-1}} \alpha_0.$$

Proof. We begin by noting that $\alpha \in X^* = \mathfrak{g}^{**} \cong \mathfrak{g}$ and ρ^* is the adjoint representation of G on X^*, i.e., $\rho_g^* = \operatorname{Ad}_{g^{-1}}$ for any $g \in G$. We also have $\rho'_\xi = -\operatorname{ad}_\xi^*$ for any $\xi \in \mathfrak{g}$, and it follows that $\xi_X(x) = -\operatorname{ad}_\xi^* x$ for any $x \in X$. From (8), we have

$$\begin{aligned}\langle \mathbf{J}_X(x, \alpha), \xi \rangle &= \langle \alpha, -\operatorname{ad}_\xi^* x \rangle \\ &= \langle -\operatorname{ad}_\xi \alpha, x \rangle \\ &= \langle x, \operatorname{ad}_\alpha \xi \rangle \\ &= \langle \operatorname{ad}_\alpha^* x, \xi \rangle,\end{aligned}$$

which gives $\mathbf{J}_X(x, \alpha) = \operatorname{ad}_\alpha^* x$. ■

The Euler–Poincaré equation (9) is not particularly feasible for describing the time evolution of u and λ because the equations for them are combined into a single equation. However, assuming an additional structure on the Lie algebra $\mathfrak{g}$, we may decouple the equations for u and λ (see [3] for a similar approach applied to the standard Euler–Poincaré equation).

Proposition 8. *Assume that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that*

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{p};$$

then the time evolution of u and λ in (9) is given by the following equations:

$$\frac{d}{dt}\frac{\partial C}{\partial u} = \operatorname{ad}_{e_0}^* \frac{\partial C}{\partial u} + \operatorname{ad}_{u-e_0}^* \lambda + \mathbf{J}_X\left(\frac{\partial V_{\text{ext}}}{\partial \alpha}, \alpha\right)\bigg|_{\mathfrak{k}},$$

$$\frac{d\lambda}{dt} = \mathrm{ad}^*_{e_0}\lambda + \mathrm{ad}^*_{u-e_0}\frac{\partial C}{\partial u} + \mathbf{J}_X\left(\frac{\partial V_{\mathrm{ext}}}{\partial \alpha}, \alpha\right)\bigg|_{\mathfrak{p}},$$

where $u - e_0 \in \mathfrak{k}$.

Proof. It is easy to verify that $\mathfrak{g}^* = \mathfrak{k}^* \oplus \mathfrak{p}^*$ such that

$$\mathrm{ad}^*_{\mathfrak{k}}\,\mathfrak{k}^* \subseteq \mathfrak{p}^*,\ \ \mathrm{ad}^*_{\mathfrak{p}}\,\mathfrak{k}^* \subseteq \mathfrak{k}^*,\ \ \mathrm{ad}^*_{\mathfrak{p}}\,\mathfrak{p}^* \subseteq \mathfrak{p}^*,\ \ \mathrm{ad}^*_{\mathfrak{k}}\,\mathfrak{p}^* \subseteq \mathfrak{k}^*. \tag{11}$$

It is also easy to verify that $\frac{\partial C}{\partial u} \in \mathfrak{k}^*$ and by definition $\lambda \in \mathfrak{p}^*$. We now have a splitting of the left-hand side of (9) in $\mathfrak{k}^*$ and $\mathfrak{p}^*$. We have

$$\mathrm{ad}^*_u\frac{\partial C}{\partial u} = \mathrm{ad}^*_{e_0}\frac{\partial C}{\partial u} + \mathrm{ad}^*_{u-e_0}\frac{\partial C}{\partial u},$$
$$\mathrm{ad}^*_u\lambda = \mathrm{ad}^*_{e_0}\lambda + \mathrm{ad}^*_{u-e_0}\lambda,$$

where $u - e_0 \in \mathfrak{k}$. By using (11), we have $\mathrm{ad}^*_{e_0}\frac{\partial C}{\partial u} \in \mathfrak{k}^*$, $\mathrm{ad}^*_{u-e_0}\frac{\partial C}{\partial u} \in \mathfrak{p}^*$, $\mathrm{ad}^*_{e_0}\lambda \in \mathfrak{p}^*$, and $\mathrm{ad}^*_{u-e_0}\lambda \in \mathfrak{k}^*$. We can also split $\mathbf{J}_X(\frac{\partial V_{\mathrm{ext}}}{\partial\alpha}, \alpha) \in \mathfrak{g}^*$ in $\mathfrak{k}^*$ and $\mathfrak{p}^*$. We now have a splitting of the right-hand side of (9) in $\mathfrak{k}^*$ and $\mathfrak{p}^*$. So, (9) splits into the following equations:

$$\frac{d}{dt}\frac{\partial C}{\partial u} = \mathrm{ad}^*_{e_0}\frac{\partial C}{\partial u} + \mathrm{ad}^*_{u-e_0}\lambda + \mathbf{J}_X\left(\frac{\partial V_{\mathrm{ext}}}{\partial \alpha}, \alpha\right)\bigg|_{\mathfrak{k}},$$
$$\frac{d\lambda}{dt} = \mathrm{ad}^*_{e_0}\lambda + \mathrm{ad}^*_{u-e_0}\frac{\partial C}{\partial u} + \mathbf{J}_X\left(\frac{\partial V_{\mathrm{ext}}}{\partial \alpha}, \alpha\right)\bigg|_{\mathfrak{p}}. \qquad \blacksquare$$

Remark 9. Note that semisimple Lie algebras admit a Cartan decomposition; i.e., if $\mathfrak{g}$ is semisimple, then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{p},\ \ [\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{k},\ \ [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{p},$$

where $\mathfrak{k} = \{x \in \mathfrak{g} \mid \theta(x) = -x\}$ is the -1 eigenspace of the Cartan involution θ and $\mathfrak{p} = \{x \in \mathfrak{g} \mid \theta(x) = x\}$ is the $+1$ eigenspace of the Cartan involution θ. In addition, the Killing form is positive definite on $\mathfrak{k}$ and negative definite on $\mathfrak{p}$. So, connected semisimple Lie groups are potential candidates that satisfy the assumption of Proposition 8. Conversely, a Cartan decomposition determines a Cartan involution θ (see, e.g., [21]). For more details, see [13, 18, 21]. Also, note that the roles of $\mathfrak{k}$ and $\mathfrak{p}$ can be reversed in Proposition 8.

3.2. Legendre transformation. If we assume that the reduced Lagrangian ℓ_a is hyperregular, then we can obtain the reduced Hamiltonian $h_a : \mathfrak{g}^* \times \mathfrak{p}^* \times X^* \to \mathbb{R}$ (by using a Legendre transformation) given by

$$h_a(\mu, \lambda, \alpha) = \langle \mu, u\rangle - \ell_a(u, \lambda, \alpha),$$

where $\mu = \frac{\partial \ell_a}{\partial u} = (\frac{\partial C}{\partial u} + \lambda)$, with $\mu(\cdot) \in C^1([0, T], \mathfrak{g}^*)$. The Euler–Poincaré equations (9)–(10) can now be written as the Lie–Poisson equations (see, e.g., [15, 19]), which are given by

$$\dot{\mu} = \mathrm{ad}^*_u\mu + \mathbf{J}_X\left(\frac{\partial V_{\mathrm{ext}}}{\partial \alpha}, \alpha\right), \tag{12}$$

$$\dot{\alpha} = \rho'^*_u(\alpha),\ \ \alpha(0) = \rho^*_{g_0}(\alpha_0). \tag{13}$$

Remark 10. The Lie–Poisson equations (12)–(13) are also obtained in [7, 8, 9] using Pontryagin's maximum principle. For more details, see [7, 8, 9].

3.3. Discrete Lagrange–Pontryagin principle. Consider the discrete-time version of **(P1)** given by

$$\min_{\{u_k\}_{k=0}^{N-1}} J_d := \min_{\{u_k\}_{k=0}^{N-1}} \sum_{k=0}^{N-1} [C_d(g_k, u_k - e_0) + V_d(g_k)]$$

subject to **(P2)**

$$g_{k+1} = g_k \tau(h u_k), \text{ with given boundary conditions } g_0 \text{ and } g_N,$$

where $h \in \mathbb{R}_{>0}$ is the time step, $\tau : \mathfrak{g} \to G$ is the retraction map (see, e.g., [10, 11, 22, 23, 24]), and the functions C_d and V_d satisfy assumptions (i)–(v).

We can solve **(P2)** using a discrete analogue of the Lagrange–Pontryagin variational principle. Define the augmented cost function as

$$J_{d,a} = \sum_{k=0}^{N-1} L_{d,a}(g_k, g_{k+1}, u_k, \mu_k, \alpha),$$

where the augmented Lagrangian $L_{d,a} : G \times E \times \mathfrak{g}^* \times X^* \to \mathbb{R}$ is defined as

$$\begin{aligned} L_{d,a}(g_k, g_{k+1}, u_k, \mu_k, \alpha) &= C_d(g_k, u_k - e_0) + V_{d,\text{ext}}(g_k, \alpha) \\ &\quad + h \left\langle \mu_k, \frac{1}{h}\tau^{-1}(g_k^{-1} g_{k+1}) - (u_k - e_0) \right\rangle. \end{aligned} \tag{14}$$

Define the left action of G on $G \times E \times \mathfrak{g}^* \times X^*$ as follows:

$$\begin{aligned} \Gamma : G \times (G \times E \times \mathfrak{g}^* \times X^*) &\longrightarrow G \times E \times \mathfrak{g}^* \times X^*, \\ (g, (h_1, h_2, u, \mu, \alpha)) &\longmapsto (L_g(h_1), L_g(h_2), u, \mu, \rho^*_{g^{-1}}(\alpha)). \end{aligned} \tag{15}$$

Under assumption (v), it follows that $L_{d,a} : G \times E \times \mathfrak{g}^* \times X^* \to \mathbb{R}$ is G-invariant under (15); i.e., $L_{d,a} \circ \Gamma_g = L_{d,a}$ for any $g \in G$. In particular, under assumptions (iii)–(v), it follows that the augmented Lagrangian $L_{d,a}(\cdot,\cdot,\cdot,\cdot,\alpha_0) =: L_{d,a,\alpha_0} : G \times E \times \mathfrak{g}^* \to \mathbb{R}$ is G_{α_0}-invariant under (15); i.e., $L_{d,a,\alpha_0} \circ \Gamma_g = L_{d,a,\alpha_0}$ for any $g \in G_{\alpha_0}$. We can now obtain the reduced augmented Lagrangian $\ell_{d,a} : E \times \mathfrak{g}^* \times X^* \to \mathbb{R}$, which is given by

$$\begin{aligned} \ell_{d,a}(\pi(g_k, g_{k+1}), u_k, \mu_k, \bar{\alpha}_k) &:= L_{d,a}(e, \pi(g_k, g_{k+1}), u_k, \mu_k, \bar{\alpha}_k) \\ &= C_d(u_k - e_0) + V_{d,\text{ext}}(\bar{\alpha}_k) \\ &\quad + h \left\langle \mu_k, \frac{1}{h}\tau^{-1}(\pi(g_k, g_{k+1})) - (u_k - e_0) \right\rangle, \end{aligned}$$

where $\pi(g_k, g_{k+1}) := g_k^{-1} g_{k+1} \in G$, $\bar{\alpha}_k = \rho^*_{g_k}(\alpha_0)$, and with a slight abuse of notation, we write $C_d(e, u_k - e_0) = C_d(u_k - e_0)$ and $V_{d,\text{ext}}(\bar{\alpha}_k) = V_{d,\text{ext}}(e, \bar{\alpha}_k)$. A normal extremal for **(P2)** satisfies the following discrete-time Lie–Poisson equations (see Theorem 12 below for a proof):

$$(d\tau^{-1}_{hu_k})^* \mu_k = (d\tau^{-1}_{-hu_{k-1}})^* \mu_{k-1} + \mathbf{J}_X \left(\frac{\partial V_{d,\text{ext}}}{\partial \bar{\alpha}_k}, \bar{\alpha}_k \right), \tag{16}$$

$$\bar{\alpha}_{k+1} = \rho^*_{\tau(hu_k)}(\bar{\alpha}_k), \;\; \bar{\alpha}_0 = \rho^*_{g_0}(\alpha_0), \tag{17}$$

where $\mu_k = \frac{1}{h}\frac{\partial C_d}{\partial u_k} \in \mathfrak{g}^*$. To summarize, we have the following theorem, but before proceeding further, we recall a lemma which will be used in its proof.

Lemma 11 ([10, 11, 24]). *The following properties hold:*

$$\frac{1}{h}\delta\tau^{-1}(g_k^{-1}g_{k+1}) = \frac{1}{h}\delta\tau^{-1}(\pi(g_k, g_{k+1})) = \frac{1}{h}d\tau^{-1}_{hu_k}(-\eta_k + \mathrm{Ad}_{\tau(hu_k)}\,\eta_{k+1}), \tag{18}$$

where $\eta_k = T_{g_k}L_{g_k^{-1}}(\delta g_k) \in \mathfrak{g}$, *and*

$$(d\tau^{-1}_{-hu_k})^*\mu_k = \mathrm{Ad}^*_{\tau(hu_k)}(d\tau^{-1}_{hu_k})^*\mu_k, \tag{19}$$

where $\mu_k \in \mathfrak{g}^*$ *and* $d\tau^{-1}$ *is the inverse right trivialized tangent of* τ.

Theorem 12. *A normal extremal for* **(P2)**, *under assumptions* (i)–(v), *satisfies the following discrete-time Lie–Poisson equations:*

$$(d\tau^{-1}_{hu_k})^*\mu_k = (d\tau^{-1}_{-hu_{k-1}})^*\mu_{k-1} + \mathbf{J}_X\left(\frac{\partial V_{d,\mathrm{ext}}}{\partial\bar{\alpha}_k}, \bar{\alpha}_k\right), \tag{20}$$

$$\bar{\alpha}_{k+1} = \rho^*_{\tau(hu_k)}(\bar{\alpha}_k), \;\; \bar{\alpha}_0 = \rho^*_{g_0}(\alpha_0), \tag{21}$$

where $\mu_k = \frac{1}{h}\frac{\partial C_d}{\partial u_k} \in \mathfrak{g}^*$.

Proof. Consider the following variational principles:

(a) The variational principle

$$\delta\sum_{k=0}^{N-1} L_{d,a,\alpha_0}(g_k, g_{k+1}, u_k, \mu_k) = 0$$

holds for all variations of g_k (vanishing at the endpoints), $\tau^{-1}(g_k^{-1}g_{k+1})$ (induced by the variations of g_k), and u_k.

(b) The variational principle

$$\delta\sum_{k=0}^{N-1} \ell_{d,a}(\pi(g_k, g_{k+1}), u_k, \mu_k, \bar{\alpha}_k) = 0$$

holds for all variations of $\tau^{-1}(\pi(g_k, g_{k+1}))$ (induced by the variations of g_k vanishing at the endpoints), u_k, and $\bar{\alpha}_k$ of the form

$$\delta\bar{\alpha}_k = \rho'^*_{\eta_k}(\bar{\alpha}_k),$$

where $\eta_k \in \mathfrak{g}$ vanishes at the endpoints.

We will first show that the variational principle (a) implies the variational principle (b). We begin by noting that $L_{d,a} : G\times E\times\mathfrak{g}^*\times X^* \to \mathbb{R}$ is G-invariant under (15), i.e., $L_{d,a}\circ\Gamma_g = L_{d,a}$ for any $g \in G$ and $\bar{\alpha}_k = \rho^*_{g_k}(\alpha_0)$, so the summand in the variational principle (a) is equal to the summand in the variational principle (b). In addition, we have $\delta\bar{\alpha}_k = \rho'^*_{\eta_k}(\bar{\alpha}_k)$. Hence, the variational principle (a) implies the variational principle (b).

A normal extremal for **(P2)**, satisfies the variational principle (a) and hence, the variational principle (b). So, we have

$$\begin{aligned}
0 &= \delta \sum_{k=0}^{N-1} \ell_{d,a}(\pi(g_k, g_{k+1}), u_k, \mu_k, \bar{\alpha}_k) \\
&= \sum_{k=0}^{N-1} \Bigg[\left\langle \frac{\partial C_d}{\partial u_k}, \delta u_k \right\rangle - h\langle \mu_k, \delta u_k\rangle + \left\langle \delta\bar{\alpha}_k, \frac{\partial V_{d,\text{ext}}}{\partial \bar{\alpha}_k} \right\rangle \\
&\qquad + h \left\langle \mu_k, \frac{1}{h} d\tau^{-1}_{hu_k}(-\eta_k + \operatorname{Ad}_{\tau(hu_k)} \eta_{k+1}) \right\rangle \Bigg] \\
&= \sum_{k=0}^{N-1} \Bigg[\left\langle \frac{\partial C_d}{\partial u_k} - h\mu_k, \delta u_k \right\rangle + \left\langle \delta\bar{\alpha}_k, \frac{\partial V_{d,\text{ext}}}{\partial \bar{\alpha}_k} \right\rangle \\
&\qquad + h \left\langle \mu_k, \frac{1}{h} d\tau^{-1}_{hu_k}(-\eta_k + \operatorname{Ad}_{\tau(hu_k)} \eta_{k+1}) \right\rangle \Bigg] \\
&= \sum_{k=1}^{N-1} \Bigg[\left\langle \frac{\partial C_d}{\partial u_k} - h\mu_k, \delta u_k \right\rangle + \left\langle \rho'^*_{\eta_k}(\bar{\alpha}_k), \frac{\partial V_{d,\text{ext}}}{\partial \bar{\alpha}_k} \right\rangle \\
&\qquad + \left\langle (d\tau^{-1}_{-hu_{k-1}})^* \mu_{k-1} - (d\tau^{-1}_{hu_k})^* \mu_k, \eta_k \right\rangle \Bigg] \\
&= \sum_{k=1}^{N-1} \Bigg[\left\langle \frac{\partial C_d}{\partial u_k} - h\mu_k, \delta u_k \right\rangle \\
&\qquad + \left\langle (d\tau^{-1}_{-hu_{k-1}})^* \mu_{k-1} + \mathbf{J}_X \left(\frac{\partial V_{d,\text{ext}}}{\partial \bar{\alpha}_k}, \bar{\alpha}_k \right) - (d\tau^{-1}_{hu_k})^* \mu_k, \eta_k \right\rangle \Bigg],
\end{aligned}$$

where we have used Lemma 11, the analogue of integration by parts in the discrete-time setting, and the fact that $\eta_0 = \eta_N = 0$, and so we have

$$(d\tau^{-1}_{hu_k})^* \mu_k = (d\tau^{-1}_{-hu_{k-1}})^* \mu_{k-1} + \mathbf{J}_X \left(\frac{\partial V_{d,\text{ext}}}{\partial \bar{\alpha}_k}, \bar{\alpha}_k \right).$$

Finally, after a simple calculation, we have

$$\bar{\alpha}_{k+1} = \rho^*_{\tau(hu_k)}(\bar{\alpha}_k), \;\; \bar{\alpha}_0 = \rho^*_{g_0}(\alpha_0). \qquad \blacksquare$$

As in the continuous-time case, there are are two special cases of Theorem 12 that are of particular interest, and we state them as corollaries.

Corollary 13. *Let $X = \mathfrak{g}$, and let ρ be the adjoint representation of G on X, i.e., $\rho_g = \operatorname{Ad}_g$ for any $g \in G$. Then, the discrete-time Lie–Poisson equations (20)–(21) give the following equations:*

$$(d\tau^{-1}_{hu_k})^* \mu_k = (d\tau^{-1}_{-hu_{k-1}})^* \mu_{k-1} - \operatorname{ad}^*_{\frac{\partial V_{d,\text{ext}}}{\partial \bar{\alpha}_k}} \bar{\alpha}_k,$$

$$\bar{\alpha}_{k+1} = \mathrm{Ad}^*_{\tau(hu_k)} \bar{\alpha}_k, \;\; \bar{\alpha}_0 = \mathrm{Ad}^*_{g_0} \alpha_0,$$

where $\mu_k = \frac{1}{h}\frac{\partial C_d}{\partial u_k} \in \mathfrak{g}^*$.

Proof. See Corollary 6. ■

Corollary 14. *Let* $X = \mathfrak{g}^*$, *and let* ρ *be the coadjoint representation of* G *on* X, *i.e.,* $\rho_g = \mathrm{Ad}^*_{g^{-1}}$ *for any* $g \in G$. *Then, the discrete-time Lie–Poisson equations* (20)–(21) *give the following equations:*

$$(d\tau^{-1}_{hu_k})^* \mu_k = (d\tau^{-1}_{-hu_{k-1}})^* \mu_{k-1} + \mathrm{ad}^*_{\bar{\alpha}_k} \frac{\partial V_{d,\mathrm{ext}}}{\partial \bar{\alpha}_k},$$
$$\bar{\alpha}_{k+1} = \mathrm{Ad}_{\tau(hu_k)^{-1}} \bar{\alpha}_k, \;\; \bar{\alpha}_0 = \mathrm{Ad}_{g_0^{-1}} \alpha_0,$$

where $\mu_k = \frac{1}{h}\frac{\partial C_d}{\partial u_k} \in \mathfrak{g}^*$.

Proof. See Corollary 7. ■

Remark 15. Note that the discrete-time Euler–Poincaré equations (and hence, the discrete-time Lie–Poisson equations) are obtained in [5, 6] in the case when the symmetry group is a subgroup of a Lie group. In the case when the symmetry group is the Lie group itself, one recovers the discrete-time Lie–Poisson equations given in [27]. The discrete-time Lie–Poisson equations obtained in this paper come from a Lagrangian that is defined for a special class of discrete-time OCPs (which satisfy some additional assumptions) rather than an actual physical Lagrangian, and so, they can be thought of as the optimal control counterpart of those in [5, 6].

4. Examples.

We will now consider two examples to illustrate the theory developed in the paper.

4.1. The heavy top as an optimal control problem.

We will now show that the heavy top equations can also be obtained from OCPs for left-invariant control affine systems on Lie groups, with partial symmetry breaking cost functions.

We now consider **(P1)**, with $G = \mathrm{SO}(3)$, $e_0 = 0_{3\times 3}$, and with the cost function given by

$$C(g, u - e_0) = \frac{1}{2}\langle u, \mathbb{I}u\rangle,$$
$$V(g) = -m\mathbf{g}l\langle \mathbf{e}_3, g\boldsymbol{\chi}\rangle,$$

where $\langle \xi, \xi\rangle := \mathrm{tr}(\xi^T\xi)$ for any $\xi \in \mathfrak{g} = \mathfrak{so}(3)$; $u(\cdot) = \sum_{i=1}^3 u^i(\cdot)e_i$, with the elements of the basis of $\mathfrak{g}$ (which is semisimple) given by

$$e_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \;\; e_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \;\; e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which satisfy

$$[e_1, e_2] = e_3, \;\; [e_2, e_3] = e_1, \;\; [e_3, e_1] = e_2;$$

$\mathbb{I} : \mathfrak{g} \to \mathfrak{g}^* = \mathfrak{so}(3)^*$ is the inertia tensor of the top;[1] m is the mass of the body; $\mathbf{g}$ is the acceleration due to gravity; $\mathbf{e}_3$ is the vertical unit vector; $\boldsymbol{\chi} \in \mathbb{R}^3$ is the unit vector from the point of support to the direction of the body's center of mass (constant) in body coordinates; and l is the length of the line segment between these two points.

Under the dual pairing, where $\langle \alpha, \xi \rangle := \operatorname{tr}(\alpha\xi)$ for any $\xi \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}^*$, the elements of the basis of $\mathfrak{g}^*$ are given by

$$e^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{bmatrix}, \quad e^2 = \begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}, \quad e^3 = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to verify that the potential function is SO(2)-invariant but not G-invariant, and so the potential function breaks the symmetry partially. We will now show how the theory developed in the paper can be applied to the OCP under consideration.

Let $X = \mathfrak{g}$, and let ρ be the adjoint representation of G on X, i.e., $\rho_g = \operatorname{Ad}_g$ for any $g \in G$. So, ρ^* is the coadjoint representation of G on $X^* = \mathfrak{g}^*$, i.e., $\rho^*_g = \operatorname{Ad}^*_g$ for any $g \in G$. Let the extended potential function $V_{\text{ext}} : G \times X^* \to \mathbb{R}$ be given as follows:

$$V_{\text{ext}}(g, \breve{\alpha}) = -m\mathbf{g}l\langle g^{-1}\alpha, \boldsymbol{\chi}\rangle,$$

where $\alpha \in \mathbb{R}^3$ is identified with $\breve{\alpha} \in \mathfrak{g}^*$ (see, e.g., [20, section 5.3]). If we set $\breve{\alpha}_0 = -2e^3$ in the extended potential function, we recover our original potential function. It is easy to verify that the extended potential function is G-invariant under (3); i.e., $V_{\text{ext}} \circ \Phi_g = V_{\text{ext}}$ for any $g \in G$. It is also easy to verify that the potential function is invariant under the left action of the isotropy group

$$G_{\alpha_0} := \{ g \in G \mid \operatorname{Ad}^*_g \alpha_0 = \alpha_0 \} \cong \mathrm{SO}(2),$$

i.e., rotations about the vertical axis $\mathbf{e}_3$. We can now see that assumptions (i)–(v) are satisfied. By Corollary 6, the Euler–Poincaré equations (9)–(10) (under the identifications $\mathfrak{g} \cong \mathbb{R}^3$ and $\mathfrak{g}^* \cong \mathbb{R}^3$) give the following equations:

$$\mathbf{I}\dot{u} = \operatorname{ad}^*_u \mathbf{I}u - \operatorname{ad}^*_{\frac{\partial V_{\text{ext}}}{\partial \alpha}} \alpha, \tag{22}$$

$$\dot{\alpha} = \operatorname{ad}^*_u \alpha, \;\; \alpha(0) = \operatorname{Ad}^*_{g_0} \alpha_0, \tag{23}$$

where

$$\operatorname{ad}^*_u \mathbf{I}u = \mathbf{I}u \times u, \;\; \operatorname{ad}^*_{\frac{\partial V_{\text{ext}}}{\partial \alpha}} \alpha = -m\mathbf{g}l\, \alpha \times \boldsymbol{\chi}, \;\; \operatorname{ad}^*_u \alpha = \alpha \times u, \;\; \operatorname{Ad}^*_{g_0} \alpha_0 = g_0^{-1}\alpha_0,$$

and $\mathbf{I} \in \mathbb{R}^{3\times 3}$ is the inertia matrix. For more details, see [20, 28]. Thus, (22)–(23) give the following equations:

$$\mathbf{I}\dot{u} = \mathbf{I}u \times u - m\mathbf{g}l\, \boldsymbol{\chi} \times \alpha,$$

[1] It is calculated with respect to the pivot, which is not, in general, the center of mass.

$$\dot{\alpha} = \alpha \times u, \;\; \alpha(0) = g_0^{-1}\alpha_0.$$

The Lie–Poisson equations (12)–(13) are given by

$$\dot{\mu} = \mu \times \mathbf{I}^{-1}\mu - mg l \, \boldsymbol{\chi} \times \alpha,$$
$$\dot{\alpha} = \alpha \times \mathbf{I}^{-1}\mu, \;\; \alpha(0) = g_0^{-1}\alpha_0,$$

which are the well-known heavy top equations (see, e.g., [20, section 7.4], [28, section 1.4]).

4.2. Motion planning of a unicycle with an obstacle. We study the OCP for the motion planning of a unicycle with an obstacle. To avoid the obstacle, we use the barrier function approach (see, e.g, [17, 32]), which plays the role of the potential function in the cost function of the OCP.

Our model for a unicycle is a homogeneous disk rolling on a horizontal plane maintaining its vertical position (see, e.g., [2, section 1.4]). The configuration of the unicycle at any given time is completely determined by the element $g \in \mathrm{SE}(2) \cong \mathbb{R}^2 \times \mathrm{S}^1 \cong \mathbb{R}^2 \times \mathrm{SO}(2)$ (as a set) given by

$$g = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix},$$

where $[x \;\; y]^T \in \mathbb{R}^2$ represents the point of contact of the wheel with the ground and $\theta \in \mathrm{S}^1$ represents the angular orientation of the overall system (see Figure 1). For more details, see [2, 26]. The controlled equations for the unicycle are given by

$$\dot{x} = u^2 \cos\theta, \tag{24}$$
$$\dot{y} = u^2 \sin\theta, \tag{25}$$
$$\dot{\theta} = u^1. \tag{26}$$

We can view (24)–(26) as a left-invariant control affine system on SE(2) (see [26]). Note that (24)–(25) are equivalent to the nonholonomic constraint $\dot{x}\sin\theta - \dot{y}\cos\theta = 0$.

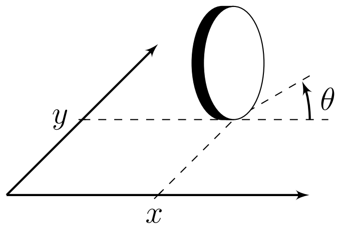

Figure 1. *The unicycle.*

Let the obstacle be circular in shape and be located in the x-y plane. Without loss of generality, assume that the obstacle has unit radius, with its center located at the point $(0, 0)$. Let the potential function $V : \mathbb{R}^2 \setminus \mathrm{S}^1 \to \mathbb{R}$ be given by

$$V(x, y) = \frac{\kappa}{2(x^2 + y^2 - 1)},$$

where $\kappa \in \mathbb{R}_{\geq 0}$. Let us now equip $\mathfrak{se}(2)$ with the inner product $\langle \xi, \xi \rangle := \mathrm{tr}(\xi^T \xi)$ for any $\xi \in \mathfrak{se}(2)$, and hence, the norm $\|\xi\| := \langle \xi, \xi \rangle^{1/2} = \sqrt{\mathrm{tr}(\xi^T \xi)}$ for any $\xi \in \mathfrak{se}(2)$. Equivalently, the potential function $V : \mathrm{SE}(2) \setminus S \to \mathbb{R}$ is given by

$$V(g) = \frac{\kappa}{2(\| \mathrm{Ad}_{g^{-1}} e_1 \|^2 - 3)},$$

where $S := \{g \in \mathrm{SE}(2) \mid \| \mathrm{Ad}_{g^{-1}} e_1\|^2 = 3\}$ and e_1 is given below. We now consider **(P1)**, with $G = \mathrm{SE}(2)$, $e_0 = 0_{3\times 3}$ and with the cost function given by

$$C(g, u - e_0) = \frac{1}{2}\langle u, u \rangle,$$
$$V(g) = \frac{\kappa}{2(\| \mathrm{Ad}_{g^{-1}} e_1 \|^2 - 3)},$$

where $u(\cdot) = \sum_{i=1}^{2} u^i(\cdot) e_i$, with the elements of the basis of $\mathfrak{g} = \mathfrak{se}(2)$ (which is not semisimple) given by

$$e_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

which satisfy

$$[e_1, e_2] = e_3, \;\; [e_2, e_3] = 0_{3\times 3}, \;\; [e_3, e_1] = e_2.$$

Under the dual pairing, where $\langle \alpha, \xi \rangle := \mathrm{tr}(\alpha \xi)$ for any $\xi \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}^* = \mathfrak{se}(2)^*$, the elements of the basis of $\mathfrak{g}^*$ are given by

$$e^1 = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad e^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

It is easy to verify that the potential function is $\mathrm{SO}(2)$-invariant but not G-invariant, and so the potential function breaks the symmetry partially. We will now show how the theory developed in the paper can be applied to the OCP under consideration.

Let $X = \mathfrak{g}^*$, and let ρ be the coadjoint representation of G on X, i.e., $\rho_g = \mathrm{Ad}^*_{g^{-1}}$ for any $g \in G$. So, ρ^* is the adjoint representation of G on $X^* = \mathfrak{g}^{**} \cong \mathfrak{g}$, i.e., $\rho^*_g = \mathrm{Ad}_{g^{-1}}$ for any $g \in G$. Let the extended potential function $V_{\mathrm{ext}} : G \times X^* \to \mathbb{R}$ be given as follows:

$$V_{\mathrm{ext}}(g, \alpha) = \frac{\kappa}{2(\| \mathrm{Ad}_{g^{-1}} \alpha \|^2 - 3)}.$$

If we set $\alpha_0 = e_1$ in the extended potential function, we recover our original potential function. It is easy to verify that the extended potential function is G-invariant under (3); i.e., $V_{\mathrm{ext}} \circ \Phi_g =$

V_{ext} for any $g \in G$. It is also easy to verify that the potential function is invariant under the left action of the isotropy group

$$G_{\alpha_0} := \{g \in G \mid \operatorname{Ad}_{g^{-1}} \alpha_0 = \alpha_0\} = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| \ \theta \in \mathrm{S}^1 \right\} \cong \mathrm{SO}(2).$$

We can now see that assumptions (i)–(v) are satisfied and we also have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{p}.$$

By Corollary 7 and Proposition 8, the Euler–Poincaré equations (9)–(10) give the following equations:

$$\dot{u} = \operatorname{ad}_u^* \lambda + \operatorname{ad}_\alpha^* \left.\frac{\partial V_{\text{ext}}}{\partial \alpha}\right|_{\mathfrak{k}}, \tag{27}$$

$$\dot{\lambda} = \operatorname{ad}_u^* u + \operatorname{ad}_\alpha^* \left.\frac{\partial V_{\text{ext}}}{\partial \alpha}\right|_{\mathfrak{p}}, \tag{28}$$

$$\dot{\alpha} = -\operatorname{ad}_u \alpha, \quad \alpha(0) = \operatorname{Ad}_{g_0^{-1}} \alpha_0, \tag{29}$$

where

$$\operatorname{ad}_u^* \lambda = \begin{bmatrix} 0 & -\dfrac{u^2\lambda_3}{2} & 0 \\ \dfrac{u^2\lambda_3}{2} & 0 & 0 \\ u^1\lambda_3 & 0 & 0 \end{bmatrix}, \quad \operatorname{ad}_\alpha^* \frac{\partial V_{\text{ext}}}{\partial \alpha} = \frac{\kappa\alpha^1}{(\|\alpha\|^2 - 3)^2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\alpha^3 & \alpha^2 & 0 \end{bmatrix},$$

$$\operatorname{ad}_u^* u = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -u^1u^2 & 0 \end{bmatrix}, \quad \operatorname{ad}_u \alpha = \begin{bmatrix} 0 & 0 & -u^1\alpha^3 \\ 0 & 0 & u^1\alpha^2 - u^2\alpha^1 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$\operatorname{Ad}_{g_0^{-1}} \alpha_0 = \begin{bmatrix} 0 & -1 & x_0\sin\theta_0 - y_0\cos\theta_0 \\ 1 & 0 & x_0\cos\theta_0 + y_0\sin\theta_0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For more details, see [28, section 14.6]. So, (27)–(29) give the following equations:

$$\begin{aligned} \dot{u}^1 &= -\frac{u^2\lambda_3}{2}, \\ \dot{u}^2 &= u^1\lambda_3 - \frac{\kappa\alpha^1\alpha^3}{(\|\alpha\|^2 - 3)^2}, \\ \dot{\lambda}_3 &= -u^1u^2 + \frac{\kappa\alpha^1\alpha^2}{(\|\alpha\|^2 - 3)^2}, \\ \dot{\alpha}^1 &= 0, \quad \alpha^1(0) = 1, \end{aligned}$$

$$\dot{\alpha}^2 = u^1\alpha^3, \;\; \alpha^2(0) = x_0 \sin\theta_0 - y_0 \cos\theta_0,$$
$$\dot{\alpha}^3 = -u^1\alpha^2 + u^2\alpha^1, \;\; \alpha^3(0) = x_0 \cos\theta_0 + y_0 \sin\theta_0.$$

The Lie–Poisson equations (12)–(13) are given by

$$\begin{aligned}
\dot{\mu}_1 &= -\mu_2\mu_3,\\
\dot{\mu}_2 &= \frac{\mu_1\mu_3}{2} - \frac{\kappa\alpha^1\alpha^3}{(\|\alpha\|^2-3)^2},\\
\dot{\mu}_3 &= -\frac{\mu_1\mu_2}{2} + \frac{\kappa\alpha^1\alpha^2}{(\|\alpha\|^2-3)^2},\\
\dot{\alpha}^1 &= 0, \;\; \alpha^1(0) = 1,\\
\dot{\alpha}^2 &= \frac{\mu_1\alpha^3}{2}, \;\; \alpha^2(0) = x_0\sin\theta_0 - y_0\cos\theta_0,\\
\dot{\alpha}^3 &= -\frac{\mu_1\alpha^2}{2} + \mu_2\alpha^1, \;\; \alpha^3(0) = x_0\cos\theta_0 + y_0\sin\theta_0.
\end{aligned}$$

Remark 16. Let us consider the case when the OCP under consideration has no potential function. In this case, the Euler–Poincaré equations (9) give the following equations:

$$\dot{u}^1 = -\frac{u^2\lambda_3}{2}, \tag{30}$$
$$\dot{u}^2 = u^1\lambda_3, \tag{31}$$
$$\dot{\lambda}_3 = -u^1u^2. \tag{32}$$

We will now show that the Euler–Poincaré equations (30)–(32) are exactly the Euler–Lagrange equations that we have when we view the OCP under consideration as a constrained variational problem. The OCP under consideration is equivalent to the following constrained variational problem:

$$\min_{([x(\cdot)\;\; y(\cdot)]^T,\theta(\cdot))} J := \min_{([x(\cdot)\;\; y(\cdot)]^T,\theta(\cdot))} \frac{1}{2}\int_0^T [\dot{x}^2(t) + \dot{y}^2(t) + 2\dot{\theta}^2(t)]dt$$

subject to **(P3)**

$\dot{x}(t)\sin\theta(t) - \dot{y}(t)\cos\theta(t) = 0$, with given boundary conditions $([x(0)\;\; y(0)]^T, \theta(0))$ and $([x(T)\;\; y(T)]^T, \theta(T))$.

The Lagrangian for **(P3)** is given by

$$L_a(\theta, \dot{x}, \dot{y}, \dot{\theta}, \lambda) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + 2\dot{\theta}^2) + \lambda(\dot{y}\cos\theta - \dot{x}\sin\theta),$$

where λ is the Lagrange multiplier. A solution for **(P3)** must satisfy the following Euler–Lagrange equations:

$$\ddot{x} = \dot{\lambda}\sin\theta + \lambda\dot{\theta}\cos\theta, \tag{33}$$

$$\ddot{y} = -\dot{\lambda}\cos\theta + \lambda\dot{\theta}\sin\theta, \tag{34}$$

$$\ddot{\theta} = -\frac{\lambda}{2}(\dot{x}\cos\theta + \dot{y}\sin\theta). \tag{35}$$

Using the facts that $\dot{x}\sin\theta - \dot{y}\cos\theta = 0$, $u^1 = \dot{\theta}$, and $u^2 = \dot{x}\cos\theta + \dot{y}\sin\theta$, after a few simple calculations, the Euler–Lagrange equations (33)–(35) give the following equations:

$$\begin{aligned} \dot{u}^1 &= -\frac{u^2\lambda}{2}, \\ \dot{u}^2 &= u^1\lambda, \\ \dot{\lambda} &= -u^1u^2, \end{aligned}$$

which are exactly the Euler–Poincaré equations (30)–(32).

Remark 17. For the above example, in the discrete-time setting, one would choose

$$\begin{aligned} C_d(g_k, u_k - e_0) &= \frac{h}{2}\langle u_k, u_k\rangle, \\ V_d(g_k) &= \frac{h\kappa}{2(\|\operatorname{Ad}_{g_k^{-1}} e_1\|^2 - 3)}, \end{aligned}$$

where

$$g_k = \begin{bmatrix} \cos\theta_k & -\sin\theta_k & x_k \\ \sin\theta_k & \cos\theta_k & y_k \\ 0 & 0 & 1 \end{bmatrix} \in G$$

and $u_k = \sum_{i=1}^{2} u_k^i e_i \in \mathfrak{g}$. Also, in the discrete-time setting, the extended potential function $V_{d,\text{ext}} : G \times X^* \to \mathbb{R}$ can be constructed in exactly the same way as in the above example and is given by

$$V_{d,\text{ext}}(g_k, \alpha) = \frac{h\kappa}{2(\|\operatorname{Ad}_{g_k^{-1}} \alpha\|^2 - 3)}.$$

Again, following the same procedure as described in the above example, it can be verified that assumptions (i)–(v) are satisfied. By Corollary 14, the discrete-time Lie–Poisson equations (20)–(21) give the following equations:

$$\begin{aligned} (d\tau_{hu_k}^{-1})^*\mu_k &= (d\tau_{-hu_{k-1}}^{-1})^*\mu_{k-1} + \operatorname{ad}^*_{\bar{\alpha}_k}\frac{\partial V_{d,\text{ext}}}{\partial \bar{\alpha}_k}, \\ \bar{\alpha}_{k+1} &= \operatorname{Ad}_{\tau(hu_k)^{-1}} \bar{\alpha}_k, \ \ \bar{\alpha}_0 = \operatorname{Ad}_{g_0^{-1}} \alpha_0, \end{aligned}$$

where

$$\operatorname{ad}^*_{\bar{\alpha}_k}\frac{\partial V_{d,\text{ext}}}{\partial \bar{\alpha}_k} = \frac{h\kappa\bar{\alpha}_k^1}{(\|\bar{\alpha}_k\|^2 - 3)^2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\bar{\alpha}_k^3 & \bar{\alpha}_k^2 & 0 \end{bmatrix}$$

and $\mu_k = u_k \in \mathfrak{g}^*$. For numerical purposes, one first chooses an appropriate retraction map, such as the Cayley map or the exponential map (see, e.g., [10, 11, 22, 23, 24]), and then computes the quantities $(d\tau^{-1}_{hu_k})^*\mu_k$ and $(d\tau^{-1}_{-hu_{k-1}})^*\mu_{k-1}$. As an example, if we choose the Cayley map cay : $\mathfrak{se}(2) \to \mathrm{SE}(2)$ as the retraction map, then we have

$$[d\mathrm{cay}^{-1}_{hu_k}]^*\mu_k = \begin{bmatrix} -\frac{h^2(u_k^1)^2u_k^2}{4} + \frac{h(u_k^2)^2}{2} & u_k^1 + \frac{h^2(u_k^1)^3}{4} & 0 \\ -u_k^1 - \frac{hu_k^1u_k^2}{2} & 0 & 0 \\ -\frac{h(u_k^1)^2}{2} + u_k^2 & 0 & 0 \end{bmatrix}$$

and

$$[d\mathrm{cay}^{-1}_{-hu_{k-1}}]^*\mu_{k-1} = \begin{bmatrix} -\frac{h^2(u_{k-1}^1)^2u_{k-1}^2}{4} - \frac{h(u_{k-1}^2)^2}{2} & u_{k-1}^1 + \frac{h^2(u_{k-1}^1)^3}{4} & 0 \\ -u_{k-1}^1 + \frac{hu_{k-1}^1u_{k-1}^2}{2} & 0 & 0 \\ \frac{h(u_{k-1}^1)^2}{2} + u_{k-1}^2 & 0 & 0 \end{bmatrix}.$$

Note that for $v = \sum_{i=1}^3 v^i e_i \in \mathfrak{g}$, the matrix representation for $d\mathrm{cay}_v^{-1}$ is given by

$$[d\mathrm{cay}_v^{-1}] = \begin{bmatrix} 1 + \frac{(v^1)^2}{4} & 0 & 0 \\ \frac{v^1v^2}{4} - \frac{v^3}{2} & 1 & \frac{v^1}{2} \\ \frac{v^1v^3}{4} + \frac{v^2}{2} & -\frac{v^1}{2} & 1 \end{bmatrix}.$$

For more details, see [10, 11, 22, 23, 24].

5. Conclusions. We studied symmetry reduction of OCPs for left-invariant control affine systems on Lie groups, with partial symmetry breaking cost functions, and we obtained the Euler–Poincaré equations. Furthermore, by using a Legendre transformation, we obtained the Lie–Poisson equations, and in the discrete-time setting, we obtained the discrete-time Lie–Poisson equations. The theory was also illustrated with some practical examples.

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