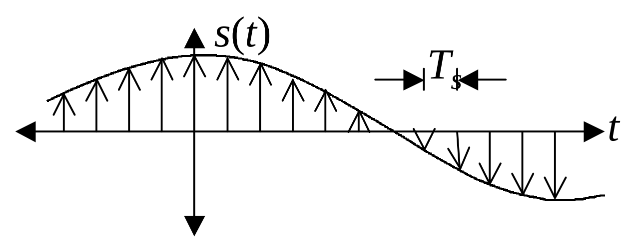
Analog-to-Digital Conversion

- Analog-to-Digital (A/D) conversion consists of 2 steps
  - **Sampling** converts the continuous-time signal, $s(t)$, to a discrete-time sequence, $s(k)$.

$$s[k] = s(kT_s)$$

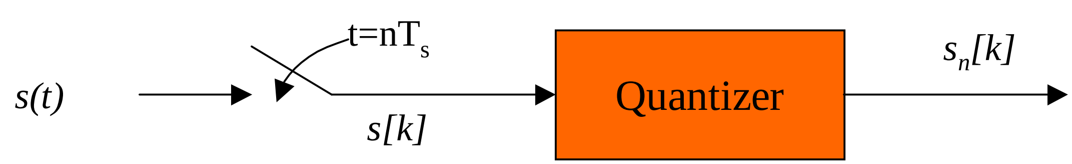
$$s_{sampled}(t) = \sum_{k=-\infty}^{\infty} \underbrace{s(kT_s)}_{s[k]} \underbrace{\delta(t - kT_s)}_{\text{impulse train}}$$



Sampled analog waveform

  - **Quantization** converts the continuous amplitude number to one of M possible numbers.

$$s_n[k] = Q(s[k]) \qquad s_n \in \{s_0, s_1, \ldots, s_{M-1}\}$$

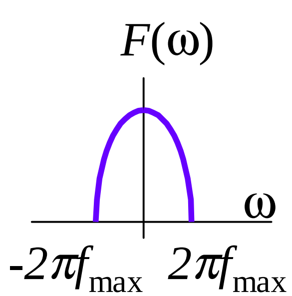


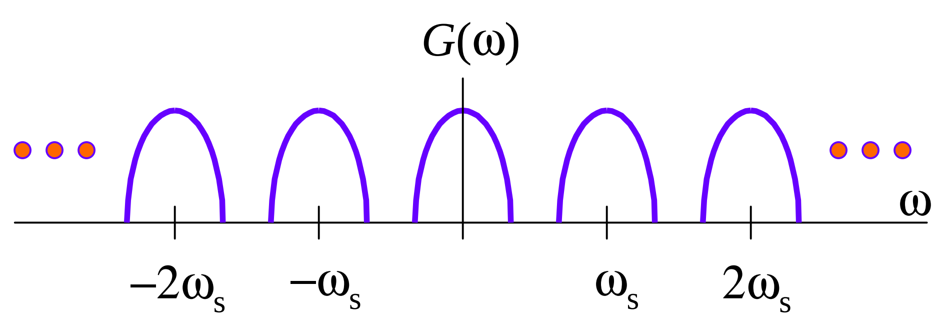
Sampling: Frequency Domain

- Spectrum of continuous-time signal replicated at integer multiples of sampling frequency
- Fourier series of impulse train where $w_s = 2\pi f_s$

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\,T_s) = \frac{1}{T_s}\left(1 + 2\cos(\omega_s\,t) + 2\cos(2\,\omega_s\,t) + \cdots\right)$$

$$g(t) = f(t)\,\delta_T(t) = \frac{1}{T_s}\left(f(t) + \underbrace{2\,f(t)\cos(\omega_s\,t)}_{\text{Modulation by }\cos(\omega_S\,t)} + \underbrace{2\,f(t)\cos(2\,\omega_s\,t)}_{\text{Modulation by }\cos(2\,\omega_S\,t)} + \cdots \right)$$





Sampling: Shannon's Theorem

- A continuous-time signal $s(t)$ with frequencies no higher than $\boldsymbol{f_{max}}$ can be reconstructed from its samples $x[k] = x(k\,T_s)$ if the samples are taken at a rate $\boldsymbol{f_s}$ which is greater than $2\,f_{max}$

  Nyquist rate = $2\,f_{max}$

  Nyquist frequency = $\boldsymbol{f_s/2}$
- What happens if $f_s = 2\,f_{max}$?
- Consider a sinusoid $sin(2\pi\,f_{max}\,t)$

  Use a sampling period of $T_s = 1/f_s = 1/2f_{max}$

  Sketch: sinusoid with zeros at $t = 0,\ 1/2f_{max},\ 1/f_{max}, \ldots$

Sampling Theorem Assumptions

- Continuous-time signal has no frequency content above the f_{max}
- Sampling time is exactly the same between any two samples
- Sequence of numbers obtained by sampling is represented in exact precision
- Conversion of the sequence of numbers to continuous-time is ideal

Why 44.1 kHz for Audio CDs?

- Sound is audible in 20 Hz to 20 kHz range:

  f_{max} = 20 kHz and the Nyquist rate 2 f_{max} = 40 kHz
- What is the extra 10% of the bandwidth used?

  Rolloff from passband to stopband in the magnitude response of the anti-aliasing filter
- Okay, 44 kHz makes sense. Why 44.1 kHz?

  At the time the choice was made, only recorders capable of storing such high rates were VCRs.
  - NTSC: 490 lines/frame, 3 samples/line, 30 frames/s = 44100 samples/s
  - PAL: 588 lines/frame, 3 samples/line, 25 frames/s = 44100 samples/s

Generalized Sampling Theorem

- Sampling rate must be greater than twice the bandwidth.
  - Bandwidth is defined as the non-zero extent of the spectrum of the continuous-time signal in positive frequencies.
  - For a lowpass signal with maximum frequency f_{max}, the bandwidth is f_{max}.
  - For a bandpass signal with frequency content on the interval $[f_1, f_2]$, the bandwidth is $f_2 - f_1$.

Sampling and Oversampling

- As sampling rate increases, sampled waveform looks more like original
- In some applications, e.g. touchtone decoding, frequency content matters not waveform shape
- Zero crossings: frequency content of a sinusoid
  - Distance between two zero crossings: one half period.
  - With the sampling theorem satisfied, sampled sinusoid crosses zero the right number of times even though its waveform shape may be difficult to recognize
- *DSP First*, Ch. 4, Sampling & interpolation demo

Aliasing

- Analog sinusoid

  $x(t) = A\cos(2\pi f_0 t + \phi)$
- Sample at $T_s = 1/f_s$ and $x[n] = x(T_s n) = A\cos(2\pi f_0 T_s n + \phi)$
- Keeping the sampling period same
  - Sample $y(t) = A\cos(2\pi(f_0 + lf_s)t + \phi)$ where l is an integer
- Here, $f_s T_s = 1$, $\cos(x + 2\pi l) = \cos(x)$

$$
\begin{aligned}
y[n] = y(T_s n) \\
&= A\cos(2\pi(f_0 + lf_s)T_s n + f) \\
&= A\cos(2\pi f_0 T_s n + 2\pi l f_s T_s n + f) \\
&= A\cos(2\pi f_0 T_s n + 2\pi l n + f) \\
&= A\cos(2\pi f_0 T_s n + f) \\
&= x[n]
\end{aligned}
$$

  - $y[n]$ indistinguishable from $x[n]$

Folding

- Second source of aliasing frequencies
- From negative frequency component of a sinusoid, $-f_0 + l\, f_s$, where l is any integer, f_s is the sampling rate
  f_0 is sinusoid frequency
- Sampling $w(t)$ with a sampling period of $T_s = 1/f_s$

$$w[n] = x[n] = x(T_s\, n)$$

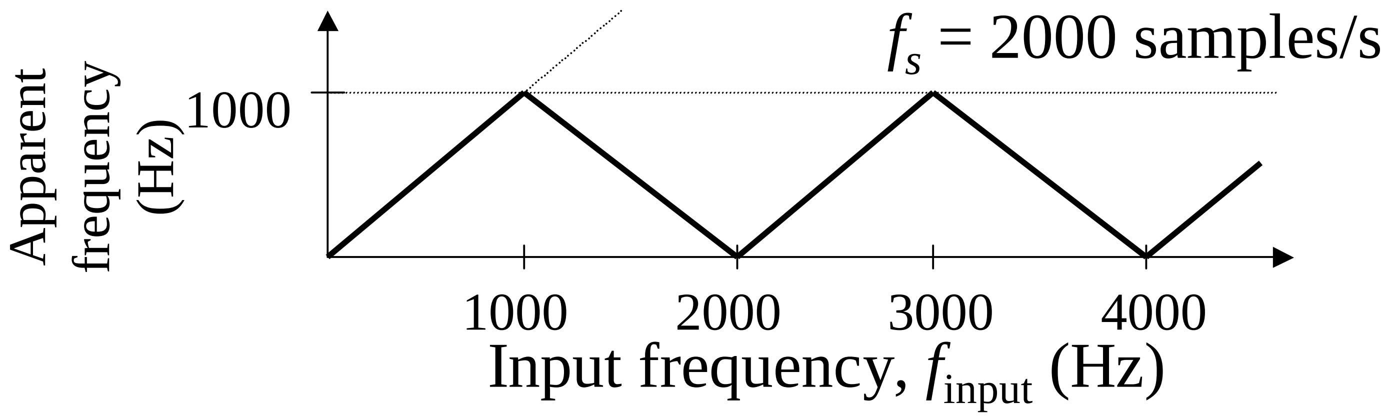
$$x(t) = A\cos(2\pi f_0 t + \phi)$$

$$w(t) = A\cos(2\pi(-f_0 + l\, f_s)\, t - \phi)$$

$$\begin{aligned} w[n] &= w(T_s\, n) \\ &= A\cos(2\pi(-f_0 + l\, f_s)\, T_s\, n - \phi) \\ &= A\cos(-2\pi\, f_0\, T_s\, n + 2\pi\, l\, f_s\, T_s\, n - \phi) \\ &= A\cos(-2\pi\, f_0\, T_s\, n + 2\pi\, l\, n - \phi) \\ &= A\cos(-2\pi\, f_0\, T_s\, n - \phi) \\ &= A\cos(2\pi\, f_0\, T_s\, n + \phi) \end{aligned}$$

Aliasing and Folding

- Aliasing and folding of a sinusoid $\sin(2 \pi f_{\text{input}} t)$ sampled at $f_s = 2000$ samples/s with f_{input} varied



- Mirror image effect about $f_{\text{input}} = ½ f_s$ gives rise to name of folding

DSP First Demonstrations

- Aliasing and folding
- Strobe demonstrations
- Disk attached to a shaft rotating at 750 rpm
  - Keep strobe light flash rate F_s the same
  - Increase rotation rate F_m (positive means counter-clockwise)
- Case I: Flash rate is equal to rotation rate
  - Vector appears to stand still
  - When else does this phenomenon occur? $F_m = l\,F_s$
  - For F_m = 750 rpm, occurs at F_s = { 375, 250, 187.5, …} rpm

Strobe Movies

- Fixes the strobe flash rate
- Increases rotation rate of shaft linearly with time
- Strobe initial keeps up with the increasing rotation rate until $F_m = \frac{1}{2} F_s$
- Then, disk appears to slow down (folding).
- Then, disk stops and appears to rotate in the other direction at an increasing rate (aliasing).
- Then, disk appears to slow down (folding) and stop