Shannon Capacity

- Defined as the maximum MI of channel
- Maximum error-free data rate a channel can support.
- Theoretical limit (not achievable)
- Channel characteristic
  - Not dependent on design techniques
- Capacity in AWGN

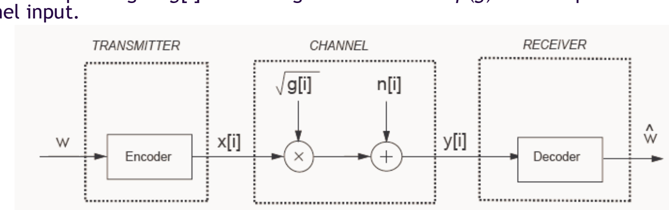
Capacity in AWGN

- Consider a discrete-time additive white Gaussian noise (AWGN) channel with channel input/output relationship
- $y[i] = x[i] + n[i]$, where $x[i]$ is the channel input at time i,
  - $y[i]$ is the corresponding channel output, and
  - $n[i]$ is a white Gaussian noise random process. Assume a channel bandwidth B and transmit power P.
- The channel SNR, the power in $x[i]$ divided by the power in $n[i]$, is constant and given by $\gamma = P/(N_0B)$, where N_0 is the power spectral density of the noise.
- The capacity of this channel is given by Shannon's well-known formula

$$C = B\log_2(1+\gamma)$$

System and Channel Model

- We assume a discrete-time channel with stationary and ergodic time-varying gain $g[i]$, $0 \le g[i]$, and AWGN $n[i]$.
- The channel power gain $g[i]$ follows a given distribution $p(g)$ and independent of the channel input.



- The channel gain $g[i]$ can change at each time i, either as an i.i.d. process or with some correlation over time.
- In a **block fading channel** $g[i]$ is constant over some blocklength T after which time $g[i]$ changes to a new independent value based on the distribution $p(g)$.
- Let P denote the average transmit signal power, $N_0/2$ denote the noise power spectral density of $n[i]$, and B denote the received signal bandwidth.
- The instantaneous received signal-to-noise ratio (SNR) is then $\gamma[i] = Pg[i]/(N_0B)$, $0 \le \gamma[i] < \infty$, and its expected value over all time is $\bar{\gamma} = P\bar{g}/(N_0B)$.
- Since $P/(N_0B)$ is a constant, the distribution of $g[i]$ determines the distribution of $\gamma[i]$ and vice versa.

Capacity of Flat-Fading Channels

- Capacity defines theoretical rate limit
  - Maximum error free rate a channel can support
- Depends on what is known about channel
- Unknown Fading:
  - Worst-case channel capacity
- **Channel Distribution Information (CDI):** The distribution of $g[i]$ is known to the transmitter and receiver.
- **Receiver Channel State Information (CSI):** The value of $g[i]$ is known at the receiver at time i, and both the transmitter and receiver know the distribution of $g[i]$
- **Transmitter and Receiver CSI:** The value of $g[i]$ is known at the transmitter and receiver at time i, and both the transmitter and receiver know the distribution of $g[i]$.

Channel State Information at Receiver

- The value of $g[i]$ is known at the receiver at time i, and both the transmitter and receiver know the distribution of $g[i]$
- There are two channel capacity definitions that are relevant to system design: Shannon capacity, also called **ergodic capacity**, and **capacity with outage**.
- Shannon capacity the rate transmitted over the channel is constant: the transmitter cannot adapt its transmission strategy relative to the CSI.
  - Poor channel states typically reduce Shannon capacity since the transmission strategy must incorporate the effect of these poor states.
- An alternate capacity definition for fading channels with receiver CSI is capacity with outage.
  - Capacity with outage is defined as the maximum rate that can be transmitted over a channel with some outage probability corresponding to the probability that the transmission cannot be decoded with negligible error probability.

Shannon (Ergodic) Capacity

- Shannon capacity is equal to Shannon capacity for an AWGN channel with SNR γ, given by B log2(1 + γ), averaged over the distribution of γ.

$$C = \int_0^{\infty} B \log_2(1+\gamma)p(\gamma)d\gamma$$

- The capacity-achieving code must be sufficiently long so that a received codeword is affected by all possible fading states. This can result in significant delay.

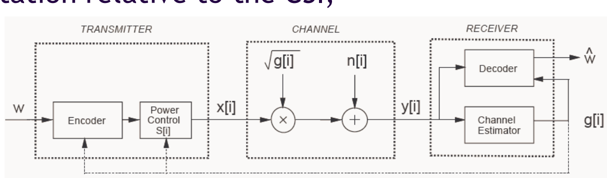
Capacity with Outage

- Capacity with outage allows bits sent over a given transmission burst to be decoded at the end of the burst with some probability that these bits will be decoded incorrectly.
- Specifically, the transmitter fixes a minimum received SNR γ_{min} and encodes for a data rate $C = B \log_2(1 + \gamma_{\min})$. The data is correctly received if the instantaneous received SNR is greater than or equal to $\gamma_{\min}$
- If the received SNR is below γ_{min} then the bits received over that transmission burst cannot be decoded correctly with probability approaching one, and the receiver declares an outage.
  - The probability of outage is thus $p_{out} = p(\gamma < \gamma_{\min})$.
- The average rate correctly received over many transmission bursts is

$$C_o = (1 - p_{out}) B \log_2(1 + \gamma_{min})$$

Fading Known at Transmitter and Receiver

- When both the transmitter and receiver have CSI, the transmitter can adapt its transmission strategy relative to this CSI
- In this case there is no notion of capacity versus outage where the transmitter sends bits that cannot be decoded, since the transmitter knows the channel and thus will not send bits unless they can be decoded correctly.
- Derive Shannon capacity assuming optimal power and rate adaptation relative to the CSI,



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Shannon Capacity

- Consider the Shannon capacity when the channel power gain $g[i]$ is known to both the transmitter and receiver at time i.

$$C = \int_0^\infty C_\gamma p(\gamma) d\gamma = \int_0^\infty B \log_2(1+\gamma) p(\gamma) d\gamma$$

- Using Jensen inequality, we can show

$$C = \int_0^\infty B \log_2(1+\gamma) p(\gamma) d\gamma \le B \log_2(1+\overline{\gamma})$$

- Let us now allow the transmit power $P(\gamma)$ to vary with γ, subject to an average power constraint $\overline{P}$:

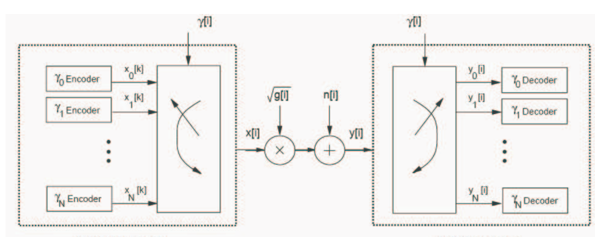
$$\int_0^\infty P(\gamma) p(\gamma) d\gamma \le \overline{P}$$

"Time Diversity" System

- Multiplexed coding and decoding to achieve the capacity given by

$$C = \max_{P(\gamma): \int P(\gamma) p(\gamma) d\gamma = \overline{P}} \int_0^\infty B \log_2\left(1 + \frac{P(\gamma)\gamma}{\overline{P}}\right) p(\gamma) d\gamma.$$

- The proof is "time diversity" system with multiplexed input and demultiplexed output as shown below



Optimal Power Allocation

- To find the optimal power allocation $P(\gamma)$, we form the Lagrangian

$$J(P(\gamma)) = \int_0^\infty B\log_2\left(1+\frac{\gamma P(\gamma)}{\overline{P}}\right)p(\gamma)d\gamma - \lambda\int_0^\infty P(\gamma)p(\gamma)d\gamma$$

- Next we differentiate the Lagrangian and set the derivative equal to zero:

$$\frac{\partial J(P(\gamma))}{\partial P(\gamma)} = \left[\left(\frac{B/\ln(2)}{1+\gamma P(\gamma)/\overline{P}}\right)\frac{\gamma}{\overline{P}} - \lambda\right]p(\gamma) = 0$$

- Solving for $P(\gamma)$ with the constraint that $P(\gamma) > 0$ yields the optimal power adaptation that maximizes the capacity as

$$\frac{P(\gamma)}{\overline{P}} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma} & \gamma \geq \gamma_0 \\ 0 & \gamma < \gamma_0 \end{cases}$$

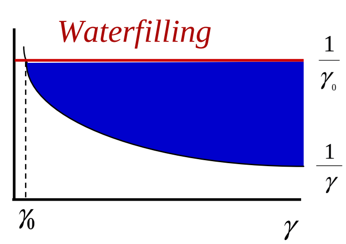
Optimal Adaptive Scheme

- Power Adaptation

$$\frac{P(\gamma)}{\overline{P}} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma} & \gamma \geq \gamma_0 \\ 0 & \gamma < \gamma_0 \end{cases}$$

- Capacity

$$\frac{C}{B} = \int_{\gamma_0}^{\infty} \log_2\left(\frac{\gamma}{\gamma_0}\right)p(\gamma)d\gamma.$$



- Note that the optimal power allocation policy only depends on the fading distribution $p(\gamma)$ through the cutoff value γ_0.
  - This cutoff value is found from the power constraint.
  - Specifically, by rearranging the power constraint and replacing the inequality with equality (since using the maximum available power will always be optimal) yields the power constraint

Optimal Power Control

- Note that the optimal power allocation policy only depends on the fading distribution $p(\gamma)$ through the cutoff value γ_0.
- This cutoff value is found from the power constraint.
- Specifically, by rearranging the power constraint and replacing the inequality with equality (since using the maximum available power will always be optimal) yields the power constraint

$$\int_0^\infty \frac{P(\gamma)}{P} p(\gamma) d\gamma = 1 \qquad \int_{\gamma_0}^\infty \left(\frac{1}{\gamma_0} - \frac{1}{\gamma}\right) p(\gamma) d\gamma = 1$$

- Note that this expression only depends on the distribution $p(\gamma)$. The value for γ_0 cannot be solved for in closed form for typical continuous pdfs $p(\gamma)$ and thus must be found numerically

Zero-outage Capacity Channel Inversion

- Fading inverted to maintain constant SNR
- Simplifies design (fixed rate)
- Channel Inversion
  - This power adaptation is given by $P(\gamma)/P = \sigma/\gamma$, where σ equals the constant received SNR that can be maintained with the transmit power constraint. σ satisfies $\sigma = 1/E[1/\gamma]$.
- Zero-outage channel capacity
- Greatly reduces capacity
  - Capacity is zero in Rayleigh fading

$$C = B\log_2[1+\sigma] = B\log_2\left[1 + \frac{1}{\mathbf{E}[1/\gamma]}\right]$$

Outage Capacity and Truncated Inversion

- Truncated inversion
  - Invert channel above cutoff fade depth
  - Constant SNR (fixed rate) above cutoff
  - Cutoff greatly increases capacity
    - Close to optimal
- Outage capacity is achieved with a truncated channel inversion policy for power adaptation that only compensates for fading above a certain cutoff fade depth γ_0:

$$\frac{P(\gamma)}{\bar{P}} = \begin{cases} \frac{\sigma}{\gamma} & \gamma \geq \gamma_0 \\ 0 & \gamma < \gamma_0 \end{cases} \qquad \mathbf{E}_{\gamma_0}[1/\gamma] \triangleq \int_{\gamma_0}^{\infty} \frac{1}{\gamma} p(\gamma) d\gamma$$

  - where γ_0 is based on the outage probability: $p_{out} = p(\gamma < \gamma_0)$. Since the channel is only used when $\gamma \geq \gamma_0$, $\sigma = 1/E_{\gamma 0}[1/\gamma]$, where

Outage Capacity with Truncation

- The outage capacity associated with a given outage probability p_{out} and corresponding cutoff γ_0 is given by

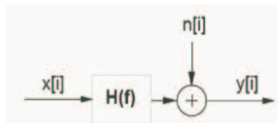
$$C(p_{out}) = B \log_2 \left(1 + \frac{1}{\mathbf{E}_{\gamma_0}[1/\gamma]}\right) p(\gamma \geq \gamma_0)$$

- We can also obtain the maximum outage capacity by maximizing outage capacity over all possible γ_0

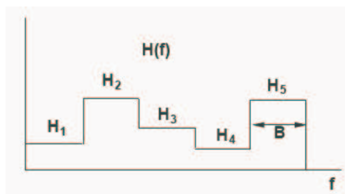
$$C = \max_{\gamma_0} B \log_2 \left(1 + \frac{1}{\mathbf{E}_{\gamma_0}[1/\gamma]}\right) p(\gamma \geq \gamma_0)$$

Capacity of Time-Invariant Frequency Selective Channels

- Consider a time-invariant channel with frequency response $H(f)$.
  - Assume a total transmit power constraint P.



- Assume $H(f)$ is block-fading
  - frequency is divided into subchannels of bandwidth B, where $H(f) = H_j$ is constant over each block.



Time-Invariant Frequency-Selective Fading Channel

- The frequency-selective fading channel thus consists of a set of AWGN channels in parallel with SNR $|H_j|2P_j/(N_0B)$ on the jth channel, where P_j is the power allocated to the jth channel in this parallel set, subject to the power constraint $jP_j \le P$.

$$C = \max_{P_j:\sum_j P_j \le P} \sum B \log_2\left(1 + \frac{|H_j|^2 P_j}{N_0 B}\right)$$

$$\frac{P_j}{P} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma_j} & \gamma_j \ge \gamma_0 \\ 0 & \gamma_j < \gamma_0 \end{cases}$$

for some cutoff value γ_0, where $\gamma_j = |H_j|2P/(N_0B)$ is the SNR associated with the jth channel assuming it is allocated the entire power budget.

- γ_0 must satisfy and the capacity becomes

$$\sum_j \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_j}\right) = 1.$$

$$C = \sum_{j:\gamma_j \ge \gamma_0} B \log_2(\gamma_j/\gamma_0).$$

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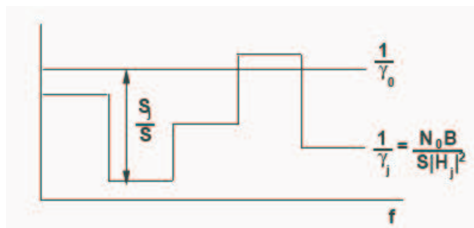
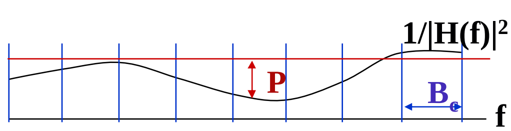
Waterfilling Block FSF

- Watefilling in frequency selective channels



Frequency Selective Fading Channels

- For TI channels, capacity achieved by water-filling in frequency
- Capacity of time-varying channel unknown
- Approximate by dividing into subbands
  - Each subband has width B_c (like MCM).
  - Independent fading in each subband
  - Capacity is the sum of subband capacities



$$C = \max_{\{\overline{P}_j\}:\sum_j \overline{P}_j \le \overline{P}} \sum_j C_j(\overline{P}_j)$$