Distributed Formation Control via Mixed Barycentric Coordinate and Distance-Based Approach

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Abstract—We present a distributed control strategy for a team of agents to autonomously achieve a desired planar formation. Our control strategy is based on combining the barycentric coordinate-based (BCB) and the distance-based (DB) approach. In the BCB approach, the almost global convergence of the agents to the desired formation shape is guaranteed, however, the formation scale cannot be controlled. In the DB method, the scale of the achieved formation is controlled, however, the convergence is local and in general stable undesired equilibria exist. By combining these methods via imposing a timescale separation between their respective dynamics, our proposed control strategy retains the advantages of each approach and avoids their shortcomings. We analyze the stability properties of the proposed control and prove that the desired formation is an almost globally stable equilibrium. We provide simulations to typify the theoretical results and compare our method with a leader-follower BCB (LF-BCB) approach that can be used to control the formation scale in the BCB strategy. In particular, we demonstrate that unlike the LF-BCB approach, our method is far more robust to measurement inaccuracies.

Index Terms—Multi-agent systems, distributed formation control, agent-based systems.

Supplementary Material

Video of paper summary and simulations is available at https://youtu.be/cYQ6ALUF83s. Simulations code can be download from https://goo.gl/QH5qhw.

I. INTRODUCTION

A formation of agents is a fundamental building block in applications such as the search and rescue missions [1], environmental mapping/monitoring [2], and cooperative object manipulation [3]. There exists a large body of work on formation control of autonomous agents [4]. While many methods rely on a centralized motion planning scheme or a global positioning/communication paradigm [5–8], fully distributed formation control strategies, such as the distance-based (DB) [9–11], bearing-based (BB) [12–15], and barycentric coordinate-based (BCB) [16–25] approach only require local relative measurements. Compared to the centralized methods, these distributed strategies have better scalability, naturally parallelized computation, and resiliency to global positioning signal jamming or loss.

In this work, we combine the BCB and the DB approach to derive a novel control strategy that has the advantages of both methods and avoids their shortcomings. A characteristic of the BCB approach is that the convergence of agents to the desired formation shape is guaranteed for almost all initial conditions. However, in the BCB (and the BB) approach, the scale of the formation cannot be controlled. In the existing DB approach, on the other hand, the scale of the formation is controllable, however, the convergence is not global and agents may attain an undesired shape. To control the scale of the formation in the BCB approach, the control can be modified to take a leader-follower (LF) form [16], in which by controlling the distance between a subset of agents (leaders) the scale of the remaining agents (followers) is controlled. However, as we will demonstrate by a simulation example, the LF-BCB approach can be very sensitive to inaccuracies and noise in the position measurements. Compared to the aforementioned methods, in our proposed approach the formation scale can be controlled, convergence is almost global, and the control is more robust to disturbances and inaccuracies in the measurements.

The main contribution of this work is the rigorous stability analysis of the proposed control, based on the mathematical machinery of slow-fast systems with timescale separation, which shows the almost global convergence of agents to the desired formation. The fast and slow dynamics in our proposed control correspond to the BCB and DB controls, respectively. To this day, no DB control strategy (for arbitrary numbers of agents and all feasible sensing graphs) with global or almost global convergence property is known in literature due to the existence of undesired stable equilibria discussed in [26–28]. Hence, this paper can be considered as the first result toward showing the existence of augmented DB controls with almost global convergence\(^1\).

The paper is organized as follows. The notation and assumptions are introduced in Section II. In Section III, the BCB, DB, and LF-BCB control strategies are introduced and their properties are discussed. The proposed control strategy is presented in Sections IV, followed by the simulations in Section V. Additional remarks and future work are discussed in Section VI.

\(^1\)The global convergence in [29] is attained via inter-agent communication, which is not an assumption of this work or other DB literature.
II. Notation and Assumptions

We consider a team of $n \in \mathbb{N}$ agents with the inter-agent sensing topology described by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \mathbb{N}_n := \{1, 2, \ldots, n\}$ is the set of vertices, and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges. Each vertex of the graph represents an agent. An edge from vertex $i \in \mathcal{V}$ to $j \in \mathcal{V}$ indicates that agents $i$ and $j$ can measure the relative position of each other in their local coordinate frames. In such a case, agents $i$ and $j$ are called neighbors. The set of neighbors of agent $i$ is denoted by $\mathcal{N}_i := \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}$.

Throughout this paper, we assume that the desired formation and the sensing topology are such that achieving the formation is physically feasible. In particular, we assume that the sensing topology is undirected and universally rigid. This assumption is both necessary and sufficient [17], [30] for guaranteeing the existence of control gains that lead to the desired formation.

III. Preliminaries

In this section, we review the BCB, DB, and LF-BCB formation control strategies for agents with the single-integrator dynamics. The motion of agents with single-integrator dynamics can be expressed as

$$\dot{q}_i = u_i,$$

where $q_i := [x_i, y_i]^\top \in \mathbb{R}^2$ represents the coordinates of agent $i \in \mathbb{N}_n$ in a global coordinate frame (unknown to the agent), and $u_i \in \mathbb{R}^2$ is the control law that is specified for agents by each strategy to achieve a desired formation. The control strategies discussed in this section are a cornerstone for our proposed control strategy that is discussed in the subsequent section.

A. The Barycentric Coordinate-Based Control Strategy

The BCB control strategy introduced by Lin et al. [16] is defined as

$$u_i := \sum_{j \in \mathcal{N}_i} A_{ij} (q_j - q_i),$$

where $q_j - q_i$ represent the relative position of agent $j$ with respect to agent $i$, and $A_{ij} \in \mathbb{R}^{2 \times 2}$ are constant control gain matrices that are designed and provided to agents before the mission and have the form

$$A_{ij} := \begin{bmatrix} a_{ij} & b_{ij} \\ -b_{ij} & a_{ij} \end{bmatrix}, \quad a_{ij}, b_{ij} \in \mathbb{R}. \tag{3}$$

Note that the diagonal elements of $A_{ij}$ are identical, and off-diagonal elements only differ in sign. From the commutativity property of $A_{ij}$ matrices (which holds due to their special structure) it follows that the closed-loop dynamics is invariant to expressing the coordinates in the global or local frames\(^2\). The geometric intuition behind the algebraic for-

\(^2\)Replacing global coordinates $q_i^{\text{global}}$ with local coordinates $q_i^{\text{local}} = Rq_i^{\text{global}} + T$ in (1) and (2) with $R, T$ representing the relative rotation and translation between the local and global frames does not affect the dynamics since $R, T$ are canceled.

mulation of control strategy (2) is explained in the following example.

Example 1. Consider three agents in Fig. 1, and assume that agents 2 and 3 are neighbors of agent 1. That is, agent 1 can measure the position of agents 2 and 3 in its local coordinate frame. Denote by $q_2 = [2, 3]^\top$ and $q_3 = [3, 1]^\top$ the positions of the neighbors in agent 1’s local coordinate frame, which itself is located at $q_1 = [0, 0]^\top$. Assume that control gain matrices

$$A_{12} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}, \tag{4}$$

are provided to agent 1 before the mission. From (2), the control vector for agent 1 in the current instance of time is computed as

$$u_1 = A_{12} (q_2 - q_1) + A_{13} (q_3 - q_1) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \tag{5}$$

which is plotted in the figure. From the single-integrator dynamics (1) it follows that agent 1 moves along the vector $u_1$ with a speed equal to its magnitude. Matrices $A_{12}, A_{13}$ can be interpreted as scaled rotation matrices that rotate and scale vectors that connect agent 1 to its neighbors. One can see that these actions are independent of the local coordinate frame’s position and orientation in the global frame. Hence, $q_2$ and $q_3$ can be represented in either global or local coordinate frames.

By substituting (2) in (1), the closed-loop dynamics of the agents can be collectively expressed as

$$\dot{q} = Aq,$$  

where $q := [q_1^\top, q_2^\top, \ldots, q_n^\top]^\top \in \mathbb{R}^{2n}$ denote the aggregate position vector, and $A \in \mathbb{R}^{2n \times 2n}$ is the aggregate gain matrix given by

$$A = \begin{bmatrix} -\sum_{j \neq 1} A_{1j} & A_{12} & \cdots & A_{1n} \\ A_{21} & -\sum_{j \neq 2} A_{2j} & \cdots & A_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nj} & -\sum_{j \neq n} A_{nj} \end{bmatrix}, \tag{7}$$

in which for $j \notin \mathcal{N}_i$ (i.e., when agents are not neighbors) the $A_{ij}$ blocks are defined as zeros. Note that the $2 \times 2$ diagonal blocks of $A$ are the negative sum of the rest of the blocks on the same row. Hence, $A$ has a block Laplacian structure,
Theorem 1. Consider agents with dynamics (1) and control (2). Assume \( A_{ij} \) are such that in (7)

(i) vectors \( \mathbf{1}, \bar{\mathbf{1}} \) form a basis for \( \ker(A) \).

(ii) all nonzero eigenvalues of \( A \) have negative real parts, then agents almost globally converge to the desired formation shape (up to a rotation and a translation).

A formal proof for the case where \( A \) is symmetric is given in the Appendix. By noting that the closed-loop system (6) is a linear system, it is straightforward to see that under the conditions of Theorem 1 each trajectory converges to a point in \( \ker(A) \), which consists of all rotations, translations, and scale factors of the desired formation coordinates. To give a geometric intuition, consider the unit square desired formation for 4 agents with a complete sensing graph. The trajectories of the agents under the BCB control (2) starting from two randomly generated initial conditions are shown in Fig. 2. As can be seen, the desired formation shape is achieved in both cases, however, the scale of the formation differs for each case.

Remark 1. The null vectors \( \mathbf{1}, \bar{\mathbf{1}} \) correspond to the case where all agents coincide. It can be shown that the set of initial conditions that converge to this coinciding equilibrium

is measure zero. Hence, the convergence results in Theorem 1 is “almost” global. In practice, the trajectory of agents cannot remain on a measure zero set (due to noise, disturbances, etc.), and such cases are not of practical concern.

Remark 2. For the existence of \( A_{ij} \) that satisfy the conditions of Theorem 1, it is both necessary and sufficient that the sensing graph among agents is 2-rooted. In addition, for matrix \( A \) in (7) to be symmetric the sensing graph must be undirected and universally rigid. These conditions have been derived in [18, Thm. 3.2], and we assume they hold throughout this paper\(^5\).

We emphasize that the strength of the BCB approach is the almost global convergence of the agent to the desired formation shape, while the downside is that the formation scale cannot be controlled and depends on the initial positions of the agents. Lastly, gain matrices for which \( A \) is symmetric and conditions of Theorem 1 are satisfied can be computed by solving a semidefinite program (SDP) that is presented in our previous work [31], [32].

B. The Distance-Based Control Strategy

An alternative approach to achieve the desired formation is the DB strategy defined by

\[
\begin{align*}
  u_i := \sum_{j \in \mathcal{N}_i} (d_{ij} - d_{ij}^*) (q_j - q_i),
\end{align*}
\]

where \( d_{ij} := \|q_j - q_i\| \) denotes the distance between agents \( i \) and \( j \), and \( d_{ij}^* \in \mathbb{R} \) denotes its corresponding value in the desired formation. Intuitively, the role of each \( (d_{ij} - d_{ij}^*) (q_j - q_i) \) term in (10) is to pull agent \( i \) toward its neighbor \( j \) when the distance \( d_{ij} \) between them is larger than the desired distance \( d_{ij}^* \), and vice versa.

Proposition 1. Consider agents with dynamics (1) and control (10). If the desired formation is infinitesimally rigid [33], then the desired formation is a locally stable equilibrium (up to a rotation and a translation).

The proof of Proposition 1 is given in [10, Thm. 13]. Note that unlike the BCB approach, in the DB control strategy convergence to the desired formation is only guaranteed locally since in general undesired stable equilibria are present.

\(^3\)The kernel or null space of a matrix \( A \in \mathbb{R}^{n \times n} \) is defined as \( \ker(A) := \{ v \in \mathbb{R}^n | A v = 0 \} \).

\(^4\)That is, if \( q_i^* = [x_i, y_i]^{\top} \), then \( q_i^* := [-y_i, x_i]^{\top} \).

\(^5\)These results are for general formation shapes. For some special shapes, the gains may exist even if the rigidity conditions are not satisfied.
The control for the remaining follower agents $u$ dynamics is the same as the BCB strategy (2). To avoid the loss of generality, that agents 1 and 2 are the leaders. The formation with the desired scale. Let us assume, without loss of generality, that agents 1 and 2 are the leaders. The formation scale in the BCB approach. The main idea is achieved only in the left figure, and the agents converge to an undesired equilibrium from their initial positions in the right figure.

C. The Leader-Follower BCB Formulation

The LF-BCB formulation has been proposed in [16] to fix the formation scale in the BCB approach. The main idea consists of fixing the distance between two leader agents and hence forcing the remaining agents (followers) to converge to the formation with the desired scale. Let us assume, without loss of generality, that agents 1 and 2 are the leaders. The corresponding LF-BCB control is given by

$$
\begin{align*}
    u_1 &:= (d_{12} - d_{12}^*) (q_1 - q_2), \\
    u_2 &:= (d_{12} - d_{12}^*) (q_2 - q_1), \\
    u_i &:= \sum_{j \in N_i} A_{ij} (q_j - q_i), \quad i \geq 3.
\end{align*}
$$

(11)

Similar to the DB strategy, the role of $u_1$ (or $u_2$) is to bring the distance $d_{12}$ between the leaders to its desired value $d_{12}^*$. The control for the remaining follower agents $u_i$, $i \geq 3$ is the same as the BCB approach (2).

Substituting (11) in (1) gives the aggregate closed-loop dynamics

$$
\dot{q} = A q + \begin{bmatrix} g(q_1, q_2) \\ 0 \end{bmatrix},
$$

(12)

where $g : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^4$ maps the coordinates of the leaders to control vectors $u_1$, $u_2$ according to (11), and the first 4 rows of $A$ associated to the leader agents are zeros. Note that in (12), matrix $A$ is not symmetric (in contrast to our proposed approach).

Proposition 2. Consider agents with dynamics (1) and control (11). If $A$ in (12) satisfies the conditions of Theorem 1, then, agents almost globally converge to the desired formation (up to a rotation and a translation).

Proof of Proposition 2 is given in [16, Thm. 4.4] and hence is omitted here. As we will illustrate by an example in Section V, additive disturbances in the leader positions, which model measurement inaccuracies that exist in practice, can propagate and get amplified by the follower agents, hence, severely affecting the formation. Therefore, ideally, all agents should participate in controlling the distance to their neighbors and the LF-BCB approach should be avoided.

IV. THE PROPOSED CONTROL STRATEGY

We now present the main results of this paper. To achieve the desired formation shape with the desired scale, we combine the BCB (2) and the DB (10) control strategies according to

$$
\dot{u}_i = \sum_{j \in N_i} A_{ij} (q_j - q_i) + \epsilon f(d_{ij} - d_{ij}^*) (q_j - q_i),
$$

(13)

where $\epsilon > 0$ is a real scalar, and the bounded smooth map $f : \mathbb{R} \rightarrow \mathbb{R}$ is chosen such that $x f(x) > 0$ for $x \neq 0$, $f(0) = 0$, and $f'(0) > 0$, as illustrated in Fig. 4. Possible choices for $f$ are $f : x \mapsto \arctan(x)$ or $f : x \mapsto \tanh(x)$. The role of $f$ in (13) is to bound the magnitude of the DB control strategy.

The intuition behind the control strategy (13) is as follows. By choosing $\epsilon$ small, the BCB control drives the agents to the desired formation shape (fast dynamics). Once the agents are in a small neighborhood of the desired shape, the DB control contracts/expands the formation to achieve the desired scale (slow dynamics).

Theorem 2. Consider agents with dynamics (1) and control (13). Assume $A_{ij}$ matrices are such $A$ in (7) is symmetric and satisfies the conditions of Theorem 1. Then, there exists $\epsilon^* > 0$ such that for all $0 < \epsilon < \epsilon^*$ the desired formation is an almost globally exponentially stable equilibrium.

Proof of Theorem 2 is presented in the Appendix. The term “equilibrium” should be understood as the set of equilibria, which consists of all rotations and translations of the desired formation. Furthermore, the almost global exponential stability implies that almost every trajectory converges to a point in this set with a rate that depends on the initial condition. As a combination of the BCB and DB control approach, the control strategy (13) has the advantages of both methods. Namely, convergence to the desired formation is almost global and the achieved formation has the desired scale.

Remark 3. Theorem 2 ensures the existence of an $\epsilon$, however, the value of $\epsilon^*$ is not specified. From the proof of Theorem 2, one can see that $\epsilon$ must be chosen small enough such that matrix $G$ in (29) is negative definite. By considering the worst case scenario and using the Gershgorin circle theorem [34, Thm. 6.1.1], an upper bound for the largest eigenvalue of $F$ in (21) can be derived as $\mu_{\text{max}} := 2(n - 1) f_{\text{max}}$, where $n$ is the number of agents and $f_{\text{max}} > 0$ is an upper bound for $|f(\cdot)|$ in (13). If $\lambda_{\text{max}} < 0$ denote the largest (nonzero) eigenvalue of $A$, then by choosing

$$
\epsilon < |\lambda_{\text{max}}| / \mu_{\text{max}}
$$

(14)

the negative definiteness of $G$ is guaranteed from the Weyl Theorem [34, Thm. 4.3.1].

Lastly, it can be shown that under (13) the closed-loop dynamics is globally stable (i.e., trajectories are bounded) for all values of $\epsilon$ and the only consequence of choosing a large $\epsilon$ that does not satisfy (14) is the possibility of convergence to an undesired equilibrium associated with the DB approach.
V. SIMULATIONS

We present two simulations to validate the proposed approach and compare the sensitivity of the discussed methods to disturbances. Links to the Matlab simulation code and videos are provided in the Supplementary Material section.

The desired formation considered in the simulations is a rectangular formation with 8 agents (see Fig. 5(b)), where the desired distance among all neighboring agents is defined as 1 unit of measurements. The gain matrices that satisfy the conditions of Theorem 1 for the corresponding desired formation are computed using the SDP approach presented in our previous work [31], [32]. The algorithm discussed in [16] is used to compute the gain matrices that satisfy the conditions of Proposition 2 for the LF-BCB approach. To have a fair comparison, the aggregate gain matrices computed from both approaches are designed to have the same eigenvalues. In particular, the eigenvalues of matrix $A$ for both methods are $\{-0.72, -2.68, -5.36, -5.36, -8.04, -10\}$, each with multiplicity two. The value of $\epsilon$ in the proposed control strategy (13) is set to 0.5.

Figs. 5(a) and (b) show the trajectories of agents starting from the same randomly chosen initial positions under the LF-BCB control strategy (11) and the proposed strategy (13), respectively. The corresponding distance errors defined as $d_{ij} - d_{ij}^*$ between agents and their neighbors are plotted versus time in Figs. 6(a), (b). As can be seen from the figures, the agents achieve the desired formation with the desired scale under both control strategies. However, the traversed trajectories are much shorter under our proposed control strategy.

To test the sensitivity of control strategies to noise and measurement inaccuracies that often exist in a practical implementation, periodic disturbances of $0.2 \begin{bmatrix} \sin(t), & \cos(t) \end{bmatrix}^T$ and $0.1 \begin{bmatrix} \cos(t), & \sin(t) \end{bmatrix}^T$ are added to the position vectors $q_1$ and $q_2$, respectively. The effect of these additive disturbances on the formation shape are shown in Figs. 5(c), (d) for the LF-BCB and the proposed control strategy, respectively. The corresponding inter-agent disturbance errors are plotted in Figs. 6(c), (d). As can be seen from the figures, in the LF-BCB control approach the additive disturbance is amplified by the follower agents. In particular, the further a follower agent is from the leaders, the more severe is the effect of the disturbance on the agent’s trajectory. Under the proposed control, on the other hand, the disturbance is attenuated and agents that are further from the disturbed agents are less affected.

VI. CONCLUDING REMARKS AND FUTURE WORK

By combining the BCB and DB control strategies using the slow-fast dynamics, we presented a novel distributed formation control that has the advantages of both methods and avoids their shortcomings. In particular, similar to the BCB and unlike the DB approach, the agents almost globally

We point out that simulation with $\epsilon < 0.5$ did not lead to a noticeably different trajectory. Although the convergence to the desired scale was slower, the length of the traversed trajectories remained roughly the same.

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6 All numbers are rounded up to two decimal digits.
coverage to the desired formation. On the other hand, similar to the DB and unlike the BCB approach, the formation scale can be controlled. Moreover, under the proposed strategy the formation is far more robust to disturbances that affect the LF-BCB approach.

As discussed in our previous work [23], [31], [32], the BCB control with gains designed from the SDP method enjoys several robustness properties. Specifically, any positive scaling or rotation (up to ±90°) of the control vector, saturation of the control magnitude, and switching among the feasible sensing topologies do not affect the convergence of the agents to the desired formation. Furthermore, a fully distributed collision avoidance algorithm emerges naturally from these robustness properties. Since the fast dynamics associated with the control proposed in this work is based on the BCB approach, one can expect the same properties to follow. Detailed analysis of these properties will be a topic of future work. We further point out that although the focus of this paper was on agents with the single-integrator dynamics, one can leverage [31], [32] to extend the results to higher-order dynamics.

The results of this work were established for planar formations and undirected sensing graphs. Additional future work includes extending the results to 3D formations and directed graphs. The observation that disturbances and measurement inaccuracies in leader positions can strongly affect the shape of the formation in the LF-BCB approach was demonstrated using a simulation example. Quantifying the effect of disturbances in the LF-BCB approach analytically is another topic of research.

APPENDIX

We first prove Theorem 1 for symmetric $A$ since a similar procedure is used subsequently to prove Theorem 2.

Proof (Theorem 1). Let $N := [q^*, \tilde{q}^*, 1, 1] \in \mathbb{R}^{2n \times 4}$ denote the bases for $\ker(A)$. Further, let $U \Sigma V^T = N$ be the (full) singular value decomposition (SVD) of $N$, where

$$U = [\tilde{Q}, Q] \in \mathbb{R}^{2n \times 2n},$$

with columns of $\tilde{Q} \in \mathbb{R}^{2n \times 4}$ denoting orthonormal vectors that span $\ker(A)$, and columns of $Q \in \mathbb{R}^{2n \times (2n-4)}$ denoting orthonormal vectors that are the orthogonal complement of $\tilde{Q}$. Consider the change of coordinates defined by $p := U^T q$, in which the closed-loop dynamics $\tilde{q} = A q$ is

$$\dot{p} = U^T A U p = \begin{bmatrix} Q^T \bar{Q} & Q^T A Q \end{bmatrix} \begin{bmatrix} Q \bar{Q} & Q^T A Q \end{bmatrix} p.$$  

(16)

Let $p := [p_1, p_2]^T$, where $p_1 \in \mathbb{R}^4$ represents the first 4 elements of $p$, and $p_2 \in \mathbb{R}^{2n-4}$ is the vector of remaining elements. Since $\tilde{Q}$ is a basis for $\ker(A)$ and $A$ is symmetric, we have $\tilde{Q}^T A = A \tilde{Q} = 0$, and from (16)

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & Q^T A Q \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}.$$  

(17)

The eigenvalues of $Q^T A Q \in \mathbb{R}^{(2n-4) \times (2n-4)}$ are the same as the eigenvalues of $A$ that are not associated to the kernel vectors. From condition (ii) in the theorem, these eigenvalues have negative real parts, and therefore $p_2 \to 0$ as $t \to \infty$, while $p_1$ remains constant. Hence,

$$\lim_{t \to \infty} q = \lim_{t \to \infty} U p = [\tilde{Q} \ Q] \begin{bmatrix} p_1 \\ 0 \end{bmatrix} = \tilde{Q} p_1,$$  

(18)

which shows each trajectory converges to a point in $\ker(A)$.

The following Lemma is used in the proof of Theorem 2.

Lemma 1. [35, Thm. 11.4] Consider the singularly perturbed system

$$\begin{bmatrix} \dot{x} \\ \dot{\epsilon} \end{bmatrix} = \begin{bmatrix} f(t, x, \epsilon) \\ g(t, x, \epsilon) \end{bmatrix},$$  

(19)

where $x \in B_r \subset \mathbb{R}^n$, with $B_r$ representing the ball of radius $r$ about the origin, and $z \in \mathbb{R}^m$. Assume that for all $(t, x, \epsilon) \in [0, \infty) \times B_r \times [0, \epsilon_0]$ it holds that

(i) $f(t, 0, 0, \epsilon) = 0$ and $g(t, 0, 0, \epsilon) = 0$.

(ii) The equation $g(t, x, 0) = 0$ has an isolated root $z = h(t, x)$ such that $h(t, 0) = 0$.

(iii) The functions $f$, $g$, $h$, and their partial derivatives up to the second order are bounded for $z = h(t, x) \in B_r \subset \mathbb{R}^m$.

(iv) The origin of the reduced system $\dot{x} = f(t, x, h(t, x), 0)$ is exponentially stable.

(v) The origin of the boundary-layer system $\frac{dy}{dt} = g(t, x, y + h(t, x), 0)$, where $\tau = \epsilon t$, is exponentially stable, uniformly in $(t, x)$.

Then, there exists $\epsilon^* > 0$ such that for all $\epsilon < \epsilon^*$, then origin of (19) is exponentially stable.

Remark 4. In Lemma 1, if $B_r = \mathbb{R}^n$, $B_\rho = \mathbb{R}^m$, and the reduced and the boundary-layer systems are globally stable, from the proof presented in [35, Thm. 11.4] one can conclude that the stability results hold globally, i.e., all trajectories converge to the origin. This observation is used in the following analysis.

Based on the procedure discussed in the proof of Theorem 1, we now present the proof of Theorem 2.

Proof (Theorem 2). The closed-loop dynamics of single-integrator agents under control law (13) is given by

$$\dot{q} = A q + \epsilon F(q) q$$  

(20)

where $F(q) : \mathbb{R}^{2n} \to \mathbb{R}^{2n \times 2n}$ is defined as

$$F(q) = \begin{bmatrix} -\sum_{j \neq 1} F_{1j} \\ F_{21} & -\sum_{j \neq 2} F_{2j} & \cdots & F_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ F_{n1} & F_{n2} & \cdots & -\sum_{j \neq n} F_{nj} \end{bmatrix},$$  

(21)

with $2 \times 2$ off-diagonal block elements

$$F_{ij} := \begin{cases} f(d_{ij} - d_{ij}^*) I & \text{if } j \in N_i \\ 0 & \text{otherwise} \end{cases}$$  

(22)
where $I \in \mathbb{R}^{2 \times 2}$ is the identity matrix.

Consider the change of coordinates $p := U^T q$ defined in the proof of Theorem 1, where $U = [\bar{Q}, \bar{Q}]$ with columns of $\bar{Q} \in \mathbb{R}^{2n \times 2}$ denoting orthonormal vectors that span ker($A$). In particular, $\bar{Q}$ can be chosen as $\bar{Q} = [\bar{Q}_1, \bar{Q}_2]$ with

$$\bar{Q}_1 := \left[ \frac{q_1}{\|q_1\|}, \frac{q_2}{\|q_2\|} \right] \in \mathbb{R}^{2n \times 2}, \quad \bar{Q}_2 := [v_1, v_2] \in \mathbb{R}^{2n \times 2},$$

(23)

where $v_1, v_2 \in \mathbb{R}^{2n}$ are orthonormal vectors that span $1, \bar{1}$ and are orthogonal to $Q_1$. In the new coordinates, the closed-loop dynamics (20) is given by

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & Q^T A Q \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \epsilon \begin{bmatrix} Q^T F & Q^T F \\ Q^T F & Q^T F \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix},$$

(24)

where $F(q)$ is denoted by $F$ for brevity. Due to the block Laplacian structure of $F$, which implies $F \bar{1} = F \bar{1} = 0$, by using the bases vectors defined in (23) it follows that

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \epsilon \begin{bmatrix} \lambda I & 0 \\ 0 & H^T \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \epsilon \begin{bmatrix} Q^T F & Q^T F \\ Q^T F & Q^T F \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix},$$

(25)

where $I \in \mathbb{R}^{2 \times 2}$ is the identity matrix, and $\lambda(p) : \mathbb{R}^{2n} \rightarrow \mathbb{R}, H(p) : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ are defined as

$$\lambda := \frac{1}{\|q^*\|^2} q^* T F q^* = \frac{1}{\|q^*\|^2} \bar{q}^* T F \bar{q}^*,$$

(26)

$$H := Q^T F \bar{Q}_1,$$

(27)

where for brevity we omitted the dependency of $\lambda$ and $H$ on $\bar{p}_2$. Let $p_1$ denote the components of $\bar{p}_1 \in \mathbb{R}^4$ by $p_1 = [p_{11}, p_{12}]^T$, $p_{11}, p_{12} \in \mathbb{R}^2$, where $p_{11}$ is the component of state along the desired shape, and $p_{12}$ is the translational component. From substituting (25) in (24) we get

$$\begin{bmatrix} \dot{p}_{11} \\ \dot{p}_{12} \end{bmatrix} = \epsilon \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix} + \epsilon \begin{bmatrix} H^T \\ 0 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix} + \epsilon \begin{bmatrix} Q^T F & Q^T F \\ Q^T F & Q^T F \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix},$$

(28)

where $G$ is defined as

$$G := Q^T A Q + \epsilon Q^T F Q \in \mathbb{R}^{(2n-4) \times (2n-4)}.$$  

(29)

Since $Q^T A Q$ is negative definite by assumption, for $\epsilon$ small enough from the boundedness of $\|F\|$ (which holds due to the boundedness of its elements) $G$ remains negative definite [34, Thm. 6.3.2]. Furthermore, from (28) we have $\dot{p}_{12} = 0$, which shows that $p_{12}$ remains constant. Hence, by disregarding $p_{12}$ in (28) the remaining dynamics can be written as

$$\dot{p}_{11} = \epsilon \lambda p_{11} + \epsilon H^T p_{2},$$

(30)

$$\dot{p}_{2} = \epsilon H p_{11} + G p_{2}.$$

By representing (30) in the new time scale $\tau = \epsilon t$, which implies $\frac{d}{dt} = \epsilon \frac{d}{d\tau}$, we get

$$\epsilon \frac{d p_{11}}{d\tau} = \epsilon \lambda p_{11} + \epsilon H^T p_{2},$$

(31)

$$\epsilon \frac{d p_{2}}{d\tau} = \epsilon H p_{11} + G p_{2}.$$

Canceling $\epsilon$ from both sides of the first equation yields

$$\dot{p}_{11} = \lambda p_{11} + H^T p_{2},$$

$$\epsilon \dot{p}_{2} = \epsilon H p_{11} + G p_{2},$$

(32)

where for brevity $\dot{p}_{11}, \dot{p}_{2}$ denote the time derivatives in the new time $\tau$.

System (32) is in the singularly perturbed format of (19) in Lemma 1. Before applying the Lemma, we make the following remarks: 1) Our analysis is based on the desired formation equilibrium instead of the origin used in the Lemma9. Furthermore, this equilibrium represents all rotations of the desired formation on the plane. 2) The origin $q = 0$ (i.e., coinciding agents) is an equilibrium of (20). By linearizing (20) about this equilibrium, it can be shown that the corresponding Jacobian matrix, and hence the equilibrium, is unstable. It is straightforward to formulate a modification of Lemma 1 and Remark 4 to include the above remarks. Having this point in mind, we apply Lemma 1 directly on the system (32).

At the desired formation $q = q^*$, from $p = U^T q$ we have $p_{11} = p_{11}^* := \bar{Q}_1^T q^*$ and $p_{2} = \bar{Q}_2^T q^* = 0$. Also, from (22) it follows that at the desired formation $F = 0$. Taking these into account, we proceed by showing that (32) satisfies the conditions of Lemma 1:

(i) Since at the desired formation $F = 0$, from (26) and (27) $\lambda$ and $H$ are zero. Thus, $p_{11} = p_{11}^*, p_{2} = 0$ is an equilibrium of (32).

(ii) Setting $\epsilon = 0$ in (32) yields the quasi steady state equation $G p_{2} = 0$. Since $G$ is negative definite by assumption, $p_{2} = 0$ is the unique isolated root of the quasi steady state equation.

(iii) By direct calculation, it is straightforward to show that the right hand sides of (32) have bounded second order derivatives for all $p_{11} \in \mathbb{R}^2, p_{2} \in \mathbb{R}^{2n-4}$.

(iv) By substituting $p_{2} = 0$ in (32) the reduced order dynamics is derived as

$$\dot{p}_{11} = \lambda p_{11}.$$  

(33)

Note that $p_{2} = 0$ implies that the agents are the desired formation shape, i.e., $p_{11} = c p_{11}^*$, where $c \geq 0$ is the scale of the formation. Since

$$\begin{cases} d_{ij} - d_{ij}^* < 0 & \text{if } c < 1 \\ d_{ij} - d_{ij}^* = 0 & \text{if } c = 1 \\ d_{ij} - d_{ij}^* > 0 & \text{if } c > 1 \end{cases}$$

(34)

from the definitions of $F$ and $\lambda$ in (21) and (26), and since $c = \|p_{11}\|/\|p_{11}^*\|$, it follows that

$$\begin{cases} \lambda > 0 & \text{if } \|p_{11}\| < \|p_{11}^*\| \\ \lambda = 0 & \text{if } \|p_{11}\| = \|p_{11}^*\| \\ \lambda < 0 & \text{if } \|p_{11}\| > \|p_{11}^*\| \end{cases}$$

(35)

as illustrated in Fig. 7. Equation (35) implies that the desired formation $p_{11} = p_{11}^*$ (corresponding to $c = 1$)

9This does not change the analysis since one can use the transformation $\tilde{q} := q - q^*$ to shift the equilibrium to the origin.
Consequently, from Lemma 1 it follows that for $\epsilon$ small enough the desired formation is an exponentially stable equilibrium of system (20). From Remark 4, and due to the existence of an undesired but unstable equilibrium at the origin, the (nonuniform) exponential stability holds almost globally.

References


