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Stabilization of Stiff Formations with a Mix of Direction and Distance Constraints

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***Abstract*— Heterogenous formation shape control with a mix of inter-agent distance and bearing constraints involves the design of distributed control laws that ensure the formation moves such that these inter-agent constraints are achieved and maintained. This paper looks at the design of a distributed control scheme to solve the mixed constraint formation control problem with an arbitrary number of agents. A gradient control law is proposed based on the mathematical notion of a stiff formation structure and a corresponding stiff constraint matrix (which has origins in graph theory). This work provides an interesting and novel contrast to much of the existing work in formation control where distance-only or bearing-only constraints are typically maintained. A stability analysis is sketched and a number of other technical results are given.**

I. Introduction

The general distributed formation control problem involves a group of agents which are tasked with maintaining a prescribed geometrical formation described in terms of relative distance and/or angular constraints.

Typically, when formulating any distributed formation control problem one must specify what geometrical (inter-agent) constraints are to be controlled by each individual agent and what sensing (or measurements) are available to each agent. There is a wide literature that considers distance-only constraints [1]–[7] and relative position sensing. In [8] bearing-only constraints are considered with relative position measurements. There are also a number of papers [9]–[12] that consider angular-type constraints with bearing-only measurements. In [13] distance-only constraints are considered along with distance-only sensing (to a larger set of neighbours than those to which constraints are considered). In [14] a three-agent formation control problem is considered with a mix of distance and bearing constraints and a mix of distance-only and bearing-only sensing.

In this work, the shape of a formation is controlled by actively, and in a distributed fashion, controlling a *mixed set* of inter-agent bearing and distance constraints using relative position measurements. Specifically, our contribution is the design and analysis of a novel distributed controller for an arbitrary number of agents with a heterogenous set of constraints. Our work is similar in design and analysis to the work in [5], [8]; however in [8] bearing-only constraints are solely considered and in [5] distance-only constraints are solely considered. We present a unified theory now where *an (essentially) arbitrary mix of distance and bearing only constraints between neighbours can be accommodated.*

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A stability and convergence analysis is undertaken similarly to [5]. In contrast to much work on formation control, no restriction to triangular formations applies.

Many existing formation control laws, particularly those with distance-only constraints, do not provide global stabilisation of the desired formation shape due to the existence of undesired equilibria [6], [15], [16]. Another key benefit displayed in the new work is that with a mixture of distance and bearing constraints many of these undesired equilibria sets vanish; e.g. collinear formations cannot be controlled with distance-only constraints [5] while they can be controlled with mixed constraints. Moreover, with bearing-only constraints [8] the scale of the formation is uncontrollable whereas with mixed constraints one can control the scale.

II. Stiffness Theory with Truly Mixed Direction and Distance Constraints

The idea of stiff point formations discussed herein follows [17] and is related to rigid formations [4] and parallel rigid formations [18], [19] and their graph origins [20].

Consider n agents indexed by $\mathcal{V} = \{1, 2, \ldots, n\}$ and with positions $\mathbf{p}_i \in \mathbb{R}^2$. Suppose now a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is defined on $\mathcal{V} = \{1, 2, \ldots, n\}$ where $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ defines a set of m links (i, j) between agents $i, j \in \mathcal{V}$. We define the so-called neighbour sets accordingly $(i, j) \in \mathcal{E} \Rightarrow j \in \mathcal{N}_i \Leftrightarrow i \in \mathcal{N}_j$.

Definition 1 (Formal Point Formation)**.** *A point formation $\mathcal{F}_p(\mathcal{G})$ is defined by a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and a map $p : \mathcal{V} \to \mathbb{R}^2$ which takes agent i in $\mathcal{V}$ to its respective position $\mathbf{p}_i$ in $\mathbb{R}^2$.*

Now suppose we create two (not necessarily, but possibly, disjoint) sets $\mathcal{E}_\mathcal{B} \subseteq \mathcal{E}$ and $\mathcal{E}_\mathcal{D}$ from $\mathcal{E} \subseteq \mathcal{E}$ such that $\mathcal{E}_\mathcal{B} \cup \mathcal{E}_\mathcal{D} = \mathcal{E}$. We suppose that $\mathcal{E}_\mathcal{B} \neq \emptyset$ and $\mathcal{E}_\mathcal{D} \neq \emptyset$.

Define a graph $\mathcal{G}(\mathcal{V}, \mathcal{E}_\mathcal{B}, \mathcal{E}_\mathcal{D})$ consisting of two edge sets. When $(i, j) \in \mathcal{E}_\mathcal{B}$ and $(i, j) \in \mathcal{E}_\mathcal{D}$ then $\mathcal{G}(\mathcal{V}, \mathcal{E}_\mathcal{B}, \mathcal{E}_\mathcal{D})$ has two edges between agents i and j.

Definition 2 (Formal Point Formation (Alternate Definition))**.** *A point formation $\mathcal{F}_p(\mathcal{G})$ can then be defined by a graph $\mathcal{G}(\mathcal{V}, \mathcal{E}_\mathcal{B}, \mathcal{E}_\mathcal{D})$ and a map $p : \mathcal{V} \to \mathbb{R}^2$ which takes agent i in $\mathcal{V}$ to its respective position $\mathbf{p}_i$ in $\mathbb{R}^2$.*

Let ϕ_{ij} denote the bearing to agent j at agent i and

$$\mathcal{B}^{'} = \{\phi_{ij} \in [0, 2\pi) : (i, j) \in \mathcal{E}_\mathcal{B}\} \tag{1}$$

where $\phi_{ij} \equiv (\pi + \phi_{ji}) \bmod(2\pi)$ and

$$\mathcal{D}^{'} = \{d_{ij}^2 \in \mathbb{R}^+ : (i, j) \in \mathcal{E}_\mathcal{D}\} \tag{2}$$

where $d_{ij}^2 = \|\mathbf{p}_i - \mathbf{p}_j\|^2 = d_{ji}^2$ denotes the range squared.

Assumption 1 (Global Coordinate System)**.** *There exists a global coordinate frame within which agent positions and inter-agent bearings are measured against.*

This assumption can be eliminated in the design of the control algorithm but for notational and setup simplicity we leave this assumption in place.

Definition 3 (Equivalent Formations)**.** *Two formations $\mathcal{F}_q$ and $\mathcal{F}_p$ are said to be equivalent if their underlying graphs $\mathcal{G}(\mathcal{V}, \mathcal{E}_\mathcal{B}, \mathcal{E}_\mathcal{D})$ are identical and the set of bearing measurements $\mathcal{B}'$ and distance measurements $\mathcal{D}'$ in one of the formations is equal to the same sets in the other.*

Consider a formation $\mathcal{F}_p$ and a continuously parameterised formation trajectory defined by a time-varying $\mathbf{q}_i(t)$ for all $i \in \mathcal{V}$ such that $\mathcal{F}_{q(t)}$ is defined by a time-varying map $q(t) : \mathcal{V} \rightarrow \mathbb{R}^2$. Both $\mathcal{F}_p$ and $\mathcal{F}_q(t)$ are defined by the same underlying $\mathcal{G}(\mathcal{V}, \mathcal{E})$. Suppose $\mathbf{q}_i(0) = \mathbf{p}_i$ for all i. Then for each $(i, j) \in \mathcal{E}$ consider the constraint

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{q}_i - \mathbf{q}_j) = d_{ij}^2 \tag{3}$$

The time-derivative of this constraint is then

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\dot{\mathbf{q}}_i - \dot{\mathbf{q}}_j) = 0 \tag{4}$$

If such a constraint holds for each $(i, j) \in \mathcal{E}$ then the solutions $\dot{\mathbf{q}}_i$ of the corresponding $|\mathcal{E}| = m$ homogenous linear equations defines an infinitesimal formation motion with respect to $\mathcal{F}_p$.

Similarly, consider a formation $\mathcal{F}_p$ and $\mathbf{q}_i(t)$ for all $i \in \mathcal{V}$ defining $\mathcal{F}_{q(t)}$ as before with both $\mathcal{F}_p$ and $\mathcal{F}_q(t)$ defined by the same $\mathcal{G}(\mathcal{V}, \mathcal{E})$. Suppose $\mathbf{q}_i(0) = \mathbf{p}_i$ for all i. For each $(i, j) \in \mathcal{E}$ consider the constraint

$$(\mathbf{p}_i - \mathbf{p}_j)^\perp \cdot (\mathbf{q}_i - \mathbf{q}_j) = 0 \tag{5}$$

where the operator $(\cdot)^\perp$ rotates a plane vector by $\pi/2$ counterclockwise. If such a constraint holds for each $(i, j) \in \mathcal{E}$ then $\mathcal{F}_p$ and $\mathcal{F}_q$ are said to be parallel drawings [18], [19] of each other in the sense that for each $(i, j) \in \mathcal{E}$ the vectors $(\mathbf{p}_i - \mathbf{p}_j)$ and $(\mathbf{q}_i - \mathbf{q}_j)$ are parallel. The time-derivative of this constraint is then

$$(\mathbf{p}_i - \mathbf{p}_j)^\perp \cdot (\dot{\mathbf{q}}_i - \dot{\mathbf{q}}_j) = 0 \tag{6}$$

and the solutions $\dot{\mathbf{q}}_i$ of the corresponding $|\mathcal{E}| = m$ homogenous linear equations defines an infinitesimal formation motion with respect to $\mathcal{F}_p$.

Definition 4 (Formation Shakes)**.** *Assume $\mathcal{F}_p$ is given and $\mathcal{F}_q$ is defined on the same underlying graph $\mathcal{G}(\mathcal{V}, \mathcal{E}_\mathcal{B}, \mathcal{E}_\mathcal{D})$. Then $\mathcal{F}_q$ is said to be a shake with respect to $\mathcal{F}_p$ if and only if (6) is satisfied for all $(i, j) \in \mathcal{E}_\mathcal{B}$ and (4) is satisfied for all $(i, j) \in \mathcal{E}_\mathcal{D}$.*

Thus, $\mathcal{F}_{q(t)}$ is a shake with respect to $\mathcal{F}_p$ if

$$\begin{aligned} (\mathbf{p}_i - \mathbf{p}_j) \cdot (\dot{\mathbf{q}}_i(t) - \dot{\mathbf{q}}_j(t)) &= 0, \quad (i, j) \in \mathcal{E}_\mathcal{D} \\ (\mathbf{p}_i - \mathbf{p}_j)^\perp \cdot (\dot{\mathbf{q}}_i(t) - \dot{\mathbf{q}}_j(t)) &= 0, \quad (i, j) \in \mathcal{E}_\mathcal{B} \end{aligned} \tag{7}$$

which can be written in matrix form as

$$\mathbf{R}(\mathbf{p})\dot{\mathbf{q}} = 0 \tag{8}$$

where $\mathbf{p} = [\mathbf{p}_1^\top \ \mathbf{p}_2^\top \ \dots \ \mathbf{p}_n^\top]^\top$ and similarly for $\mathbf{q}$. $\mathbf{R}(\mathbf{p}) \in \mathbb{R}^{m \times 2n}$ is called the constraint matrix for formations with distance and bearing constraints [17].

Definition 5 (Stiff Formations)**.** *A point formation $\mathcal{F}_p$ is said to be a stiff formation if all shakes of $\mathcal{F}_p$ can be obtained via translations.*

Example 1. *Consider four agents indexed by* 1*,* 2*,* 3*, and* 4*. An example of a stiff formation is illustrated in Figure 1. Conditions for testing and confirming stiffness are given subsequently.*

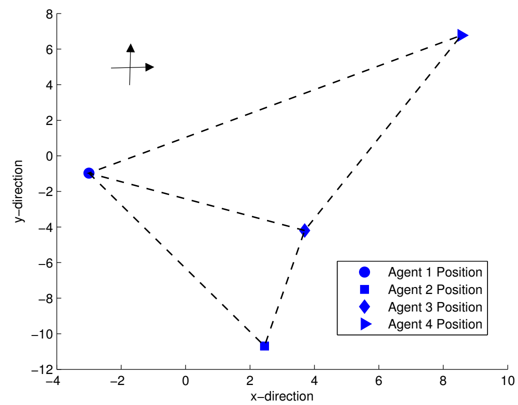

Fig. 1. A stiff formation defined by the interaction graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and a random embedding of the four agents on the plane. In this case $\mathcal{V} = \{1, 2, 3, 4\}$ and $\mathcal{E} = \{(1, 2), (1, 3), (1, 4), (2, 3), (3, 4)\}$.

The graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ associated with the formation has 5 *edges while the graph $\mathcal{G}(\mathcal{V}, \mathcal{E}_\mathcal{B}, \mathcal{E}_\mathcal{D})$ associated with the formation has* 6 *edges (two between agents* 3 *and* 4*). Edges (1,2) and (1,3) correspond to distance constraints, edges (1,4) and (2,3) correspond to bearing constraints and edge (3,4) corresponds to both a distance and bearing constraint. The edges, arranged in lexicographical order, are* $\{(1, 2), (1, 3), (1, 4), (2, 3), (3, 4)\}$*. The constraint matrix for the formation is given by (9). The constraint matrix is a* 6×8 *matrix in this example. The rows correspond to the independent constraints in the graph associated with the formation and the columns correspond to the agents.*

The convention for ordering the rows of $\mathbf{R}(\mathbf{p})$ outlined in the following assumption.

Assumption 2. *We suppose that the constraints $\mathbf{R}(\mathbf{p})\dot{\mathbf{q}} = 0$ in (8) are written such that the rows corresponding to the bearing constraints are written on top of those corresponding to the distance constraints and that within this partitioning the rows are ordered lexicographically with respect to the edge labelling in the graph $\mathcal{G}(\mathcal{V}, \mathcal{E}_\mathcal{B}, \mathcal{E}_\mathcal{D})$; see (9).*

We highlight now a test for stiffness.

Theorem 1. *A formation $\mathcal{F}_p$ of n agents is stiff if* $\mathrm{rank}(\mathbf{R}(\mathbf{p})) = 2n - 2$.

Refer to (8) and note the condition $\mathrm{rank}(\mathbf{R}(\mathbf{p})) = 2n - 2$ implies the kernel of $\mathbf{R}(\mathbf{p})$ is of dimension 2. It is easily

$$\begin{array}{r} \\ edge\ (1,4) \\ edge\ (2,3) \\ edge\ (3,4) \\ edge\ (1,2) \\ edge\ (1,3) \\ edge\ (3,4) \end{array} \begin{array}{c} \begin{array}{cccc} agent\ 1 & agent\ 2 & agent\ 3 & agent\ 4 \end{array} \\ \left[\begin{array}{c|c|c|c} ((\mathbf{p}_1-\mathbf{p}_4)^\perp)^\top & \mathbf{0} & \mathbf{0} & ((\mathbf{p}_4-\mathbf{p}_1)^\perp)^\top \\ \mathbf{0} & ((\mathbf{p}_2-\mathbf{p}_3)^\perp)^\top & ((\mathbf{p}_3-\mathbf{p}_2)^\perp)^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & ((\mathbf{p}_3-\mathbf{p}_4)^\perp)^\top & ((\mathbf{p}_4-\mathbf{p}_3)^\perp)^\top \\ (\mathbf{p}_1-\mathbf{p}_2)^\top & (\mathbf{p}_2-\mathbf{p}_1)^\top & \mathbf{0} & \mathbf{0} \\ (\mathbf{p}_1-\mathbf{p}_3)^\top & \mathbf{0} & (\mathbf{p}_3-\mathbf{p}_1)^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\mathbf{p}_3-\mathbf{p}_4)^\top & (\mathbf{p}_4-\mathbf{p}_3)^\top \end{array}\right] \end{array} = \mathbf{R}(\mathbf{p}) \tag{9}$$

shown that this is the lowest dimension the kernel can take on and it corresponds to the fact that the trajectories of the formation at $\mathbf{q}$ in (8) are free up to translations (accounting for two linearly independent solutions $\dot{\mathbf{q}}$ to (8)).

Definition 6 (Generic Formations)**.** *A formation is said to be in generic position* $\mathbf{p}$ *in* $\mathbb{R}^{2n}$ *if the set of its coordinates are not algebraically dependent; e.g. see [4] for more details.*

Theorem 2. *Consider two formations* $\mathcal{F}_p$ *and* $\mathcal{F}_q$ *in generic positions defined on the same underlying graph* $\mathcal{G}(\mathcal{V},\mathcal{E})$*. Then* $\mathcal{F}_p$ *is stiff if and only if* $\mathcal{F}_q$ *is stiff.*

This theorem underpins the following definition.

Definition 7 (Generically Stiff Graph)**.** *When* $\mathcal{F}_p$ *is stiff for all generic points* $\mathbf{p}$ *then we say the graph* $\mathcal{G}(\mathcal{V},\mathcal{E})$ *associated with* $\mathcal{F}_p$ *is generically stiff.*

We often refer also to the formation $\mathcal{F}_p$ whose graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ is generically stiff as a generically stiff formation.

III. The Stiff Formation Control Problem

Consider again the n agents indexed by $\mathcal{V}=\{1,2,\ldots,n\}$ and with positions $\mathbf{z}_i\in\mathbb{R}^2$. Let $t\in[0,\infty)$ denote time. The motion of agent i is governed by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}_i=\dot{\mathbf{z}}_i=\mathbf{u}_i \tag{10}$$

where $\mathbf{u}_i$ is a control vector to be determined. The combined motion of the formation is $\dot{\mathbf{z}}=\mathbf{u}$ where $\mathbf{z}=[\mathbf{z}_1^\top\ \mathbf{z}_2^\top\ \ldots\ \mathbf{z}_n^\top]^\top$ etc.

Suppose agent i can measure the bearing and range (or relative position) to agent j iff $j\in\mathcal{N}_i\Leftrightarrow i\in\mathcal{N}_j$ where $\mathcal{N}_i$ is the set of neighbours of i. The sets $\mathcal{V}=\{1,2,\ldots,n\}$ and $\mathcal{N}_i$, $\forall i\in\mathcal{V}$ define a graph that represents the measurements between the agents. Denote this sensing graph by $\mathcal{G}_\mathcal{M}(\mathcal{V},\mathcal{E}_\mathcal{M})$ where $\mathcal{E}_\mathcal{M}\subseteq\mathcal{V}\times\mathcal{V}$ is the set of m (measurement) links (i,j) where (i,j) exists iff $j\in\mathcal{N}_i\Leftrightarrow i\in\mathcal{N}_j$.

The set of bearing measurements $\mathcal{B}$ is

$$\mathcal{B}(\mathbf{z})=\{\phi_{ij}\in[0,2\pi):i<j,\ (i,j)\in\mathcal{E}_\mathcal{M}\} \tag{11}$$

where $\phi_{ij}\equiv(\pi+\phi_{ji})\bmod(2\pi)$ and $|\mathcal{B}|=m$.

Define also a set of range measurements

$$\mathcal{D}(\mathbf{z})=\{d_{ij}^2\in\mathbb{R}^+:i<j,\ (i,j)\in\mathcal{E}_\mathcal{M}\} \tag{12}$$

where $|\mathcal{D}|=m$ and $d_{ij}^2=\|\mathbf{z}_i-\mathbf{z}_j\|^2=d_{ji}^2$. If an agent actually measures d_{ij} then it also knows d_{ij}^2.

Suppose now a constraint graph $\mathcal{G}_\mathcal{C}(\mathcal{V},\mathcal{E}_\mathcal{B},\mathcal{E}_\mathcal{D})$ is defined on $\mathcal{V}=\{1,2,\ldots,n\}$ and where $\mathcal{E}_\mathcal{B}\subseteq\mathcal{E}_\mathcal{M}$ and $\mathcal{E}_\mathcal{D}\subseteq\mathcal{E}_\mathcal{M}$ and $\mathcal{E}_\mathcal{B}\cup\mathcal{E}_\mathcal{D}=\mathcal{E}_\mathcal{M}$. We suppose that $\mathcal{E}_\mathcal{B}\neq\emptyset$ and $\mathcal{E}_\mathcal{D}\neq\emptyset$.

We define some so-called neighbour sets accordingly $(i,j)\in\mathcal{E}_\mathcal{B}\Rightarrow j\in\mathcal{N}_i^\mathcal{B}\Leftrightarrow i\in\mathcal{N}_j^\mathcal{B}$ and $(i,j)\in\mathcal{E}_\mathcal{D}\Rightarrow j\in\mathcal{N}_i^\mathcal{D}\Leftrightarrow i\in\mathcal{N}_j^\mathcal{D}$.

Now define a set of desired bearing values

$$\mathcal{B}_c=\{\phi_{ij}^c\in[0,2\pi):i<j,\ (i,j)\in\mathcal{E}_\mathcal{B}\} \tag{13}$$

where $\phi_{ij}^c\equiv(\pi+\phi_{ji}^c)\bmod(2\pi)$ and $\mathcal{B}_c\subseteq\mathcal{B}(\mathbf{z})$. Similarly,

$$\mathcal{D}_c=\{d_{ij}^c\in\mathbb{R}^+:i<j,\ (i,j)\in\mathcal{E}_\mathcal{D}\} \tag{14}$$

where $d_{ij}^c=\|\mathbf{z}_i-\mathbf{z}_j\|^2=d_{ji}^c$ and $\mathcal{D}_c\subseteq\mathcal{D}(\mathbf{z})$. Note d_{ij}^c is a squared distance.

Assumption 3. *The formation* $\mathcal{F}_p(\mathcal{G}_\mathcal{C}(\mathcal{V},\mathcal{E}_\mathcal{B},\mathcal{E}_\mathcal{D}))$ *is generically stiff and* $\mathbf{z}_i\neq\mathbf{z}_j$ *at* $t=0$*,* $\forall i,j\in\mathcal{V}$*.*

This assumption does not imply $\operatorname{rank}(\mathbf{R}(\mathbf{z}))=2n-2$ since $\mathbf{z}$ at $t=0$ may not be a generic point.

Definition 8 (Realizable Constraint Sets)**.** *Assume a formation* $\mathcal{F}_p$ *is given. Then a pair of sets* $\mathcal{B}'$ *and* $\mathcal{D}'$ *of bearings and distances are realizable if and only if each* $\phi_{ij}\in\mathcal{B}'$ *and* $d_{ij}\in\mathcal{D}'$ *can exist between the respective* $\mathbf{p}_i$ *and* $\mathbf{p}_j$ *simultaneously.*

Assumption 4. *The set of desired bearing values* $\mathcal{B}_c$ *and desired distance values* $\mathcal{D}_c$ *that define the desired formation shape, scale and orientation is realizable. Moreover, there is a value* ϕ_{ij}^c *for each* $(i,j)\in\mathcal{E}_\mathcal{B}$ *and a value* d_{ij}^c *for each* $(i,j)\in\mathcal{E}_\mathcal{D}$ *and due to Assumption 3 the desired formation is generically stiff.*

Suppose now that one has a set $\mathcal{B}'$ of bearings indexed by edges in some subset of $\mathcal{E}_\mathcal{M}$ which includes $\mathcal{E}_\mathcal{B}$. Then

$$\mathbf{b}=\operatorname{column}(\mathcal{B}';\mathcal{E}_\mathcal{B}) \tag{15}$$

defines a $|\mathcal{E}_\mathcal{B}|\times 1$ column vector by stacking the bearings in $\mathcal{B}'$ that are indexed by edges in $\mathcal{E}_\mathcal{B}$. The bearings are stacked according to a lexicographical ordering such that ϕ_{ij} is above ϕ_{il} if $j<l$ and ϕ_{ij} is above ϕ_{kl} if $i<k$. Similarly, the same idea applies given a set $\mathcal{D}'$ of distances and $\operatorname{column}(\mathcal{D}';\mathcal{E}_\mathcal{D})$.

Thus define $\mathbf{b}(\mathbf{z})=\operatorname{column}(\mathcal{B}(\mathbf{z});\mathcal{E}_\mathcal{B})$ and similarly define $\mathbf{d}(\mathbf{z})=\operatorname{column}(\mathcal{D}(\mathbf{z});\mathcal{E}_\mathcal{D})$. Similarly define $\mathbf{b}_c=\operatorname{column}(\mathcal{B}_c;\mathcal{E}_\mathcal{B})$ and similarly define $\mathbf{d}_c=\operatorname{column}(\mathcal{D}_c;\mathcal{E}_\mathcal{B})$.

Both $\mathbf{b}_c$ and $\mathbf{d}_c$ are formed by stacking all the constraints in (13) and (14) respectively into column vectors. On the other hand $\mathbf{b}(\mathbf{z})$ and $\mathbf{d}(\mathbf{z})$ are formed by stacking (typically a subset of) measurements such that they correspond row-wise with $\mathbf{b}_c$ and $\mathbf{d}_c$ in terms of their respective indexing.

Note $\mathbf{b}(\mathbf{z})$ is determined by the bearing measurements and is a function of $\mathbf{z}$ whereas $\mathbf{b}_c$ is a vector of desired bearing constraints and is constant. Similarly for $\mathbf{d}(\mathbf{z})$ and $\mathbf{d}_c$.

Now it possible to define an error vector as

$$\mathbf{e} = [\mathbf{b}(\mathbf{z})^\top \ \mathbf{d}(\mathbf{z})^\top]^\top - [\mathbf{b}_c^\top \ \mathbf{d}_c^\top]^\top \tag{16}$$

and note $\mathbf{e} \to 0$ for some formation $\mathcal{F}_z$ implies the formation is equivalent to the desired formation.

Problem 1. *The formation control problem is to design a control input $\mathbf{u}_i$, $\forall i \in \mathcal{V}$, as a function of at most ϕ_{ij}, d_{ij}, d_{ij}^c and ϕ_{ij}^c, for all $j \in \mathcal{N}_i$, such that $\mathbf{e} \to 0$.*

Before outlining the control law proposed to solve Problem 1 we note that the Jacobian of $\mathbf{e} \in \mathbb{R}^m$ evaluated at a point $\mathbf{p} \in \mathbb{R}^{2n}$ is given by

$$\begin{aligned} \mathbf{J_e}(\mathbf{p}) &= \nabla \mathbf{e} \\ &= \frac{\partial}{\partial \mathbf{z}}\left(\begin{bmatrix} \mathbf{b}(\mathbf{z}) \\ \mathbf{d}(\mathbf{z}) \end{bmatrix} - \begin{bmatrix} \mathbf{b}_c \\ \mathbf{d}_c \end{bmatrix} \right)\Bigg|_{\mathbf{z}=\mathbf{p}} \\ &= \frac{\partial}{\partial \mathbf{z}} \begin{bmatrix} \mathbf{b}(\mathbf{z}) \\ \mathbf{d}(\mathbf{z}) \end{bmatrix}\Bigg|_{\mathbf{z}=\mathbf{p}} \end{aligned} \tag{17}$$

where $\mathbf{J_e}(\mathbf{p}) \in \mathbb{R}^{m \times 2n}$.

Let $\mathbf{D} = \mathbf{d}(\mathbf{z})^\top \mathbf{I}$ where $\mathbf{I}$ is an identity matrix. The ℓ^{th} element of a m-vector $\mathbf{x}$ is $(\mathbf{x} : \ell)$. We then have

$$\begin{aligned} \mathbf{J_e}(\mathbf{p}) &= \frac{\partial}{\partial \mathbf{z}} \mathbf{b}(\mathbf{z})\Bigg|_{\mathbf{z}=\mathbf{p}} \\ &= \begin{bmatrix} \frac{\partial(\mathbf{b}(\mathbf{z}):1)}{\partial \mathbf{z_1}}\Big|_{\mathbf{z}_1=\mathbf{p}_1} & \cdots & \frac{\partial(\mathbf{b}(\mathbf{z}):1)}{\partial \mathbf{z_n}}\Big|_{\mathbf{z}_n=\mathbf{p}_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial(\mathbf{b}(\mathbf{z}):|\mathcal{B}_c|)}{\partial \mathbf{z_1}}\Big|_{\mathbf{z}_1=\mathbf{p}_1} & \cdots & \frac{\partial(\mathbf{b}(\mathbf{z}):|\mathcal{B}_c|)}{\partial \mathbf{z_n}}\Big|_{\mathbf{z}_n=\mathbf{p}_n} \\ \frac{\partial(\mathbf{d}(\mathbf{z}):1)}{\partial \mathbf{z_1}}\Big|_{\mathbf{z}_1=\mathbf{p}_1} & \cdots & \frac{\partial(\mathbf{d}(\mathbf{z}):1)}{\partial \mathbf{z_n}}\Big|_{\mathbf{z}_n=\mathbf{p}_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial(\mathbf{d}(\mathbf{z}):|\mathcal{D}_c|)}{\partial \mathbf{z_1}}\Big|_{\mathbf{z}_1=\mathbf{p}_1} & \cdots & \frac{\partial(\mathbf{d}(\mathbf{z}):|\mathcal{D}_c|)}{\partial \mathbf{z_n}}\Big|_{\mathbf{z}_n=\mathbf{p}_n} \end{bmatrix} \\ &= \begin{bmatrix} -\mathbf{D}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \mathbf{R}(\mathbf{p}) \end{aligned} \tag{18}$$

where $\mathbf{R}(\mathbf{p})$ is the constraint matrix for the formation $\mathcal{F}_z|_{z=p}$.

Example 2. *Consider four agents indexed by 1, 2, 3, and 4 and the stiff formation illustrated in Figure 1 of Example 1. Again, the edges, arranged in lexicographical order, are $\{(1,2),(1,3),(1,4),(2,3),(3,4)\}$. The constraints associated with each edge are as in Example 1.*

$$\left[\mathbf{b}(\mathbf{z})^\top \ \mathbf{d}(\mathbf{z})^\top\right]^\top = [\phi_{14} \ \phi_{23} \ \phi_{34} \ d_{12}^2 \ d_{13}^2 \ d_{34}^2]^\top \tag{19}$$

where each measured ϕ_{ij} or d_{ij} is a function of $\mathbf{z}_i$ and $\mathbf{z}_j$. The constraint matrix for the formation is given by (9). The Jacobian $\mathbf{J_e}(\mathbf{p})$ of the error vector $\mathbf{e}$ is given by (20) and is of the same dimension as (9). The rows correspond to the edges in the graph associated with the formation and the columns correspond to the agents. We note again that an agent i that knows ϕ_{ij} and d_{ij} also knows ϕ_{ji} and d_{ji} and vice versa. Thus, given the measurements ϕ_{ij} and d_{ij} at agent i for $j \in \mathcal{N}_i$ it follows that the rows of $\mathbf{J_e}(\mathbf{p})$ corresponding to an edge incident on i are known locally at agent i and the two columns corresponding to the agent itself are also known locally.

The proof is immediate from (18). In particular, the sparsity pattern of both $\mathbf{R}(\mathbf{p})$ and $\mathbf{J_e}(\mathbf{p})$ is identical for an arbitrary formation $\mathcal{F}_p$.

A. The Proposed Control Law

The control law proposed is a gradient-type control law, associated with the function $\frac{1}{2}\mathbf{e}^\top \mathbf{e}$, and can be written as

$$\begin{aligned} \mathbf{u} &\triangleq -(\nabla \mathbf{e})^\top \mathbf{e} \\ &= -\mathbf{J_e}(\mathbf{z})^\top \mathbf{e} \\ &= \mathbf{R}^\top(\mathbf{z}) \begin{bmatrix} \mathbf{D}^{-1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \mathbf{e} \end{aligned} \tag{21}$$

from (18), and there results

$$\begin{aligned} \dot{\mathbf{z}} = \mathbf{u} &= -\mathbf{J_e}(\mathbf{z})^\top \mathbf{e} \\ &= \mathbf{R}^\top(\mathbf{z}) \begin{bmatrix} \mathbf{D}^{-1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \mathbf{e} \end{aligned} \tag{22}$$

More specifically, the control law for an individual agent is

$$\begin{aligned} \dot{\mathbf{z}}_i &= \mathbf{u}_i \\ &= -\sum_{j \in \mathcal{N}_i^{\mathcal{D}}} \begin{bmatrix} \cos\phi_{ij} \\ \sin\phi_{ij} \end{bmatrix} (d_{ij}^2 - d_{ij}^c) \\ &\quad + \sum_{j \in \mathcal{N}_i^{\mathcal{B}}} \frac{1}{d_{ij}} \begin{bmatrix} \cos\phi_{ij} \\ -\sin\phi_{ij} \end{bmatrix} (\phi_{ij} - \phi_{ij}^c) \end{aligned} \tag{23}$$

The first summation is a superposition of $|\mathcal{N}_i^{\mathcal{D}}|$ vectors pointing away from the neighbours of agent i and with which there is a distance constraint between agent i and that neighbour. The second summation is a superposition of $|\mathcal{N}_i^{\mathcal{B}}|$ vectors pointing perpendicular to those links leaving agent i and corresponding to a bearing constraint. Each vector is scaled by an appropriate error term (which may be negatively signed) and those corresponding to bearing-only constraints are also scaled by the inverse range between the agents.

The controller proposed in this work is similar in principle to the controller proposed in [5] for formation control with range-only constraints and that proposed in [8] for bearing-only constraints. More generally, there is a close connection between this work and that in [1], [4]–[6], [8], [21] due to the relationship between rigidity, parallel drawings and the theory of stiffness used herein [17], [20].

The novelty of this work, compared to e.g. [5], [8], is that we allow for a truly heterogenous constraint set made up of both distance and bearing constraints. The desired formation is invariant in this case only up to translation.

The existence and uniqueness of the coupled system of differential equations (23) is guaranteed using standard arguments [5], [22] if the trajectories over $t \in [0, \infty)$ or as $t \to \infty$ are such that $d_{ij} > 0$ for the subset of inter-agent distances $(i,j) \in \mathcal{E}_{\mathcal{M}}$.

$$\begin{array}{c} \\ edge\ (1,4) \\ edge\ (2,3) \\ edge\ (3,4) \\ edge\ (1,2) \\ edge\ (1,3) \\ edge\ (3,4) \end{array} \begin{array}{c} \quad agent\ 1 \qquad\qquad agent\ 2 \qquad\qquad agent\ 3 \qquad\qquad agent\ 4 \\ \left[\begin{array}{cc|cc|cc|cc} -\frac{\cos\phi_{14}}{d_{14}^2} & \frac{\sin\phi_{14}}{d_{14}^2} & 0 & 0 & 0 & 0 & \frac{\cos\phi_{14}}{d_{14}^2} & -\frac{\sin\phi_{14}}{d_{14}^2} \\ 0 & 0 & -\frac{\cos\phi_{23}}{d_{23}^2} & \frac{\sin\phi_{23}}{d_{23}^2} & \frac{\cos\phi_{23}}{d_{23}^2} & -\frac{\sin\phi_{23}}{d_{23}^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\cos\phi_{34}}{d_{34}^2} & \frac{\sin\phi_{34}}{d_{34}^2} & \frac{\cos\phi_{34}}{d_{34}^2} & -\frac{\sin\phi_{34}}{d_{34}^2} \\ \cos\phi_{12} & \phi_{12} & -\cos\phi_{12} & -\sin\phi_{12} & 0 & 0 & 0 & 0 \\ \cos\phi_{13} & \sin\phi_{13} & 0 & 0 & -\cos\phi_{13} & -\sin\phi_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\phi_{34} & \sin\phi_{34} & -\cos\phi_{34} & -\sin\phi_{34} \end{array}\right] \end{array} = \mathbf{J}_{\mathbf{e}}(\mathbf{p}) \quad (20)$$

Lemma 1. *Let* $\bar{\mathbf{z}} = \frac{1}{|\mathcal{V}|}\sum_{i\in\mathcal{V}} \mathbf{z}_i$. *Then* $\dot{\bar{\mathbf{z}}} = 0$.

The next lemma concerns the controller and its invariance to the global coordinate system chosen.

Lemma 2. *For all* $\mathbf{w} \in \mathbb{R}^2$ *it follows that* $\mathbf{J_e}(\mathbf{z}) = \mathbf{J_e}(\mathbf{z} + (\mathbf{1} \otimes \mathbf{w}))$ *where* $\mathbf{1}$ *is an* n*-dimensional column vector of all* 1*'s. Moreover, for every orthogonal matrix* $\mathbf{X} \in \mathbb{R}^{2\times 2}$ *it follows that* $\mathbf{J_e}(\mathbf{z})(\mathbf{I_n} \otimes \mathbf{X})^\top = \mathbf{J_e}((\mathbf{I_n} \otimes \mathbf{X})\mathbf{z})$ *where* $\mathbf{I}_n$ *is a* $n \times n$ *identity matrix.*

IV. Stability Results

A. Minimally and Generically Stiff Formations

We know that a generically stiff formation $\mathcal{F}_z$ is one that can be characterized entirely by the associated graph $\mathcal{G}(\mathcal{V}, \mathcal{E}_\mathcal{B}, \mathcal{E}_\mathcal{D})$ defining the formation interactions; e.g. see Theorem 1. It is also the case that a necessary condition for the formation to be (generically) stiff is that $|\mathcal{E}_\mathcal{B} \cup \mathcal{E}_\mathcal{D}| = m \geq 2|\mathcal{V}| - 2 = 2n - 2$ and $\mathcal{E}_\mathcal{B} \neq \emptyset$ and $\mathcal{E}_\mathcal{D} \neq \emptyset$.

Definition 9 (Minimally Stiff)**.** *Suppose* $\mathcal{E}_\mathcal{B} \neq \emptyset$ *and* $\mathcal{E}_\mathcal{D} \neq \emptyset$*. A formation* $\mathcal{F}_z$ *with* $|\mathcal{E}_\mathcal{B} \cup \mathcal{E}_\mathcal{D}| = m = 2|\mathcal{V}| - 2 = 2n - 2$ *at* $\mathbf{z} \in \mathbb{R}^{2n}$ *is called a minimally stiff formation if* $\text{rank}(\mathbf{R}(\mathbf{z})) = 2n - 2$*. A formation* $\mathcal{F}_z$ *with* $|\mathcal{E}_\mathcal{B} \cup \mathcal{E}_\mathcal{D}| = m = 2|\mathcal{V}| - 2 = 2n - 2$ *is called a minimally and generically stiff formation if and only if it is generically stiff.*

Definition 10 (Fundamental Cycles [23])**.** *Consider a connected graph* $\mathcal{G}(\mathcal{V}, \mathcal{E})$ *with a spanning tree* $\mathcal{T}_\mathcal{G}(\mathcal{V}, \mathcal{E}(\mathcal{T}_\mathcal{G})) \subset \mathcal{G}$*. Then for every edge* $(i, j) \in \mathcal{E} \setminus \mathcal{E}(\mathcal{T}_\mathcal{G})$ *there is a unique cycle* $\mathcal{C}$ *in* $\mathcal{T}_\mathcal{G}(\mathcal{V}, \mathcal{E}(\mathcal{T}_\mathcal{G}) \cup (i, j))$ *and these cycles are called* fundamental cycles *of* $\mathcal{G}$ *with respect to* $\mathcal{T}_\mathcal{G}$*.*

There are $(|\mathcal{E}| - \mathcal{V} + 1)$ independent cycles in a connected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$. Let $\boldsymbol{\mathcal{C}}(\mathcal{T}_{\mathcal{G}(\mathcal{V},\mathcal{E})}) = \{\mathcal{C}_i(\mathcal{T}_{\mathcal{G}(\mathcal{V},\mathcal{E})})\}_{i=1}^{|\mathcal{E}|-\mathcal{V}+1}$ be the set of fundamental cycles of $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with respect to $\mathcal{T}_\mathcal{G}$.

Suppose $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is not connected but is formed by a set $\mathfrak{C}_{\mathcal{G}(\mathcal{V},\mathcal{E})}$ of subgraphs of connected graph components. The number of connected components is $|\mathfrak{C}_{\mathcal{G}(\mathcal{V},\mathcal{E})}|$ and is bounded below by 1 and above by $|\mathcal{V}|$. A spanning tree $\mathcal{T}_\mathcal{H}$ can be associated with each component $\mathcal{H} \in \mathfrak{C}_{\mathcal{G}(\mathcal{V},\mathcal{E})}$ and the set of spanning trees associated with the components in $\mathfrak{C}_{\mathcal{G}(\mathcal{V},\mathcal{E})}$ (one tree associated with each component) is denoted by $\mathfrak{T}_{\mathcal{G}(\mathcal{V},\mathcal{E})}$. In such a case let $\boldsymbol{\mathcal{C}}(\mathfrak{T}_{\mathcal{G}(\mathcal{V},\mathcal{E})}) = \{\mathcal{C}_i\}$ be the set of fundamental cycles of $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with respect to $\mathfrak{T}_{\mathcal{G}(\mathcal{V},\mathcal{E})}$.

Lemma 3. *Suppose* $\mathcal{F}_z$ *is a minimally and generically stiff formation. Let* $\boldsymbol{\mathcal{C}}(\mathfrak{T}_{\mathcal{G}(\mathcal{V},\mathcal{E}_\mathcal{B})})$ *be a set of fundamental cycles in* $\mathcal{G}(\mathcal{V}, \mathcal{E}_\mathcal{B})$ *and let* $\boldsymbol{\mathcal{C}}(\mathfrak{T}_{\mathcal{G}(\mathcal{V},\mathcal{E}_\mathcal{D})})$ *be a set of fundamental cycles in* $\mathcal{G}(\mathcal{V}, \mathcal{E}_\mathcal{D})$*. Then* $\text{rank}(\mathbf{R}(\mathbf{z}))$ *drops by* 1 *for the existence of each:*

1) *Cycle* $\mathcal{C}_i \in \boldsymbol{\mathcal{C}}(\mathfrak{T}_{\mathcal{G}(\mathcal{V},\mathcal{E}_\mathcal{B})})$ *where all agents defining* $\mathcal{C}_i$ *are collinear.*
2) *Cycle* $\mathcal{C}_i \in \boldsymbol{\mathcal{C}}(\mathfrak{T}_{\mathcal{G}(\mathcal{V},\mathcal{E}_\mathcal{D})})$ *where all agents defining* $\mathcal{C}_i$ *are collinear.*

The stability analysis in this subsection concerns minimally and generically stiff formations $\mathcal{F}_z$ and the resulting differential system (22). Consider the set

$$\mathcal{Z}^* = \{\mathbf{z} \in \mathbb{R}^{2n} : \mathbf{e} = \mathbf{0}\} \tag{24}$$

of equilibrium points corresponding to the formation $\mathcal{F}_z$ reaching the desired shape, scale and orientation defined by $\mathbf{b}_c$ and $\mathbf{d}_c$. Each formation that lives in $\mathcal{Z}^*$ is generically stiff due to Assumptions 3 and 4.

Definition 11 (Connected Space)**.** *A topological space* $\mathcal{X}$ *is said to be disconnected if there exists two open sets* $\mathcal{U} \neq \emptyset$ *and* $\mathcal{W} \neq \emptyset$ *such that* $\mathcal{U} \cap \mathcal{W} = \emptyset$ *and* $\mathcal{X} = \mathcal{U} \cup \mathcal{W}$*. If* $\mathcal{X}$ *is not disconnected than it is said to be connected.*

The maximal connected subsets of a nonempty topological space are called the connected components of the space.

Lemma 4. *The set* $\mathcal{Z}^*$ *is connected and each* $\mathbf{z}' \in \mathcal{Z}^*$ *can be obtained from* $\mathbf{z} \in \mathcal{Z}^*$ *by translation.*

Unfortunately, the set $\mathcal{Z}^*$ is not the only equilibrium set for the differential system (23) and minimally stiff formations under Assumptions 3 and 4. Consider the set

$$\mathcal{Z}_* = \left\{ \mathbf{z} \in \mathbb{R}^{2n} : \mathbf{R}^\top(\mathbf{z}) \begin{bmatrix} \mathbf{D}^{-1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \mathbf{e} = \mathbf{0} \right\} \tag{25}$$

and note that it is trivial to conclude that $\dot{\mathbf{z}} = \mathbf{0}$ if and only if $\mathbf{z} \in \mathcal{Z}_*$. A question remains as to when $\mathcal{Z}^* \equiv \mathcal{Z}_*$.

Theorem 3. *Suppose Assumptions 3 and 4 hold and the formation* $\mathcal{F}_z$ *is minimally and generically stiff. Assume that* $\text{rank}(\mathbf{R}(\mathbf{z})) = m = 2n - 2$ *for all* $t \in [0, \infty)$*. Then* $\dot{\mathbf{z}} = \mathbf{0}$ *if and only if* $\mathbf{z} \in \mathcal{Z}^*$*.*

Proof: The *if* part of the theorem is obvious from (22). If $\text{rank}(\mathbf{R}(\mathbf{z})) = m = 2n - 2$ then $\mathbf{R}(\mathbf{z})$ has full (row) rank and the kernel of $\mathbf{R}^\top(\mathbf{z})$ is trivial. Thus, $\mathcal{Z}^* \equiv \mathcal{Z}_*$. □

We know from Lemma 3 that $\text{rank}(\mathbf{R}(\mathbf{z}))$ drops by 1 for the existence of each fundamental cycle of collinear agents in the respective constraint graphs. Thus, any undesirable equilibria in $\{\mathcal{Z}_* \backslash \mathcal{Z}^*\}$ seemingly coincide with the existence of such collinear cycles. We conjecture, in the spirit of [5], that any equilibria in $\{\mathcal{Z}_* \setminus \mathcal{Z}^*\}$ are non-attractive. Note that in distance-constraint-based formation control using the rigidity matrix [5], every initial collinear formation will remain collinear. The situation here is less troublesome in this respect as collinear formations do not generically correspond to undesired equilibria when bearing-constraints are introduced (thus an advantage of the general mixed-constraint framework introduced).

Analysis concerning the state space and the equilibrium sets $\{\mathcal{Z}_*$ and $\mathcal{Z}^*\}$ is the topic of further work.

Theorem 4. *Suppose Assumptions 3 and 4 hold and the formation $\mathcal{F}_z$ is minimally and generically stiff. Then $\mathcal{Z}^*$ is locally asymptotically stable and there exists a neighbourhood $\mathcal{U}$ of $\mathcal{Z}^*$ such that for all $\mathbf{z}(0) \in \mathcal{U}$ there exists a point $\mathbf{z}^* \in \mathcal{Z}^*$ such that $\lim_{t\to\infty}\mathbf{z} = \mathbf{z}^*$.*

Proof will appear elsewhere. The theorem's validity is not surprising given the gradient-like nature of the system and the structural similarity between the differential system considered here and that considered in [5].

B. Generically Stiff Formations

The stability analysis in this subsection concerns generically stiff formations $\mathcal{F}_z$ and the resulting differential system (22). The equilibrium set considered in this subsection is the desired one (24). However, we note that in general, non-minimally stiff formations it follows that $\mathcal{Z}^* \subseteq \mathcal{Z}_*$.

Theorem 5. *Suppose Assumptions 3 and 4 hold and the formation $\mathcal{F}_z$ is generically stiff. Then $\mathcal{Z}^*$ is locally asymptotically stable and there is a neighbourhood $\mathcal{U}$ of $\mathcal{Z}^*$ such that $\forall \mathbf{z}(0) \in \mathcal{U}$ there exists a $\mathbf{z}^* \in \mathcal{Z}^*$ such that $\lim_{t\to\infty}\mathbf{z} = \mathbf{z}^*$.*

The proof of the preceding theorem will appear elsewhere due to space limitations.

V. Conclusion

This paper looks at the design of a distributed control scheme to solve the formation shape control problem with a mix of distance and bearing constraints and an arbitrary number of agents. In particular, a gradient control law is proposed based on the notion of a stiffness constraint matrix. An outline stability analysis is provided.

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