Robust Distributed Formation Control of Agents With Higher-Order Dynamics

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***Abstract*—We present a distributed control strategy for agents with a variety of holonomic dynamics to autonomously achieve a desired formation. The proposed control is fully distributed, and can be implemented locally on agents using the relative position measurements. Furthermore, agents do not need to communicate or have a common sense of orientation. Our strategy allows control gains designed for agents with single-integrator dynamics to be used directly for agents with higher-order linear (or linearizable) holonomic dynamics. We provide rigorous mathematical analysis to prove convergence of the agents to the desired formation. The proposed method is applied to quadrotors with linearized dynamics, and simulations are provided to typify the theoretical results.**

***Index Terms*—Multi-agent systems, formation control, distributed control, agent-based systems.**

I. Introduction

TECHNOLOGICAL advances in recent years have made it possible to deploy a large number of autonomous vehicles (agents) to execute tasks such as search and rescue [1], inspection [2], and monitoring [3]. In these applications, the ability of agents to autonomously achieve a desired formation is the fundamental building block upon which more sophisticated navigation capabilities can be constructed. Distributed formation control can prescribe local control laws to agents, such that the desired formation emerges as their collective team behavior. Compared to the centralized methods, distributed control strategies offer better scalability, naturally parallelized computation, resilience to communication loss and hardware failure, and robustness to uncertainty and lack of global measurements.

In the distributed formation control literature, often agents are assumed to have a simplified dynamic model, such as the single-integrator dynamics [4], [5], and the main focus is to derive the conditions in terms of the sensing topology among agents under which the desired formation is realized [6]–[8]. Since, in practice, agents often have more complicated dynamics, it is important to extend the control to agents with higher-order models and ensure the stability of the desired formation in such cases.

In this letter, we present a unified control strategy for agents with linear (or linearizable) holonomic dynamics. This strategy is distributed, only local relative position measurements are needed, and convergence to the desired formation is global. These advantages distinguish our approach from many existing works, such as the position-based or displacement-based methods [9], in which knowledge of a global coordinate frame or a common sense of orientation is required. By formulating a semidefinite programming (SDP) problem, formation control gains are initially designed for agents with the single-integrator model. We show that this design enjoys a robustness property, where if the agents move in the desired direction perturbed by a rotation up to $\pm 90°$, convergence to the desired formation is still guaranteed. Furthermore, the control can be augmented by an integrator term to reject constant input/output disturbances. Our analysis follows by considering agents with higher-order holonomic dynamics, where we show how the set of previously designed control gains can be used directly to achieve the formation. As an example, we use the proposed method for a team of quadrotors and present simulations to typify the theoretical results.

The linear formulation for the control used in this letter is inspired by [6] and [7]. The aforementioned work, and our previous work on formation control [10], [11], are based on agents with the single-integrator model. In [12], a SDP formulation has been considered, and extension of the control to agents with feedback-linearized dynamics is briefly discussed. In comparison, the SDP formulation proposed here provides robustness to the direction of motion, and the extension of the control law to higher-order dynamics is extensively studied.

In extending our work in distributed formation control, the main contributions of this letter include: 1) A novel gain design based on the SDP formulation, which provides robustness for the direction of motion; 2) Addition of integrator terms in the control to reject input/output disturbances; 3) Extension

Manuscript received March 6, 2018; revised May 8, 2018; accepted May 13, 2018. Date of publication May 29, 2018; date of current version June 14, 2018. This work was supported in part by the OSD Sponsored Autonomy Research Pilot Initiative, in part by RW Autonomy Initiative, in part by U.S. Air Force Research Laboratory under Grant FA8651-17-1-0001, and in part by U.S. Army Research Office under Grant W911NF-17-1-0058. Recommended by Senior Editor J. Daafouz. *(Corresponding author: Nicholas R. Gans.)*

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This paper has supplementary downloadable material available at http://ieeexplore.ieee.org, provided by the author.

Video of the simulation is available at https://youtu.be/POgYrZ8ymBQ, and the simulation code can be accessed at https://goo.gl/QH5qhw.

Digital Object Identifier 10.1109/LCSYS.2018.2841941

of the single-integrator control to agents with higher-order linear holonomic dynamics, or nonlinear dynamics that can be linearized via methods such as feedback linearization, or local approximation.

The organization of this letter is as follows. The notation and assumptions used in this letter are introduced in Section II. In Section III, the control strategy for agents with the single-integrator model is designed, and its robustness to perturbations and disturbances is studied. In Section IV-A, the control is extended to agents with the double-integrator and higher-order dynamics. Simulations are presented in Section V, followed by concluding remarks in Section VI.

II. Notation and Assumptions

We consider a team of $n \in \mathbb{N}$ agents with the inter-agent sensing topology described by an undirected graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} := \mathbb{N}_n$ is the set of vertices, and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges. Each vertex of the graph represents an agent. An edge from vertex $i \in \mathcal{V}$ to $j \in \mathcal{V}$ indicates that agents i and j can measure the relative position of each other in their local coordinate frames. In such a case, agents i and j are called neighbors. The set of neighbors of agent i is denoted by $\mathcal{N}_i := \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$. We denote by $\sigma(A) \subset \mathbb{C}$ the set of eigenvalues of matrix A.

The SDP gain design proposed in this letter, which allows robustness to the direction of motion, is based on the assumption that the sensing graph is undirected. However, extension of the control strategy to higher-order dynamics does not require this assumption. Gain design strategies for directed sensing topologies are presented in [7] and [12]. We assume that the desired formation is feasible and the sensing topology is such that the desired formation is realizable. The necessary and sufficient topological conditions for a formation to be realizable are studied in [7] and [13]. We do not control the scale of the formation, i.e., by "formation" we mean a desired geometric shape up to a scale factor. We assume that the sensing topology is fixed through all time, and collision avoidance is not considered. In Section VI, we provide some remarks on how to relax these assumptions.

III. Formation Control for Single-Integrator Agents

In this section, we briefly discuss the distributed formation control strategy that was introduced in [6] and [7] for agents with the single-integrator dynamics. We then propose a novel gain design approach based on formulating a SDP problem. These gains will be used to control agents with more complicated dynamics in the next section.

A. Control for Agents With the Single-Integrator Model

Consider n agents with the single integrator dynamics

$$\dot{q}_i = u_i, \qquad i \in \{1, 2, \ldots, n\}, \tag{1}$$

where $q_i := [x_i, y_i]^T \in \mathbb{R}^2$ is the coordinate of agent i in a common global coordinate frame (unknown to agents), and u_i is the control law, that is chosen as

$$u_i := \sum_{j \in \mathcal{N}_i} A_{ij} \left(q_j - q_i\right), \tag{2}$$

where $A_{ij} \in \mathbb{R}^{2\times 2}$ are constant control gain matrices to be determined. By constraining the gain matrices to the form

$$A_{ij} := \begin{bmatrix} a_{ij} & b_{ij} \\ -b_{ij} & a_{ij} \end{bmatrix}, \qquad a_{ij}, b_{ij} \in \mathbb{R}, \tag{3}$$

it can be shown that the closed-loop dynamics with coordinates q_i and q_j expressed in agents' local coordinate frames are identical to the case that coordinates are expressed in a common global frame (for more details see [11]). Therefore, while the implementation is distributed and uses the local relative position measurements, the control strategy can be designed and analyzed in a global coordinate frame.

Let $q := [q_1^\top, q_2^\top, \ldots, q_n^\top]^\top \in \mathbb{R}^{2n}$, and $u := [u_1^\top, u_2^\top, \ldots, u_n^\top]^\top \in \mathbb{R}^{2n}$ denote the aggregate state and control vectors of all agents, respectively. Using this notation, the closed-loop dynamics can be expressed as

$$\dot{q} = A\, q, \tag{4}$$

where

$$A = \begin{bmatrix} -\sum_{j=2}^{n} A_{1j} & A_{12} & \cdots & A_{1n} \\ A_{21} & -\sum_{j=1, j\neq 2}^{n} A_{2j} & \cdots & A_{2n} \\ \vdots & & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & -\sum_{j=1}^{n-1} A_{nj} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}, \tag{5}$$

is the aggregate state matrix that consists of A_{ij}'s, and has block Laplacian structure. Note that if $j \notin \mathcal{N}_i$, then A_{ij} in (5) is zero. Also, the 2×2 diagonal blocks are the negative sum of the rest of the blocks on the same row. From the block Laplacian structure of A, it follows that vectors

$$\begin{aligned} \mathbf{1} &:= [1,\ 0,\ 1,\ 0,\ \ldots,\ 1,\ 0]^\top \in \mathbb{R}^{2n} \\ \bar{\mathbf{1}} &:= [0,\ 1,\ 0,\ 1,\ \ldots,\ 0,\ 1]^\top \in \mathbb{R}^{2n} \end{aligned} \tag{6}$$

are in the kernel[1] of A.

Let $q^* \in \mathbb{R}^{2n}$ denote the coordinates of agents in an arbitrary embedding of the desired formation. That is, coordinates of agents at their desired formation in a coordinate frame that is chosen arbitrarily. Further, let $\bar{q}^* \in \mathbb{R}^{2n}$ be the coordinates of agents in this embedding when the formation is rotated by 90 degrees about the origin.

Theorem 1: Consider a team of n single-integrator agents with closed-loop dynamics (4). Assume that A is such that

- Vectors $\mathbf{1}$, $\bar{\mathbf{1}}$, q^* and $\bar{q}^*$ are in the kernel of A.
- Other than the four zero eigenvalues associated with these eigenvectors, the remaining eigenvalues of A have negative real parts.

Then, starting from any initial condition, agents converge to the desired formation up to a rotation, translation, and a non-negative scale factor.

Proof of Theorem 1 follows from the following Lemma, which is well-known from the linear systems theory (for more details on the proof see [11, Th. 1]).

Lemma 1: Suppose that nonzero eigenvalues of A have negative real parts. Then, all trajectories of $\dot{q} = A\, q$ exponentially converge to the kernel of A.

[1]If $A \in \mathbb{R}^{n\times n}$, the kernel or null space of A is defined as $\ker(A) := \{v \in \mathbb{R}^n \mid A\, v = 0\}$.

The conclusion of Theorem 1 follows from Lemma 1 and noting that the kernel of A is nothing but all rotations, translations, and non-negative scale factors of the desired formation. Note that the kernel vectors $\mathbf{1}, \bar{\mathbf{1}}$ correspond to the case where all agents coincide, which can be considered as the desired formation achieved with zero scale. Further, note that if agents are not initially coinciding, they will never converge to this coinciding equilibrium. To find a gain matrix that meets the conditions of the Theorem, one can formulate an optimization problem.

Remark 1: The topological conditions that guarantee the existence of a symmetric matrix A satisfying the conditions of Theorem 1 are studied in [12, Th. 3.2], which presents the necessary and sufficient condition that the sensing graph is universally rigid. Throughout this letter, we assume that this condition is met.

B. Control Gain Design via SDP

Given a desired formation, we proceed by showing how a stabilizing gain matrix can be designed to meet the conditions of Theorem 1. Let q^* and $\bar{q}^*$ respectively denote the coordinates of agents in an arbitrary embedding of the desired formation and 90° rotated desired formation. Define $N := [q^*, \bar{q}^*, \mathbf{1}, \bar{\mathbf{1}}] \in \mathbb{R}^{2n\times 4}$, where $\mathbf{1}, \bar{\mathbf{1}}$ are given in (6). Notice that N is a set of bases for the kernal of A. Let $U S V^\top = N$ be the (full) singular value decomposition (SVD) of N, where

$$U = \left[\bar{Q},\ Q\right] \in \mathbb{R}^{2n\times 2n}, \tag{7}$$

with $Q \in \mathbb{R}^{2n\times(2n-4)}$ being the last $2n-4$ columns of U.

Lemma 2: Using Q in (7), define

$$\bar{A} := Q^\top A\, Q \in \mathbb{R}^{(2n-4)\times(2n-4)}. \tag{8}$$

Matrices A and $\bar{A}$ have the same set of nonzero eigenvalues.

Proof of Lemma 2 follows by observing that U is an orthogonal matrix, and $\mathrm{range}(\bar{Q}) = \mathrm{range}(N)$. Therefore $\bar{A}$ is the projection of A onto the orthogonal complement of $\mathrm{range}(N)$. Effectively, the projection (8) removes the zero eigenvalues of A and allows us to formulate the stability of A in terms of $\bar{A}$.

For an undirected sensing topology, matrix A can be designed to be symmetric by imposing the constraints $a_{ij} = a_{ji}$, $b_{ij} = -b_{ji}$ in (3). In this case, $\bar{A}$ is symmetric, and its eigenvalues are real and can be ordered. Hence, A can be computed by solving the optimization problem

$$\begin{aligned} A = \underset{a_{ij},\, b_{ij}}{\mathrm{argmax}} \quad & \lambda_1\left(-\bar{A}\right) \\ \text{subject to} \quad & A\,N = 0 \end{aligned} \tag{9}$$

where $\lambda_1(\cdot)$ denote the smallest eigenvalue of a matrix.[2] Note that (9) is a concave maximization problem [14], and can be formulated as the SDP problem

$$\begin{aligned} A = \underset{a_{ij},\, b_{ij},\, \gamma}{\mathrm{argmax}} \quad & \gamma \\ \text{subject to} \quad & \bar{A} + \gamma\, I \preceq 0 \\ & A\,N = 0 \end{aligned} \tag{10}$$

where the first constraint is a linear matrix inequality. In recent years, effective algorithms for numerically solving SDPs have been developed and are now available [15]. For our simulations, we used CVX [16], which is available free online, to solve problem (9).

[2] This objective effectively maximizes convergence rate to the desired formation. However, any convex objective could be used, and there are several interesting possibilities that we are exploring in ongoing work. For example, $\mathrm{trace}(\bar{A}^{-1})$ is convex and relates closely to the average dispersion around the desired formation in the presence of an additive stochastic disturbance.

C. Robustness to Perturbations

In practice, due to noise, disturbances, unmodeled dynamics, etc., often agents do not perfectly move along the desired control direction. Thus, it is important to analyze the stability and convergence properties of a control strategy under perturbations. The following theorem provides an upper bound for the perturbations under which convergence to the desired formation is unaffected.

Theorem 2: Consider a symmetric gain matrix A designed for single-integrator agents. Let $R_i \in \mathrm{SO}(2)$ denote a rotation matrix of α_i degrees, and $c_i \in \mathbb{R}$ be a scalar. If $\alpha_i \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $c_i > 0$, under the perturbed control

$$u_i := c_i\, R_i \sum_{j\in\mathcal{N}_i} A_{ij}\, \left(q_j - q_i\right) \tag{11}$$

agents achieve the desired formation.

Proof: We will use Definition 1 and Lemmas 3, 4, 5 that are given in the Appendix. Under the perturbed control (11), the aggregate dynamics can be represented by $\dot{q} = R A\, q$, where $R := \mathrm{diag}(c_1 R_1,\, c_2 R_2,\, \ldots,\, c_n R_n) \in \mathbb{R}^{2n\times 2n}$ is a block diagonal matrix that contains the perturbation terms. Due to the special block structure of A and R, they can equivalently be represented in complex notation by denoting the 2×2 blocks $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ as a complex number $a + \iota\, b \in \mathbb{C}$. In this notation, diagonal entries of $R \in \mathbb{C}^{n\times n}$ are $\cos(\alpha_i) + \iota\, \sin(\alpha_i)$, and since $\alpha_i \in (-\frac{\pi}{2}, \frac{\pi}{2})$, have positive real parts. This, together with Lemma 4, implies that $\mathcal{F}(R)$ is contained in the right hand plane (RHP). By design, the complex representation of $A \in \mathbb{C}^{n\times n}$ is Hermitian and negative semidefinite. Thus, $-A$ is positive semidefinite, and from Lemma 5 we conclude that $\sigma(-RA)$ is contained in the union of the RHP and the imaginary axis. Thus, RA is a stable matrix, and trajectories of $\dot{q} = RA\, q$ converge to the kernel of RA. Since R is full-rank, the null space of A and RA are identical, which shows that the desired formation is achieved. ■

D. Robustness to Constant Input/Output Disturbances

In practice, the control law (2) may not be sufficient to achieve a desired formation. For example, consider a team of ground robots that experience floor friction. When the magnitude of control u_i is less than the friction forces, the robots stop moving toward the desired direction. By adding an integrator term to the control, steady state errors induced by the friction or other constant disturbances can be canceled.

This augmented control can be defined by $u = k_1 A\, q + k_0 \int_0^t A\, q(\tau)\, d\tau$, or in the state space form

$$u = k_1 A\, q + g,$$
$$\dot{g} = k_0 A\, q, \tag{12}$$

where $k_0,\ k_1 \in \mathbb{R}$ are scalar control gains, and $g \in \mathbb{R}^{2n}$ is the state of the integrator. Under (12), the close-loop dynamics is given by

$$\begin{bmatrix} \dot{q} \\ \dot{g} \end{bmatrix} = \underbrace{\begin{bmatrix} k_1 A & I \\ k_0 A & 0 \end{bmatrix}}_{\bar{A}} \begin{bmatrix} q \\ g \end{bmatrix}, \tag{13}$$

where $I \in \mathbb{R}^{2n \times 2n}$ is the identity matrix.

Theorem 3: If for all nonzero $\mu \in \sigma(A)$ roots of the quadratic equation

$$\lambda^2 - k_1 \mu\, \lambda - k_0 \mu = 0 \tag{14}$$

have negative real parts, then under the control (12), agents globally converges to the desired formation.

Proof: We need to show that the nonzero eigenvalues of the closed-loop state matrix $\bar{A}$ in (13) are stable. Eigenvalues of $\bar{A}$ are roots of the characteristic equation

$$\det(\lambda I - \bar{A}) = \det\left(\begin{bmatrix} \lambda I - k_1 A & -I \\ -k_0 A & \lambda I \end{bmatrix}\right) \tag{15}$$
$$= \det(\lambda\, I)\, \det\Big(\lambda I - k_1 A - (-I)(\lambda I)^{-1}(-k_0 A)\Big)$$
$$= \lambda^{2n} \det\left(\frac{1}{\lambda}(\lambda^2 I - \lambda\, k_1 A - k_0 A)\right)$$
$$= \det\Big(\lambda^2 I - \lambda\, k_1 A - k_0 A\Big). \tag{16}$$

If μ is an eigenvalue of A, from Lemma 6 in the Appendix and (16) it follows that the eigenvalues of $\bar{A}$ are roots of the quadratic equation $\lambda^2 - k_1 \mu \lambda - k_0 \mu = 0$, where by the condition of the theorem have negative real parts. ■

Corollary 1: If A is symmetric, for all $k_0, k_1 > 0$ roots of (14) have negative real parts.

Proof: Let $\mu = -\alpha + \iota \beta \in \mathbb{C}$, $\alpha > 0$, be a nonzero eigenvalue of A. When A is symmetric, μ is real, i.e., $\beta = 0$. From the Routh-Hurwitz stability criterion, it follows that $\forall k_0, k_1 > 0$ roots of $\lambda^2 + k_1 \alpha \lambda + k_0 \alpha = 0$ have negative real parts. ■

Theorem 4: Control (12) rejects all constant input disturbances. Moreover, it rejects constant output disturbances that are identical for all agents.[3]

Proof: The input disturbance can be modeled by

$$u = k_1 A\, q + g + d,$$
$$\dot{g} = k_0 A\, q, \tag{17}$$

where $d \in \mathbb{R}^{2n}$ is an unknown constant disturbance vector. The closed-loop system under this control is

$$\begin{bmatrix} \dot{q} \\ \dot{g} \end{bmatrix} = \begin{bmatrix} k_1 A & I \\ k_0 A & 0 \end{bmatrix} \begin{bmatrix} q \\ g \end{bmatrix} + \begin{bmatrix} d \\ 0 \end{bmatrix}, \tag{18}$$

[3]Such output disturbance can model a constant bias on relative position measurements acquired by agents.

where by setting the left hand side equal to zero, the equilibrium equations

$$k_1 A\, q + g + d = 0, \tag{19}$$
$$k_0 A\, q = 0, \tag{20}$$

are derived. From (20), we have that $q \in \ker(A)$, and hence the desired formation is achieved. Replacing $A\, q$ by 0 in (19) gives $g = -d$, which indicates that the integrator has compensated the unknown disturbance.

Let $d \in \mathbb{R}^{2n}$ denote a constant output disturbance vector that is identical for all agents. That is, if $d_i, d_j \in \mathbb{R}^2$ are output disturbances associated with agents i, j, then $d_i = d_j$. Under this disturbance, the input can be represented by

$$u = k_1 A\, (q + d) + g,$$
$$\dot{g} = k_0 A\, (q + d). \tag{21}$$

Due to the block Laplacian structure of A, we have that $d \in \ker(A)$, and thus $A\, d = 0$. This implies that (21) is equivalent to (12), and dynamics of the system remain unaffected. ■

IV. Formation Control for Agents With Higher-Order Model

In this section, we extend the control strategy to agents with the double-integrator and higher-order dynamics. By using the control gains designed for agents with the single-integrator model, we unify the design and avoid having to formulate and solve a new optimization problem.

A. Formation Control for Double-Integrator Agents

Dynamics of agents with the double-integrator model can be expressed (in the vector form) by $\dot{q} = v$, $\dot{v} = u$, where $v := [v_1^\top, v_2^\top, \ldots, v_n^\top]^\top \in \mathbb{R}^{2n}$ is the aggregate velocity vector, and q and u are respectively aggregate position and control vectors. Given matrix A designed for agents for the single-integrator model in Section III-B, the control law for double-integrator agents can be chosen as

$$u = k_0 A\, q + k_1 A\, v, \tag{22}$$

where $k_0,\ k_1 \in \mathbb{R}$ are scalar control gains. Note that due to the special structure of A, (22) can be implemented using only the local relative position and velocity measurements. The closed-loop dynamics under (22) is given by

$$\begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ k_0 A & k_1 A \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix}. \tag{23}$$

Due the similarity of (23) with (13), the proof of following theorem follows the proof of Theorem 3 and is omitted.

Theorem 5: If for all nonzero $\mu \in \sigma(A)$ roots of the quadratic equation $\lambda^2 - k_1 \mu \lambda - k_0 \mu = 0$ have negative real parts, then under the control (22), double-integrator agents globally converges to the desired formation. When A is symmetric, this condition is satisfied for all $k_0, k_1 > 0$.

B. Formation Control for Agents With Higher-Order Model

We now extend the control strategy to agents with higher-order models, where we assume that the aggregate dynamics of the agents can be expressed as

$$\begin{bmatrix} \dot{q} \\ \dot{q}^{(1)} \\ \vdots \\ \dot{q}^{(m-1)} \\ \dot{q}^{(m)} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & & I \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} q \\ q^{(1)} \\ \vdots \\ q^{(m-1)} \\ q^{(m)} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix} u. \tag{24}$$

We should point out that (24) encompasses a large class of agents with linear or nonlinear dynamics since by coordinate transformation techniques such as feedback linearization, or approximation techniques such as linearization, dynamics of the many systems can be expressed as (24).

Given the gain matrix A designed for agents with the single-integrator model, the control for agents with dynamics (24) can be chosen as

$$u = k_0\, A\, q + k_1\, A\, q^{(1)} + \cdots + k_m\, A\, q^{(m)}, \tag{25}$$

where $k_0, k_1, \ldots, k_m \in \mathbb{R}$ are scalar control gains. Under this control, the closed-loop dynamics is given by

$$\begin{bmatrix} \dot{q} \\ \dot{q}^{(1)} \\ \vdots \\ \dot{q}^{(m\text{-}1)} \\ \dot{q}^{(m)} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & & I \\ k_0A & k_1A & k_2A & \cdots & k_mA \end{bmatrix}}_{\bar{A}} \begin{bmatrix} q \\ q^{(1)} \\ \vdots \\ q^{(m\text{-}1)} \\ q^{(m)} \end{bmatrix}. \tag{26}$$

Note that (25) can be implemented locally using only the relative measurements (due to the special structure of A).

Theorem 6: If for all nonzero $\mu \in \sigma(A)$ roots of the polynomial equation

$$\lambda^{m+1} - k_m\, \mu\, \lambda^m - \cdots - k_1\, \mu\, \lambda - k_0\, \mu = 0 \tag{27}$$

have negative real parts, then under control (25), agents with dynamics (24) globally converge to the desired formation.

Proof: The closed-loop state matrix $\bar{A}$ given in (26) is in the controllable canonical form, and therefore its characteristic equation is given by

$$\lambda^{m+1}\, I - k_m\, \lambda^m\, A - \cdots - k_1\, \lambda\, A - k_0\, A = 0. \tag{28}$$

From Lemma 6 and the assumption of the theorem, the nonzero eigenvalues of $\bar{A}$ have negative real parts. ■

To find gains $k_0, k_1, \ldots, k_m$ that satisfy the condition of Theorem 6, the Routh-Hurwitz criterion can be used.

Remark 2: In above analysis, the control can alternatively be chosen as $u = k_0\, A\, q + k_1\, I\, q^{(1)} + \cdots + k_m\, I\, q^{(m)}$ to eliminate relative measurements of $q^{(1)}, \ldots, q^{(m)}$ from neighbors. Note that this control can also be implemented using only the local relative measurements.

Example 1 (Quadrotor Dynamics): The dynamics of a hovering quadrotor can be approximated using the linear model [17]

$$\begin{aligned} \delta\ddot{x}_i &= \ g\, \delta\theta_i & \delta\ddot{\theta}_i &= u_i^y \\ \delta\ddot{y}_i &= -g\, \delta\varphi_i & \delta\ddot{\varphi}_i &= u_i^x \\ \delta\ddot{z}_i &= \ u_i^a & \delta\ddot{\psi}_i &= u_i^z \end{aligned} \tag{29}$$

where δ represents a small displacement, g is the gravitational constant, x_i, y_i, z_i are the coordinates of the i'th quadrotor in the world frame, $\varphi_i, \theta_i, \psi_i$ are the Euler angles that describe the orientation of the quadrotor in the world frame, u_i^x, u_i^y, u_i^z are moment inputs applied to the airframe about corresponding body axes, and u_i^a is a mass-normalized thrust input.

Since we are interested in 2D formations, we only consider the lateral dynamics along the x-y axes, and assume that z is controlled separately to keep the quadrotors at a constant altitude. To represent the dynamics in the canonical form (24), let us define $\delta\bar{\theta}_i := g\, \delta\theta_i$, $\delta\bar{\varphi}_i := -g\, \delta\varphi_i$, $\bar{u}_i^y := g\, u_i^y$, $\bar{u}_i^x := -g\, u_i^x$. Using this notation, (29) can be described in the vector form as

$$\dot{p}_i = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{bmatrix} p_i + \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix} u_i \tag{30}$$

where $p_i := [\delta x_i, \delta y_i, \delta\dot{x}_i, \delta\dot{y}_i, \delta\bar{\theta}_i, \delta\bar{\varphi}_i, \delta\dot{\bar{\theta}}_i, \delta\dot{\bar{\varphi}}_i]^\top$ and $u_i := [\bar{u}_i^y, \bar{u}_i^x]^\top$ are respectively the state and control vectors, and $I \in \mathbb{R}^{2\times 2}$ is the identity matrix. Defining the aggregate position vector as $q = [\delta x_1, \delta y_1, \ldots, \delta x_n, \delta y_n]^\top$ yields dynamics of the form (24). This model will be used in the next section to achieve a desired formation.

V. Simulations

To validate the proposed strategy, a simulation with 9 quadrotors is performed. The desired formation is defined as a square grid and can be seen in Fig. 1(e), where the sensing graph among agents is shown by gray lines connecting the quadrotors. Notice that this sensing graph is undirected and is fixed throughout the simulation.

Stabilizing control gains associated to the desired formation are computed from the optimization routine explained in Section III-B. The nonzero eigenvalues of computed A matrix range from -0.72 to -10. The control law used for each quadrotor is chosen according to (25), where gains are set as $k_0 = 0.6$, $k_1 = 2$, $k_2 = 3$, $k_3 = 3$ to make the closed-loop state matrix $\bar{A}$ given in (26) stable. The original nonlinear quadrotor dynamics given in [17] is used in the simulation to test the control law.

The initial positions of the quadrotors are chosen randomly, as shown in Fig. 1(a). The locations of the quadrotors at other instances of time are shown in Figs. 1 (b)-(e). The proposed control strategy brings the agents to the desired formation. Notice that since the control only uses the local relative position measurements, the desired formation is achieved up to a rotation and translation. That is, the orientation of the square formation is not controlled. Furthermore, the scale of the formation is not controlled, and depends on the starting position

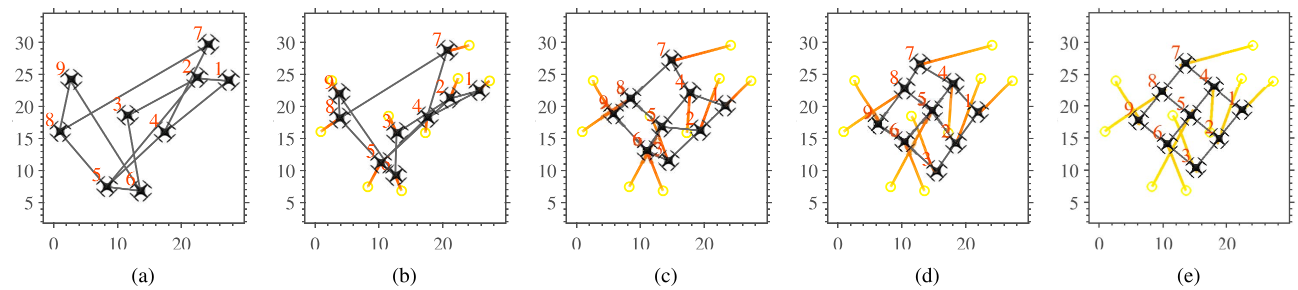

Fig. 1. Simulation of 9 quadrotors with a square grid desired formation. (a) Initial pose at $t = 0$s. (b) $t = 3$s. (c) $t = 5$s. (d) $t = 8$s. (e) $t = 20$s.

of the agents. Links to simulation code and video are provided in the Supplementary Material section.

VI. Concluding Remarks and Future Work

We presented a distributed formation control strategy for a team of agents to autonomously achieve a desired formation. We showed that formation control gains can be designed by solving a SDP problem. This design enjoys a robustness property, where positive scaling and rotation of the control vectors (up to $\pm 90°$) does not affect the stability. By augmenting the control with an integrator term, constant input/output disturbances were rejected. The control was extended to agents with higher-order linear (or linearizable) holonomic dynamics. Simulation of a quadrotor formation was presented to typify the proposed strategy.

The robustness property of SDP design allows agents to move along a rotated control direction. This property can be used to prevent collision among agents. Designing a *distributed* collision avoidance strategy based on this observation is a topic of future work. Throughout this letter, we assumed that the sensing topology is fixed, and scale of the desired formation is not controlled. We previously proposed a design strategy for time-varying sensing topologies and an augmented control to fix the scale of desired formation [11]. Incorporating these strategies for agents with higher-order dynamics will further be investigated.

Appendix

Definition 1: The *field of values* of matrix $A \in \mathbb{C}^{n\times n}$ is defined as $\mathcal{F}(A) := \{x^* A\, x \mid x \in \mathbb{C}^n,\ x^* x = 1\}$, where x^* is the conjugate transpose of x.

One can show that $\mathcal{F}(A)$ is a convex and compact subset of $\mathbb{C}$. The following Lemmas are proved in [18] and [19].

Lemma 3: Denote by $\sigma(A) \subset \mathbb{C}$ the set of eigenvalues of $A \in \mathbb{C}^{n\times n}$. Then $\sigma(A) \subset \mathcal{F}(A)$.

Lemma 4: If $A \in \mathbb{C}^{n\times n}$ is a diagonal matrix, $\mathcal{F}(A)$ is the convex hull of the diagonal entries (the eigenvalues) of A.

Lemma 5: Let $R \in \mathbb{C}^{n\times n}$ be nonsingular and $A \in \mathbb{C}^{n\times n}$ be a positive (semi-) definite matrix. Then $\sigma(RA)$ is contained in the RHP (union the imaginary axis) if and only if $\mathcal{F}(R)$ is contained in the union of the RHP and $\{0\}$.

Lemma 6: Let $p(\cdot)$ be a given polynomial. If μ is an eigenvalue of matrix A with v as the associated eigenvector, then $p(\mu)$ is an eigenvalue of the matrix $p(A)$ with v as the associated eigenvector.

References

[1] M. A. Goodrich *et al.*, "Supporting wilderness search and rescue using a camera-equipped mini UAV," *J. Field Robot.*, vol. 25, nos. 1–2, pp. 89–110, 2008.

[2] J. Zhang *et al.*, "Seeing the forest from drones: Testing the potential of lightweight drones as a tool for long-term forest monitoring," *Biol. Conserv.*, vol. 198, pp. 60–69, Jun. 2016.

[3] J. Scherer and B. Rinner, "Persistent multi-UAV surveillance with energy and communication constraints," in *Proc. Conf. Autom. Sci. Eng.*, Fort Worth, TX, USA: IEEE, 2016, pp. 1225–1230.

[4] L. Krick, M. E. Broucke, and B. A. Francis, "Stabilisation of infinitesimally rigid formations of multi-robot networks," *Int. J. Control*, vol. 82, no. 3, pp. 423–439, 2009.

[5] A. N. Bishop, T. H. Summers, and B. D. O. Anderson, "Stabilization of stiff formations with a mix of direction and distance constraints," in *Proc. IEEE Int. Conf. Control Appl.*, 2013, pp. 1194–1199.

[6] Z. Lin, L. Wang, Z. Han, and M. Fu, "Distributed formation control of multi-agent systems using complex Laplacian," *IEEE Trans. Autom. Control*, vol. 59, no. 7, pp. 1765–1777, Jul. 2014.

[7] Z. Lin, L. Wang, Z. Han, and M. Fu, "A graph Laplacian approach to coordinate-free formation stabilization for directed networks," *IEEE Trans. Autom. Control*, vol. 61, no. 5, pp. 1269–1280, May 2016.

[8] T. Han, R. Zheng, Z. Lin, and M. Fu, "A barycentric coordinate based approach to formation control of multi-agent systems under directed and switching topologies," in *Proc. IEEE Conf. Decis. Control*, Osaka, Japan, 2015, pp. 6263–6268.

[9] K.-K. Oh, M.-C. Park, and H.-S. Ahn, "A survey of multi-agent formation control," *Automatica*, vol. 53, pp. 424–440, Mar. 2015.

[10] K. Fathian, D. I. Rachinskii, M. W. Spong, and N. R. Gans, "Globally asymptotically stable distributed control for distance and bearing based multi-agent formations," in *Proc. Amer. Control Conf.*, Boston, MA, USA: IEEE, 2016, pp. 4642–4648.

[11] K. Fathian, D. I. Rachinskii, T. H. Summers, M. W. Spong, and N. R. Gans, "Distributed formation control under arbitrarily changing topology," in *Proc. Amer. Control Conf.*, Seattle, WA, USA, 2017, pp. 271–278.

[12] Z. Lin, L. Wang, Z. Chen, M. Fu, and Z. Han, "Necessary and sufficient graphical conditions for affine formation control," *IEEE Trans. Autom. Control*, vol. 61, no. 10, pp. 2877–2891, Oct. 2016.

[13] L. Wang, Z. Han, and Z. Lin, "Realizability of similar formation and local control of directed multi-agent networks in discrete-time," in *Proc. IEEE Conf. Decis. Control*, Florence, Italy, 2013, pp. 6037–6042.

[14] S. Boyd, "Convex optimization of graph Laplacian eigenvalues," in *Proc. Int. Congr. Math.*, vol. 3, 2006, pp. 1311–1319.

[15] R. H. Tütüncü, K.-C. Toh, and M. J. Todd, "Solving semidefinite-quadratic-linear programs using SDPT3," *Math. Program.*, vol. 95, no. 2, pp. 189–217, 2003.

[16] M. Grant and S. Boyd. (Mar. 2014). *CVX: MATLAB Software for Disciplined Convex Programming, Version 2.1*. [Online]. Available: http://cvxr.com/cvx

[17] F. Sabatino, "Quadrotor control: Modeling, nonlinear control design, and simulation," M.S. thesis, School Elect. Eng., KTH Roy. Inst. Technol., Stockholm, Sweden, 2015.

[18] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1990.

[19] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1994.