Performance bounds for optimal feedback control performance and robustness in networks

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***Abstract*—Important complex networks including critical infrastructure and emerging industrial automation systems are becoming increasingly intricate webs of interacting feedback control loops. A fundamental concern is to quantify the control properties and performance limitations of the network as a function of the structure of its dynamics and control architecture. We study performance bounds for networks in terms of both optimal and robust feedback control costs as a function of the system dynamics and actuator structure. For unstable network dynamics, we demonstrate a trade-off between feedback control performance and the number of control inputs, in particular showing that optimal cost can increase exponentially with the size of the network. Likewise, we demonstrate a trade-off between robustness and the numbers of control and adversarial inputs. We also derive a bound on the performance of the worst-case actuator subset for open-loop stable networks, providing insight into dynamics properties that affect the potential efficacy of actuator selection. However, we show that such a bound is generally not guaranteed even for open-loop stable networks in terms of robustness to adversarial inputs. When the open-loop network dynamics has a finite H_∞ norm with respect to the adversarial inputs, we find an analogous performance bound of the worst case actuator set. We illustrate our results with numerical experiments that analyze performance in regular and random networks.**

I. Introduction

Recent spectacular advances in computation and communication technologies are transforming our ability to control complex networked systems. Critical infrastructure, industrial automation systems and many other technological and social networks crucial to modern society are becoming increasingly intricate webs of interacting feedback loops. As this complexity increases, a fundamental concern is to quantify the control properties and performance limitations of the network as a function of the structure of its dynamics and control architecture.

A variety of metrics have been theorized to quantify notions of network controllability. Significant recent research has been devoted to studying connections between such notions and the structural properties of the network, and to studying algorithms for designing networks with good controllability and observability properties. One broad line of work has focused on classical binary controllability metrics based on Kalman rank [1]–[8]. Another line of work has focused on metrics based on the Gramian [9]–[15]. These binary and open-loop notions fail to capture essential feedback and robustness properties, and other recent work has considered more general optimal control and estimation metrics and algorithms [16]–[28].

An important part of understanding network controllability in terms of any metric is expressing fundamental performance limitations. A clear understanding of performance limitations can set practical expectations and guide the design and analysis of network control architectures. Recent work on performance limitations and network controllability include [9] in the context of the Gramian, and [21] in the context of sensor selection and Kalman filtering. However, to the best of our knowledge, no such studies have been done in a network context for more general optimal control metrics.

Here we study performance bounds for networks in terms of both optimal and robust feedback control costs as a function of the system dynamics and actuator structure. Our main contributions are as follows.

- We derive performance bounds for the linear quadratic regulator (LQR), showing that for open-loop unstable systems the optimal cost can increase exponentially with network size for any fixed-size actuator set (Theorem 1).
- For open-loop stable systems, we derive a bound on the performance of the worst-case actuator subset for open-loop stable networks, providing insight into dynamics properties that affect the potential efficacy of actuator selection (Theorem 2).
- We derive robustness bounds for open-loop unstable dynamics in the presence of adversarial inputs (Theorem 3) with a two-player zero-sum dynamic game (ZSDG) framework.
- We illustrate that there is not necessarily any guarantee for finite cost for open-loop stable networks under adversarial inputs. In this case, we derive a relationship between open-loop and closed-loop stable dynamics of a system under attack in terms of the H_∞ bounds of the system and then derive the controller performance bounds (Theorem 4).
- We present extensive numerical experiments on regular and random networks to illustrate the relationship between cardinality and position of actuators and attackers in networks, network size, open-loop stability and the optimal or equilibrium cost of the system. A key conclusion is the importance of the position of the actuators in determining network control performance and robustness, especially scaling with network size.

A preliminary version of this work appeared in [29], which featured performance bounds only for LQR performance

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metrics. We significantly extend this preliminary version by developing analogous bounds for network robustness using a two-player zero-sum dynamic game (ZSDG) framework that incorporates the effects of strategic adversarial inputs, and we significantly expand the numerical experiments. We conclude in Section VI with a brief summary of the results from the paper and possible future directions of research.

II. Network Description for Optimal Control and Robustness Performance Metrics

We begin by formulating actuator selection problems based on optimal feedback control performance and robustness for linear dynamical networks with quadratic costs.

A. Network performance for optimal control

The network dynamics are modeled by the discrete-time linear dynamical system evolving on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

$$x_{t+1} = Ax_t + B_S u_t, \quad t = 0, ..., T, \tag{1}$$

where $x_t \in \mathbf{R}^n$ is the system state at time t, $u_t \in \mathbf{R}^{|S|}$ is the input at time t, and A is the network dynamics matrix, which encodes the weighted connections in the underlying graph $\mathcal{G}$ and we assume to be invertible throughout. Let $\mathcal{B} = \{e_1, ..., e_M\}$ be a finite set of the canonical column vectors in $\mathbf{R}^n$ associated with possible locations for actuators that could be placed in the network to affect the dynamics of nodes in the graph. For any subset $S \subset \mathcal{B}$, the input matrix B_S comprises the columns indexed by S, i.e., $B_S = [e_{s_1}, ..., e_{s_{|S|}}] \in \mathbf{R}^{n \times |S|}$.

We first consider an optimal open-loop linear quadratic regulator performance index associated with an input sequence $\mathbf{u} = [u_0^\top, ..., u_{T-1}^\top]^\top$. The optimal cost function is

$$V^*_{\text{LQR}}(S, x_0) = \min_{\mathbf{u}} \sum_{t=0}^{T-1} (x_t^\top Q x_t + u_t^\top R_S u_t) + x_T^\top Q_T x_T \tag{2}$$

where $Q \succeq 0$ and $Q_T \succ 0$ are state and terminal cost matrices and $R_S \succ 0$ is an input cost matrix associated with actuator subset S.

This standard least squares problem has the solution

$$V^*_{\text{LQR}}(S, x_0) = x_0^\top \underbrace{G^\top (I + H\mathbf{B}_S \mathbf{B}_S^\top H^\top)^{-1} G}_{P_0} x_0 \tag{3}$$

where

$$H = \text{diag}(I \otimes Q^{\frac{1}{2}}, Q_T^{\frac{1}{2}}) \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ I & 0 & 0 & \cdots & 0 \\ A & I & 0 & \cdots & 0 \\ A^2 & A & I & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ A^{T-1} & A^{T-2} & \cdots & A & I \end{bmatrix},$$

$$G = \text{diag}(I \otimes Q^{\frac{1}{2}}, Q_T^{\frac{1}{2}}) \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^T \end{bmatrix}, \mathbf{B}_S = \text{diag}(B_S R_S^{-\frac{1}{2}}).$$

Alternatively, dynamic programming can be used to compute the optimal cost matrix P_0 via the backward Riccati recursion

$$P_{t-1} = Q + A^\top P_t A - A^\top P_t B_S (R_S + B_S^\top P_t B_S)^{-1} B_S^\top P_t A, \tag{4}$$

for $t = T, ..., 1$ with $P_T = Q_T$. The infinite horizon cost matrix P can be computed from the limit of the recursion, resulting in the algebraic Riccati equation

$$P = Q + A^\top PA - A^\top PB_S(R_S + B_S^\top PB_S)^{-1} B_S^\top PA. \tag{5}$$

The optimal cost function (3) also quantifies feedback control performance as a function of the actuator subset and the initial state. Our performance bounds will be expressed in terms of worst-case and average values of this cost over initial states. In particular, we define

$$\begin{aligned} \hat{J}_{\text{LQR}}(S) &= \max_{\|x_0\|=1} V^*(S, x_0) = \lambda_{\max}(P_0) \\ J^*_{\text{LQR}}(S) &= \mathbf{E}_{x_0} V^*(S, x_0) = \mathbf{tr}[P_0 X_0], \end{aligned} \tag{6}$$

where $\hat{J}_{\text{LQR}}(S)$ represents a worst-case cost and $J^*_{\text{LQR}}(S)$ represents an average cost over a distribution of initial states with zero-mean and finite covariance X_0.

Actuator selection. The mappings $J^*_{\text{LQR}} : 2^{\mathcal{B}} \rightarrow \mathbf{R}$ and $\hat{J}_{\text{LQR}} : 2^{\mathcal{B}} \rightarrow \mathbf{R}$ shown above are set functions that map actuator subsets to optimal feedback control performance. We pose set function optimization problems to select a k-element subset of actuators to optimize control performance

$$\min_{S \subset \mathcal{B},\ |S|=k} \hat{J}_{\text{LQR}}(S), \qquad \min_{S \subset \mathcal{B},\ |S|=k} J^*_{\text{LQR}}(S). \tag{7}$$

Our performance bounds will also be expressed and interpreted in terms of actuator subset selections.

B. Network robustness metrics

The network dynamics are modeled by the discrete-time linear dynamical system evolving on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

$$x_{t+1} = Ax_t + B_S u_t + F_{S_a} v_t \tag{8}$$

where here $v_t \in \mathbf{R}^{|S_a|}$ represents the input of a strategic adversary. Note that the adversary need not represent an actual malicious input, but can be viewed as a worst-case disturbance or model perturbation. Let $\mathcal{F} = \{e_1, ..., e_{M_a}\}$ be a finite set of the canonical column vectors in $\mathbf{R}^n$ associated with possible locations for the adversarial inputs to affect the dynamics. For any subset $S_a \in \mathcal{F}$, the input matrix F_{S_a} comprises of a possibly empty input matrix or columns indexed by S_a, i.e., $F_{S_a} = [e_{s_{a,1}}, ..., e_{s_{a,|S_a|}}] \in \mathbf{R}^{n \times |S_a|}$. We consider a zero-sum dynamic game between the control input and the adversary. In zero-sum dynamic games, the appropriate equilibrium concept is the saddle point, whose value is given by

$$\begin{aligned} V^*_{\text{ZSDG}}(S, S_a, x_0) = \min_{\mathbf{u}} \max_{\mathbf{v}} \sum_{t=0}^{T-1} (x_t^\top Q x_t + u_t^\top R_S u_t \\ - \gamma^2 v_t^\top v_t) + x_T^\top Q x_T, \end{aligned} \tag{9}$$

where $\mathbf{v} = [v_0^\top, ..., v_{T-1}^\top]^\top$ and γ is a parameter penalizing the adversarial input. The saddle point value in open-loop strategies can also be computed by solving least squares problems and is given by

$$\begin{aligned} V^*_{\text{ZSDG}}(S, S_a, x_0) &= x_0^\top G^\top (I + H\mathbf{B}_S\mathbf{B}_S^\top H^\top \\ &\quad - \gamma^{-2} H\mathbf{F}\mathbf{F}^\top H^\top)^{-1} G x_0 \\ &= x_0^\top Z_0 x_0 \end{aligned} \tag{10}$$

where $\mathbf{F} = \text{diag}(F_{S_a})$[1]. The optimal cost matrix Z_0 can be computed through a generalized backwards Riccati recursion

$$Z_{t-1} = Q + A^\top Z_t[I + (B_S B_S^\top - \gamma^{-2} F_{S_a} F_{S_a}^\top) Z_t]^{-1} A \tag{11}$$

with $Z_T = Q_T$ and $t = T, T-1, \dots 1$. Over an infinite horizon, the cost matrix Z can be computed from the limit of the recursion. This results in the algebraic Riccati equation

$$Z = Q + A^\top Z[I + (B_S B_S^\top - \gamma^{-2} F_{S_a} F_{S_a}^\top) Z]^{-1} A. \tag{12}$$

It is well known that the optimal open-loop input sequences and closed-loop feedback strategies yield the same state trajectories and associated saddle point equilibrium value [30].

The adversary penalty parameter, γ, determines the existence of an upper value for the game. It can be interpreted as a quantification of model uncertainty, with large γ representing small model uncertainty and smaller γ representing larger model uncertainty. We can see from (10) that as $\gamma \to \infty$ the game equilibrium value equals the optimal LQR value. As γ decreases, the equilibrium cost increases; there is a critical value, which we denote by $\gamma^*(S, S_a)$, below which the upper value of the game is infinite. This value represents the best possible closed-loop $\mathcal{H}_\infty$ norm from the adversarial input to the state, which quantifies the achievable robustness of the network in closed-loop. Clearly, this value depends on the actuator subsets S and S_a.

We thus define the following network robustness metrics

$$\begin{aligned} \hat{J}_{\text{ZSDG}}(S, S_a) &= \max_{\|x_0\|=1} V^*(S, S_a, x_0) = \lambda_{\max}(Z_0) \\ J^*_{\text{ZSDG}}(S, S_a) &= \mathbf{E}_{x_0} V^*(S, S_a, x_0) = \mathbf{tr}[Z_0 X_0] \\ J^*_{\mathcal{H}_\infty}(S, S_a) &= \gamma^*(S, S_a), \end{aligned} \tag{13}$$

where $\hat{J}_{\text{ZSDG}}(S, S_a)$ represents a worst-case equilibrium cost, $J^*_{\text{ZSDG}}(S, S_a)$ represents an average equilibrium cost over a distribution of initial states with zero-mean and finite covariance X_0, and $J^*_{\mathcal{H}_\infty}(S, S_a)$ represents a robustness level (with lower being better).

Actuator selection. The mappings J^*_{ZSDG}, $\hat{J}_{\text{ZSDG}}$, and $J^*_{\mathcal{H}_\infty}$ are set functions that map actuator subsets to network robustness. We pose set function optimization problems to select a k-element subset of actuators to optimize robustness as

$$\min_{S\subset\mathcal{B},\ |S|=k} \hat{J}_{\text{ZSDG}}(S, S_a), \qquad \min_{S\subset\mathcal{B},\ |S|=k} J^*_{\text{ZSDG}}(S, S_a), \qquad \min_{S\subset\mathcal{B},\ |S|=k} J^*_{\mathcal{H}_\infty}(S, S_a) \tag{14}$$

Our robustness bounds will also be expressed and interpreted in terms of actuator subset selections.

[1]For the dynamic game we assume $R_S = \mathbf{I}$, which is without loss of generality since $u_t^\top R_S u_t$ can be written as $\tilde{u}_t^\top \tilde{u}_t$ with $\tilde{u}_t = R_S^{\frac{1}{2}} u_t$.

Notation. The eigenvalues of a square matrix A are denoted by $\lambda_i(A)$ and ordered $|\lambda_{\max}(A)| = |\lambda_1(A)| \geq |\lambda_2(A)| \geq \cdots \geq |\lambda_n(A)| = |\lambda_{\min}(A)|$. The singular values of a matrix F are denoted and ordered as $\sigma_1(F) \geq \sigma_2(F) \geq \cdots \geq \sigma_n(F)$. The condition number of a matrix V is denoted cond(V).

III. Bounds on Optimal Feedback Control Performance

We now develop a set of complementary bounds on the optimal feedback control performance in networks as a function of the system dynamics and the actuator subset S. We start with a worst-case lower bound for the best possible actuator subset selection for unstable networks. This result shows that the optimal cost can be exponentially large even with the best fixed-size set of actuators. We then derive a worst-case upper bound for the worst possible actuator subset selection for stable networks. This result shows that even the worst set of actuators cannot have arbitrarily bad performance. Our results are inspired by bounds for the controllability Gramian [9] and an analogous bound for the Kalman filter in the context of sensor selection for state estimation [21].

A. Performance bound for unstable network dynamics

We begin with the following performance bound on optimal feedback control of networks with unstable open-loop network dynamics for a Linear Quadratic Regulator. To simplify the exposition, we will assume throughout this subsection that $\mathcal{B} = \{e_1, ..., e_n\}$, the canonical set of unit vectors (i.e., each input signal affects the dynamics of a single node), and that $R_S = I$, $\forall S$. However, it is straightforward to generalize the results to arbitrary input vectors and cost matrices. We focus here on the infinite horizon cost given by the algebraic Riccati equation (5).

Theorem 1. *Consider a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with dynamics matrix A and input set $S \subset \mathcal{B}$. Suppose that A is Schur unstable and let $\lambda_{\max}(A) > 1$ denote the eigenvalue of A with maximum magnitude. Suppose further that A is diagonalizable by the eigenvector matrix V, and for any $\eta \in (1, \lambda_{\max}(A)]$ define $\bar{n} = |\{\lambda : \lambda \in spec(A), |\lambda| \geq \eta\}|$. For all $\eta \in (1, \lambda_{\max}(A)]$ and for any $Q \succeq 0$ such that $(A, Q^{\frac{1}{2}})$ is detectable, it holds*

$$\lambda_{\max}(P) \geq cond^{-2}(V) \frac{\eta^2 - 1}{\eta^2} \eta^{2(\lceil \frac{\bar{n}}{|S|} \rceil - 1)}, \tag{15}$$

where P is the optimal closed-loop cost matrix that satisfies the algebraic Riccati equation (5).

Proof. Given in Appendix 0a. □

Discussion. Although our result is inspired by and utilizes a bound on the minimum eigenvalue of the controllability Gramian in [9], we emphasize that it is not a trivial inversion of their bound. The Gramian quantifies input energy required for state transfer from the origin, so that a limiting feature of the dynamics is stable modes. In contrast, the optimal cost matrix quantifies input energy and state regulation costs (to the origin) for feedback control, and a limiting feature of the dynamics is unstable modes. Of course, this is as expected, but one arrives at significantly different conclusions about how easy or difficult it is to control a network, depending on which quantitative notion of network controllability is used. Our bound involves a fundamental *closed-loop, feedback* notion of controllability.

The bound expresses a fundamental performance limitation for feedback control of networks with unstable dynamics. Specifically, if the number of unstable modes grows, then the feedback control costs increase exponentially for any fixed-size set of actuators, even if they are optimally placed in the network. An immediate corollary (cf. Corollary 3.3 in [9]) is that in order to guarantee a bound on the optimal control cost, the number of actuators must be a linear function of the number of unstable modes, even though a single actuator may suffice to stabilize the network dynamics in theory. As in [9] and as we will see in our numerical experiments, the bound is loose in many cases, so that large costs can be incurred even with a small number of unstable modes.

There are several ways the bound might be improved. It only accounts for the number of actuators, and not how effectively they control crucial state space dynamics. It could be improved, for example, by incorporating the angles that the input vectors make with the left eigenvectors of the dynamics matrix. Furthermore, the bound excludes the contribution of state regulation costs, so a sharper bound could be developed that includes and distinguishes both. It would also be interesting to explore possible connections with classical frequency domain performance limitations, such as Bode sensitivity theorems.

We conclude this subsection with a corollary that expresses a simplified bound for symmetric networks.

Corollary 1. *Consider a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with dynamics matrix A and input set $S \subset \mathcal{B}$. Suppose that A is Schur unstable and symmetric. Let $\lambda_{\max}(A) > 1$ denote the eigenvalue of A with maximum magnitude and $\bar{\lambda}_u(A) > 1$ denote the unstable eigenvalue of A with minimum magnitude. For any $Q \succeq 0$ such that $(A, Q^{\frac{1}{2}})$ is detectable, it holds*

$$\lambda_{\max}(P) \geq \max\left\{ \frac{\lambda_{\max}(A)^2 - 1}{\lambda_{\max}(A)^2}, \frac{\bar{\lambda}_u(A)^2 - 1}{\bar{\lambda}_u(A)^2} \bar{\lambda}_u(A)^{2(\lceil \frac{n}{|S|} \rceil - 1)} \right\}. \tag{16}$$

Proof. Given in Appendix 0b. □

B. Performance bound for stable network dynamics

Next we derive a complementary performance bound for stable network dynamics. It establishes a worst case performance bound for actuator subsets produced by any selection algorithm and quantifies how the difference between the best and worst possible actuator subsets depends on the network dynamics. This analysis is inspired by analogous results for sensor selection in the context of a state estimation metric involving the Kalman filtering error covariance matrix [21]. We focus here on the infinite horizon cost given by the solution to the algebraic Riccati equation (5), though it is also straightforward to derive for finite horizon costs.

We consider the following ratio

$$r(P) = \frac{\mathbf{tr}(P_{\text{worst}})}{\mathbf{tr}(P_{\text{opt}})}, \tag{17}$$

where P_{worst} and P_{opt} are the solutions to the algebraic Riccati equation (5) corresponding to the optimal and worst k-element selection of actuators.

Analogous to the sensor information matrix defined in [21], we also define the *actuator influence matrix* $R(S) := B_S R_S^{-1} B_S^\top$ corresponding to an actuator subset $S \subseteq \mathcal{B}$. We have the following results, whose proofs follow directly along the lines of the analogous proofs of Theorem 3 and corresponding Corollary in [21], which we omit here due to space limitations. To prove the result, we will utilize the following lemmas.

Lemma 1 ([31]). *The solution $P \succeq 0$ of (5) with $Q \succ 0$ satisfies $P \succeq A^\top (Q^{-1} + R(S))^{-1} A + Q$.*

Lemma 2 ([32]). *For symmetric matrices $Y, Z \in \mathbf{R}^{n\times n}$, there holds $\lambda_n(Y + Z) \geq \lambda_n(Y) + \lambda_n(Z)$, $\lambda_1(Y + Z) \leq \lambda_1(Y) + \lambda_1(Z)$, and $\lambda_n(Y)\mathbf{tr}(Z) \leq \mathbf{tr}(YZ) \leq \lambda_1(Y)\mathbf{tr}(Z)$.*

Lemma 3 ([33]). *A square matrix $A \in \mathbf{R}^{n\times n}$ is Schur stable if and only if there exists a nonsingular matrix T such that $\sigma_1(TAT^{-1}) < 1$.*

Based on the similarity transformation T in Lemma 3, we define a positive constant which will appear in our bound:

$$\alpha_A = \frac{\sigma_1^2(T)}{\sigma_n^2(T)(1 - \sigma_1^2(TAT^{-1}))}. \tag{18}$$

Theorem 2. *Let $\mathcal{R} = \{R(S) \mid S \subset \mathcal{B},\ |S| \leq k\}$ be the set of all actuator influence matrices for actuator subsets with k or fewer elements. Let $\lambda_1^{max} := \max\{\lambda_1(R) \mid R \in \mathcal{R}\}$. Suppose the dynamics matrix A is stable and $Q \succ 0$. Then the cost ratio satisfies*

$$r(P) \leq \frac{\alpha_A(1 + \lambda_1^{\max}\lambda_n(Q))\mathbf{tr}(Q)}{n\sigma_n^2(A)\lambda_n(Q) + (1 + \lambda_1^{\max}\lambda_n(Q))\mathbf{tr}(Q)} \tag{19}$$

Proof. Given in Appendix 0c. □

We also state the following corollary, which provides a simplified bound for stable and normal dynamics matrices.

Corollary 2. *If the system dynamics matrix A is Schur stable, then $r(P) \leq \alpha_A$, where α_A is a constant that depends only on the network dynamics matrix. Moreover, if A is also normal, i.e., $A^\top A = AA^\top$, then*

$$r(P) \leq \frac{1}{1 - \lambda_1^2(A)}. \tag{20}$$

Proof. Given in Appendix 0d. □

Discussion. Although it is not surprising that such bounds should exist for stable networks, they provide insight into the properties of the dynamics matrix A that affect the potential efficacy of actuator selection. The effect is most clearly seen in Corollary 2, where we observe that the difference between worst and optimal increases as A approaches instability, confirming intuition. The bounds complement those in the previous subsection: here, even the worst k-element actuator selection cannot have arbitrarily bad performance for stable networks, whereas even the best selection may incur large costs in unstable networks. However, even in stable networks, effective actuator set selections (perhaps obtained with greedy algorithms [16]) can greatly improve feedback control costs.

IV. Bounds on Robustness via Zero-Sum Dynamic Games

In this section, we extend the analysis of performance bounds for the optimal controller problem to a more general robust performance metric. We quantify performance in terms of the equilibrium cost of the two-player, zero-sum dynamic game using the framework described in Section II-B and the adversarial input set S_a.

A. Performance bound for unstable network dynamics

We begin with a generalization of Theorem 1 that bounds the equilibrium value of the two-player zero-sum dynamic game (9) for networks with unstable open-loop dynamics.

Theorem 3. *Consider a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with dynamics matrix A, actuator control input set $S \subseteq \mathcal{B}$, adversary input set $S_a \subseteq \mathcal{F}$ and adversary penalty parameter γ. Suppose that A is Schur unstable and let $\lambda_{\max}(A) > 1$ denote the eigenvalue of A with maximum magnitude. Suppose further that A is diagonalizable by the eigenvector matrix V, and for any $\eta \in (1, \lambda_{\max}(A)]$ define $\bar{n} = |\{\lambda : \lambda \in spec(A), |\lambda| \geq \eta\}|$. For all $\eta \in (1, \lambda_{\max}(A)]$, $\Lambda_F = \lambda_{\max}(F_{S_a} F_{S_a}^\top)$ and for any $Q \succeq 0$ such that $(A, Q^{\frac{1}{2}})$ is detectable, it holds*

$$\lambda_{\max}(Z) \geq \frac{\eta^2 - 1}{\left[\begin{array}{r} cond^2(V)\left(\eta^{-2(\lceil \frac{\bar{n}}{|S|}\rceil - 2)} - \gamma^{-2}\eta^{-2(\lceil \frac{\bar{n}}{|S_a|}\rceil - 2)}\right) \\ + \gamma^{-2}\Lambda_F(\eta^2 - 1) \end{array}\right]} \tag{21}$$

where Z is the optimal closed-loop cost matrix that satisfies the infinite-horizon algebraic Riccati equation (12).

Proof. Given in Appendix 0e. □

Discussion. As $\gamma \to \infty$ note that (21) reduces to (15). This shows that this result is a two-player, zero-sum game generalization of Theorem 1. Moreover, as the penalty on the adversarial input decreases $\gamma \to \gamma^*$, the bound increases and eventually blows up. We also note that the bound can be loose, and it is possible to have extremely large costs even with a small number of adversarial inputs and relatively large adversarial input penalty, depending on the system dynamics and relative locations of actuators and adversarial input.

B. Performance bound for open-loop robust network dynamics

This section provides a bound on the equilibrium value of a two-player zero-sum game for networks with open-loop robust dynamics. In particular, we consider networks that have bounded cost in the presence of a set of adversarial inputs with a fixed adversary cost penalty γ (i.e., the $\mathcal{H}_\infty$-norm of the open-loop system without control inputs with respect to the adversary inputs is bounded by $\sqrt{1/\gamma}$). This bound is analogous to that in Section III-B on optimal feedback control performance for stable network dynamics. However, the bound for the game requires a stronger condition on the open-loop dynamics, since even open-loop stable networks can fail to have bounded cost in the presence of an adversarial input.

Example. Consider the following instance of the model (8): $A = \begin{pmatrix} 0.95 & 0.001 \\ 0.001 & 0.1 \end{pmatrix}$, $\mathcal{B} = \{b_1 = e_1, b_2 = e_2\}$, $F_{S_a} = I_2$. The open-loop system does not have a bounded cost in the presence of the adversary for $\gamma < 1.379$. Furthermore, for $\gamma \in (1.379, 19.997)$, using only actuator b_2 results in an unbounded equilibrium cost, whereas using only actuator b_1 results in a bounded equilibrium cost. For systems with unbounded open-loop cost due to adversarial inputs, it is possible that certain actuator subsets (even ones obtained from a greedy algorithm optimizing robustness) may not guarantee bounded closed-loop equilibrium cost.

To facilitate a performance bound, we assume finite cost for a system with no actuator inputs, a fixed adversarial inputs and input penalty. Then we use a similar setup to the stable LQR system proposed in Theorem 2 and define the robustness ratio

$$r_{\text{robust}}(Z) = \frac{\mathbf{tr}(Z_{\text{worst}})}{\mathbf{tr}(Z_{\text{opt}})} \tag{22}$$

where Z_{worst} and Z_{opt} are the solutions to the algebraic Riccati equation (12) corresponding to the optimal and worst k-element selection of actuators for the system (8). We propose the following theorem.

Theorem 4. *Let $\mathcal{R} = \{R(S) | S \subset \mathcal{B}, |S| \leq k\}$ be the set of all actuator influence matrices for actuator subsets with k or fewer elements, $\lambda_1^{\max} := \max\{\lambda_1(R) | R \in \mathbf{R}\}$ and $Q > 0$. Then, the robustness ratio (22) satisfies*

$$r_{robust}(Z) \leq \frac{\alpha_{\overline{A}}(1 + \lambda_1^{\max}\lambda_n(Q))tr(Q)}{\sigma_n^2(\overline{A})\lambda_n(Q) + (1 + \lambda_1^{\max}\lambda_n(Q))tr(Q)} \tag{23}$$

where $\overline{A}$ is the closed-loop dynamics for finite-cost for the system $x_{t+1} = Ax_t + F_{S_a}v_t$ given by

$$\overline{A} = (I + F_{S_a}(\gamma^2 I - F_{S_a}^\top Z F_{S_a})^{-1} F_{S_a}^\top Z)A,$$

$\alpha_{\overline{A}} = \frac{\sigma_1^2(T)}{\sigma_n^2(T)(1 - \sigma_1^2(T\overline{A}T^{-1}))}$, *and T is a non-singular matrix that ensures $\sigma_1(T\overline{A}T^{-1}) < 1$.*

Proof. Given in Appendix 0f. □

V. Numerical Experiments

In this section, we illustrate our results with numerical experiments in regular and random undirected graphs. We compare the effect of parameters such as network model and number of nodes with changing actuator and adversary sets.

The analytic expressions derived in this paper provide best and worst case cost bounds in different contexts. In Section III-A, for systems that are relatively difficult to regulate (i.e., they have at least one unstable mode), we derive a lower bound on the cost required to regulate the system to the origin for a fixed number of actuators. In Section IV-B, we extend the result to obtain a lower bound on equilibrium cost of a dynamic game with a fixed number of actuator control and adversarial inputs. Similarly, in Section III-B, for systems that are relatively easy to regulate (i.e., all modes are stable), we identify an upper bound on the cost required to regulate the system to the origin for a fixed number of actuators. In Section IV-B, we establish an analogous bound for dynamic games with open-loop robust network dynamics. Effectively, when the system is inherently hard, we quantify the best case cost; when the system is inherently easy, we quantify the worst case cost. These relationships are informative because they reveal the scaling of cost based on the number of actuators. However, the m actuator control inputs from n nodes can be selected in many ways, the l adversary inputs may appear in various locations in the network, and these network architectures have different performance and robustness costs associated with them. Likewise, the directions associated with unstable modes can dominate the cost, and, therefore, the performance and robustness can vary depending on the initial state. These questions of input set selection and target regulation (target control) are not new, however, here we empirically demonstrate the types of variation we observe by using the generalized LQR and dynamic game cost (which has not been studied before).

A. LQR Cost Analysis in Networks

To build insight and intuition, we begin our analysis on an undirected path network, with dynamics matrix

$$A = \frac{\rho}{3}\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 & 1 \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix},$$

where $\rho > 0$ is a parameter we will used to modulate the stability of the dynamics.

Actuator Set on a Path Graph and Random Graph. Throughout this section we assume that $\mathcal{B} = \{e_1, ..., e_n\}$, so that each possible actuator injects an input into the dynamics of a single node, and that $Q = I$ and $R_S = I$, $\forall S$. Fig. 1 shows how the optimal feedback performance varies as the number of controlled nodes increases for a 50-node path network, with varying network stability properties and actuators spaced evenly throughout the path, which is empirically a near optimal actuator placement. We see that when the network becomes unstable, the optimal feedback control costs increase significantly with only a single actuator, even though a single actuator is sufficient to stabilize the network dynamics.

We first address the variation in the cost for a fixed number of actuators m. We observe this variation by selecting m nodes uniformly from n, constructing the matrix B (such that the columns of B are columns of the identity matrix), and calculating the LQR cost. We repeat this process 1000 times for each choice of $m \in \{1, 5, 10, 30\}$, constructing the sample distributions in Fig. 2 for the path graph with $n = 100$ nodes presented earlier and for the Erdős-Rényi (ER) [34] random graph (with edge probability $p = 0.1$). In both cases the

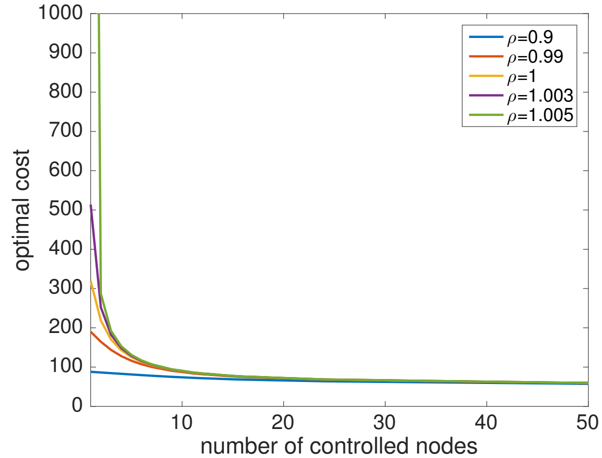

Fig. 1. Optimal cost versus the number of controlled nodes for a 50-node path graph. The controlled nodes were evenly spaced throughout the path. We see that when the dynamics are stable ($\rho = 0.9, 0.99, 1$) the optimal cost is not too large, even with only a single controlled node. When the dynamics are unstable ($\rho = 1.003, 1.005$), the optimal cost can be large.

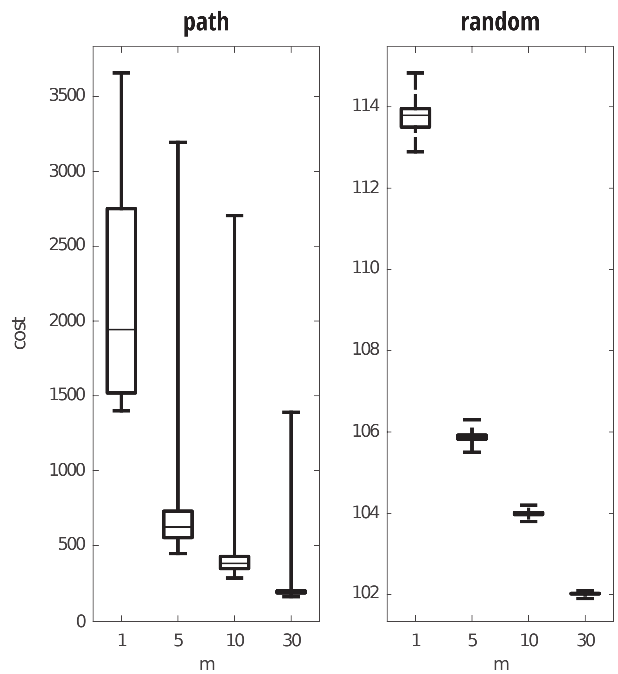

Fig. 2. For the path graph and Erdos-Renyi random graph ($p = 0.1$) with $n = 100$ nodes, $m \in \{1, 5, 10, 30\}$ actuators were selected uniformly randomly. Each of the box plots represent a sample distribution of the costs of 1000 samples (realizations of B). These box plots demarcate the mean, first and third quartiles (box), and minimum and maximum (whiskers).

adjacency matrix A has been scaled by its largest eigenvalue to make it marginally stable. While the exponential scaling related to the number of actuators can still be observed clearly, there is significant variation in the cost for a specific choice of m, most notably for lower fractions of actuators. In addition, the denser connectivity of the random graph yields not only smaller costs, but also smaller variation due to selection of B. This implies that the actuator selection problem becomes trivial as the number of actuators or the connectivity increases

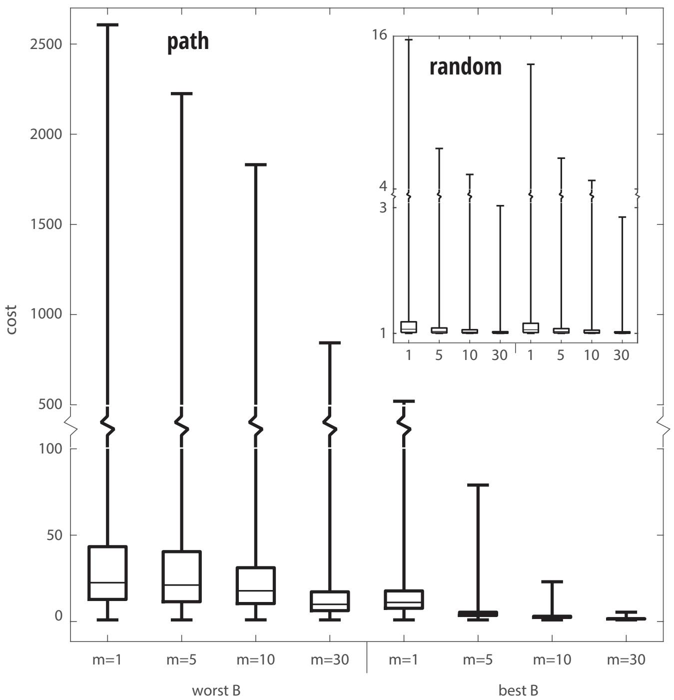

Fig. 3. For the path graph with $n = 100$ nodes, $m \in \{1, 5, 10, 30\}$ actuators were selected in both a greedy (minimizing the cost) and anti-greedy (maximizing the cost) fashion. Each of the box plots represent a sample distribution of the costs associated with 1000 (normally) randomly generated initial states x_0 with that $\|x_0\| = 1$. The inset plot shows the same results for the Erdos-Renyi random graph. These box plots demarcate the mean, first and third quartiles (box), and minimum and maximum (whiskers).

because all choices will provide roughly equivalent costs.

Optimal Actuator Sets in Path Graphs. We now turn to look at the variation in the cost caused by selectively choosing certain directions in state space to regulate. For a given number of actuators m, we pick the best selection of m actuators and also pick the worst selection of m actuators. We find these (approximate) best and worst case actuators sets by, respectively minimizing and maximizing the cost using a greedy algorithm. For each of these cases, we draw 1000 initial state vectors x_0 from a normal distribution, normalize them to lie on the $\|x_0\| = 1$ ball, and compute the cost $x_0^T P_0 x_0$ for regulating that specific direction. Fig. 3 displays these sample distributions for the $n = 100$ path graph for $m \in \{1, 5, 10, 30\}$ actuators. The inset plot shows the same for the ER random graph. By selecting the best and worst actuator choices, we have captured the extreme cases due to actuator selection; every other choice of B would fall (roughly) in between, falling in line with the results of Fig. 2. We observe that ideal actuator selection results in a system that has significantly less variation due to direction. More specifically, the optimal choice of actuators eliminates, or greatly reduces, the effect of the most unstable modes present in A.

One way to interpret the distributions in Fig. 3 is that we know the directions that are most and least costly to regulate - these are the eigenvectors (modes) of P_0 corresponding, respectively, to the largest and smallest absolute eigenvalues of P_0. For a given box and whisker, the maximum value is attained at $v_1^T P_0 v_1$, where v_1 is the eigenvector corresponding to λ_1 of P_0 and we have ordered our eigenvalues such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Likewise, the direction cheapest to regulate is v_n, which is the minimum of the distribution captured by the box plots. All other directions fall between these extremes.

Modal Analysis. To see this more clearly, in Fig. 4, we plot the first five modes of A and P_0 for the path graph (again ordering the eigenvectors according to descending absolute value of their corresponding eigenvalue) for best and worst actuator selection with $m = 1$ and $m = 10$. The eigenvectors of A (dashed) encode the modes expressed in the dynamics due to the network structure and the eigenvectors of P_0 encode the directions in state space that form the overall LQR cost. The best placed single actuator lies at the middle of the path, whereas the worst lies at one of the ends. We observe that the ideal actuator changes the modes of the path network substantially whereas the worst actuator choice does not change the modes, indicating that an actuator placed at the end of the path does not have a significant impact on the dynamics of the network. The largest eigenvalues in the best and worst case differ by approximately a factor of four. The effect is exaggerated in the $m = 10$ case, where the best actuators are evenly spaced throughout the path and the worst actuators are all aggregated at one end. A similar pattern is observed with respect to mode shape and the difference in the largest eigenvalue of P_0 is about a factor of 80.

B. *Dynamic Game Equilibrium Cost Analysis*

We now empirically explore the effects of various parameters, including network structure, open-loop stability properties, and actuator and adversarial input locations, on the dynamic game equilibrium cost and closed-loop $\mathcal{H}_\infty$-norm γ^*, which both provide related but different network robustness metrics (recall that the adversarial input need not represent an actual malicious attack input but can be regarded as a worst-case disturbance or model uncertainty). We analyze the path graph and undirected realizations of the ER and Barabási-Albert (BA) [35] graph models. The dynamics matrix of the graphs is scaled as in the LQR case to modulate the stability of the open-loop dynamics. The equilibrium cost matrix Z given by the solution to (12) is computed by value iteration with (11), and the closed-loop $\mathcal{H}_\infty$-norm γ^* is computed via bisection on γ based convergence of (11) to identify the smallest value of γ for which (11) converges.

Adversary and actuator sets and locations in a regular and random networks. The locations of adversary and actuator inputs in the network can have significant effects on the equilibrium value and $\mathcal{H}_\infty$-norm γ^*. Fig. 5 shows how γ^* equilibrium cost trace(Z) increase as the number of adversarial inputs is increased, with additional adversary inputs placed using a greedy algorithm. It can be seen that additional adversarial inputs have diminishing returns, and that the adversarial inputs have a larger effect when the open-loop dynamics have eigenvalues closer to the stability boundary.

Figure 6 shows the effects of an increasing number of actuators in the network in the presence of a fixed adversarial

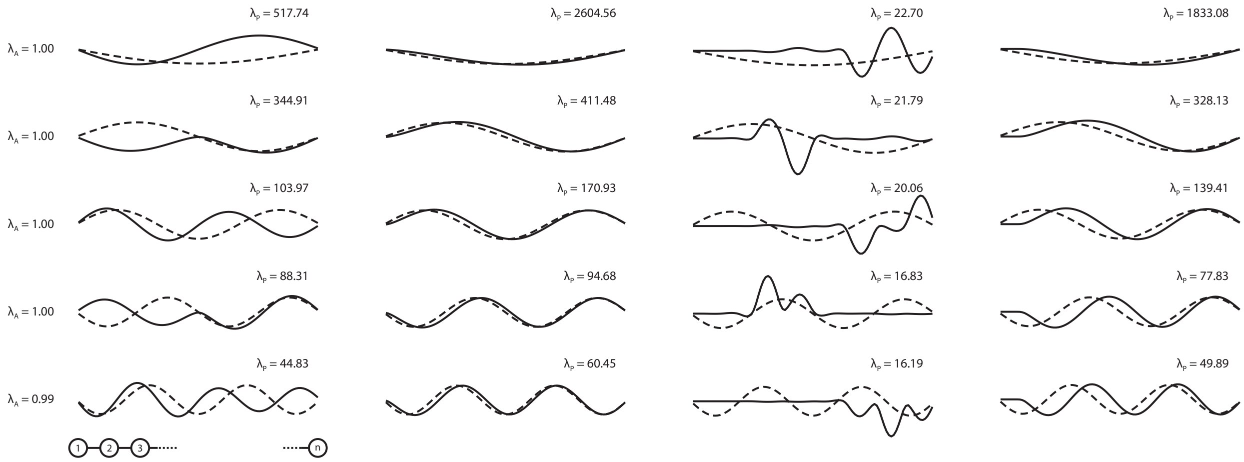

Fig. 4. For the path graph with $n = 100$ nodes, the first five modes (in decreasing absolute value of eigenvalue) of A are plotted in dashed black. Overlaid in solid black are the first five modes of P_0 corresponding to choices of B for (from left to right) the single best actuator ($m = 1$), the single worst actuator ($m = 1$), the 10 best actuators ($m = 10$), and the 10 worst actuators ($m = 10$). Here "best" and "worst" are found using a greedy method.

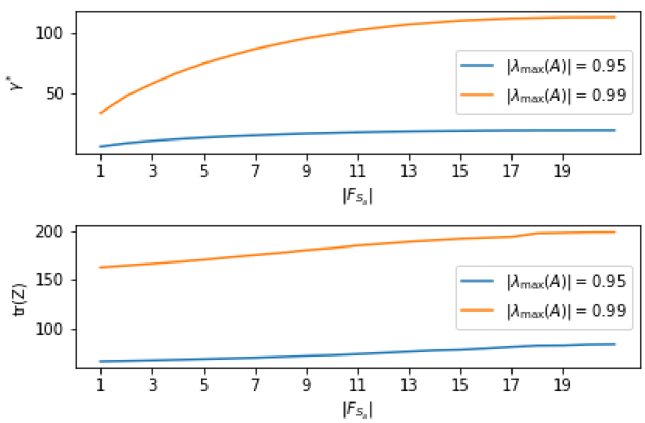

Fig. 5. The top plot shows the variation of γ^* with increasing cardinality of the adversary input set. The bottom plot shows the equilibrium cost calculated at a fixed value of γ that gives a finite value for the largest adversary set. The magnitude of the largest eigenvalue of the dynamics matrix of each model is shown in the legend.

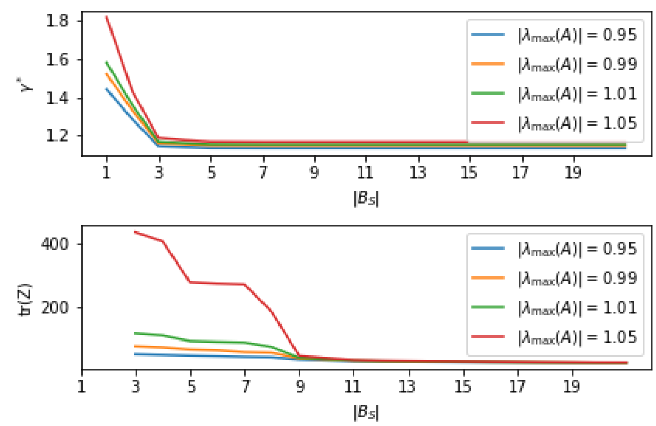

Fig. 6. The top plot shows the variation of γ^* with increasing cardinality of the control actuator set. The bottom plot shows the cost calculated at the largest value of γ for the greedy assignment of actuators. The magnitude of the largest eigenvalue of the dynamics matrix of each model is shown in the legend. The cost matrix is unbounded in the bottom plot for $|B_S| = 1$ for our choice of $\gamma = \gamma^*$ for each plot.

input set ($|F_{S_a}| = 1$). Additional actuators are placed using a greedy heuristic to optimize the marginal gain in equilibrium cost or $\mathcal{H}_\infty$-norm It can also be seen that additional actuators (roughly) have diminishing returns on the equilibrium cost and the $\mathcal{H}_\infty$-norm, and that for open-loop dynamics closer to and beyond the stability boundary, the effects of having smaller actuator subsets becomes more severe. Note that the equilibrium cost may be infinite for certain actuator subsets even when the open-loop dynamics are stable, depending on the locations of adversarial inputs and penalty parameter γ.

For a basic illustration of how the relative locations of actuators and adversarial inputs can have very large effects on the equilibrium value and $\mathcal{H}_\infty$-norm, we evaluated various placements in random graphs. In particular, we generated 100 realizations of connected Erdos-Renyi graphs of 21 nodes with diffusion dynamics, and for each network we determined the strongest (S) and weakest (W) positions of adversarial inputs (A) and defending actuator inputs (D). Figure 7 shows box plots for corresponding values of γ^* for each combination. The very large difference between the strongest adversary-weakest defender and weakest adversary-strongest defender indicate the importance of placing actuators at key locations in the network to provide robustness.

Figure 8 shows how the best position for a single actuator in a path network changes as a function of the location of a single adversarial input and the adversary penalty γ. We see that near γ^*, the optimal position of the actuator control is the same node as the adversarial position, and as γ increases, the optimal actuator position converges to the central node of the path graph. This shows clearly that locations for actuators that promote network robustness to adversarial inputs (or uncertainty) are not the same as the ones that promote optimal control performance in the absence of adversarial inputs.

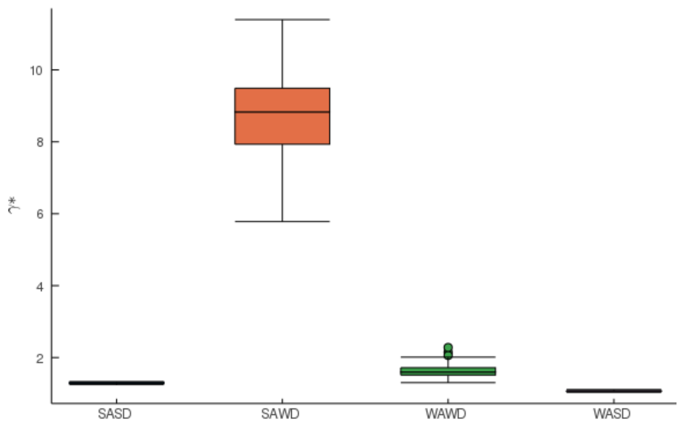

Fig. 7. The box plots compare values of γ^* for the cases of strongest (S) and weakest (W) positions of adversary inputs (A) and control actuators (D) for 100 realizations of the Erdos-Renyi graph model (edge probability $p = 0.1$). SAWD has the highest γ^* and WASD has the lowest γ^*.

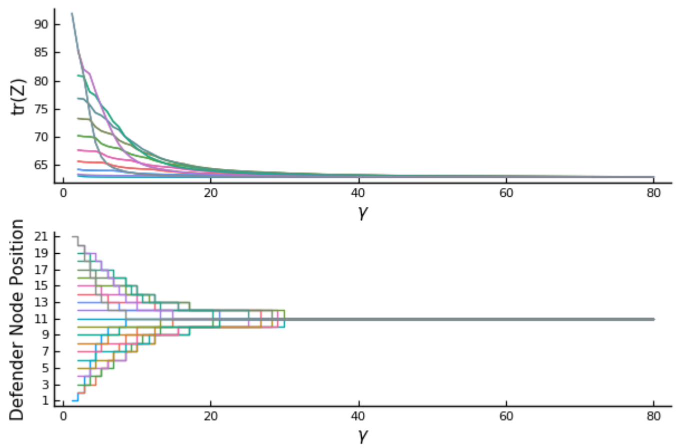

Fig. 8. These plots compare convergence of cost and position for a single actuator for all possible positions of a single adversarial input. The top plot shows the change in equilibrium cost tr(Z) with increasing adversary penalty γ. The bottom plot shows the change in position of the optimal defender as with increasing adversary penalty γ. Each line on both graphs correspond to a single fixed attacker at a different node of the path graph.

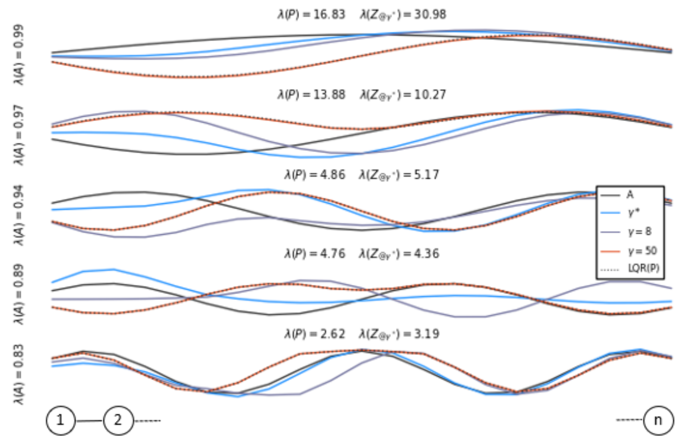

Fig. 9. The modal plot compares the first 5 normalized modes of a stable dynamics matrix for a path graph of $n = 21$ nodes, the cost matrix of LQR (P) and cost of the dynamic game with varying γ from $\gamma^* = 1.373$ to $\gamma = 50$ (nearly LQR). The eigenvalues of the dynamics matrix A and cost matrices (P for LQR and $Z_{@\gamma^*}$ for ZSDG) are annotated for each mode. This is comparable to earlier analysis The convergence of the first mode is highlighted in a following figure.

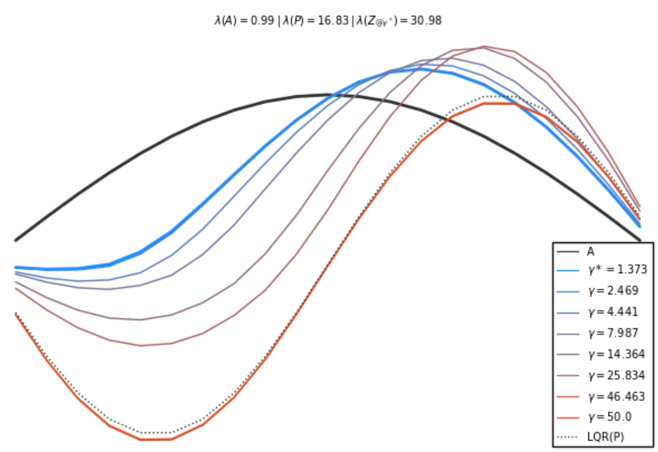

Fig. 10. The plot shows the convergence of the eigenvectors for the 1st mode of the cost matrix for a stable open-loop dynamics matrix with variation of gamma from near critical gamma to LQR conditions.

Modal Analysis. We now provide a modal analysis for the dynamic game on a 21-node path graph, analogous to that provided above for optimal LQR performance. Figure 9 shows the mode shapes and corresponding eigenvalues for the first 5 modes of the dynamics matrix A and the dynamic game equilibrium cost matrix Z for various values of γ. The first mode is emphasized in Figure 10. For the cost matrix, these represent state space directions that have the largest influence on cost. A single adversarial input ($|F_{S_a}| = 1$) is randomly located, here at node 5, and the corresponding best actuator ($|B_S| = 1$) is placed for each value of γ.

We observe that both the adversarial input location and the actuator location effect the resulting mode shapes of the cost matrix, and that as the adversary penalty γ increases, the mode shapes of the LQR cost are recovered, as expected. Figure 11 also shows how the eigenvalues and trace of the equilibrium cost matrix converge to those of the LQR cost matrix as γ is increased. The kinks in the curves correspond to changes in the location of the optimal actuator input node.

VI. Conclusions

We have derived a set of performance bounds for optimal feedback control in networks in terms of performance and robustness for both LQR and zero-sum dynamic game problems that provide insight into fundamental difficulties of network control as a function of the dynamics structure and control architecture. Ongoing and future work includes deriving tighter and more general bounds to include input effectiveness and logarithmic capacity of dynamics eigenvalues [36] and conducting case studies on more complicated network dynamics structures.

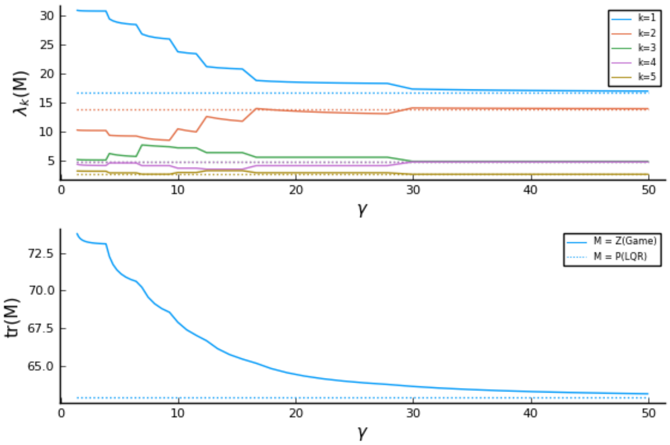

Fig. 11. The top plot compares the eigenvalues of the first 5 modes of the cost matrix for variation of γ from ZSDG to LQR values for a stable dynamics matrix. The lower plot shows the cost monotonically reduce to LQR values. In both plots, the solid lines are the ZSDG cost and eigenvalues and dotted lines are the LQR costs and eigenvalues.

APPENDIX

a) Proof of Theorem 1: We first make a connection between the optimal cost matrix for small Q and a controllability Gramian associated with the *inverse* of the dynamics matrix. Applying the Woodbury matrix identity to the Riccati recursion (4) yields

$$P_{t-1} = Q + A^\top (P_t^{-1} + B_S R_S^{-1} B_S^\top)^{-1} A. \tag{24}$$

As $Q \to 0$ the inverse cost matrix satisfies

$$P_{t-1}^{-1} = A^{-1}(P_t^{-1} + B_S R_S^{-1} B_S^\top) A^{-\top}. \tag{25}$$

Defining $X_{T-t} = P_t^{-1} + B_S B_S^\top$, setting $R_S = I$, and rearranging, we obtain the recursion

$$X_{\tau+1} = A^{-1} X_\tau A^{-\top} + B_S B_S^\top, \quad \tau = 0, ..., T-1 \tag{26}$$

with $X_0 = P_T^{-1} + B_S B_S^\top = Q_T^{-1} + B_S B_S^\top$. This gives

$$X_T = \underbrace{\sum_{\tau=0}^{T-1} (A^{-1})^\tau B_S B_S^\top (A^{-\top})^\tau}_{\bar{X}_T} + (A^{-1})^T Q_T^{-1} (A^{-\top})^T. \tag{27}$$

We see that $\bar{X}_T$ is the T-stage controllability Gramian associated with the system (A^{-1}, B_S). Then directly applying Theorem 3.1 of [9], for any $\mu \in [\lambda_{\min}(A^{-1}), 1)$ and any $T \in [1, \infty)$ it holds that

$$\lambda_{\min}(\bar{X}_T) \leq \text{cond}^2(V) \frac{\mu^{2(\lceil \frac{\bar{n}}{|S|} \rceil - 1)}}{1 - \mu^2} \tag{28}$$

where $\bar{n} = |\{\lambda : \lambda \in \text{spec}(A^{-1}), |\lambda| \leq \mu\}|$. Defining $\eta = 1/\mu$, we see that $\bar{n} = |\{\lambda : \lambda \in \text{spec}(A), |\lambda| \geq \eta\}|$ and $\eta \in (1, \lambda_{\max}(A)]$. Since $P_0^{-1} = X_T - B_S B_S^\top$, it follows that $\lambda_{\min}(P_0^{-1}) \leq \lambda_{\min}(X_T)$. Since A has at least one unstable eigenvalue, then A^{-1} has at least one stable eigenvalue, and in this direction the minimum eigenvalue of the second term in (27) approaches zero as $T \to \infty$ for any fixed $Q_T \succ 0$, so that in the limit as $T \to \infty$, $\lambda_{\min}(X_T)$ is also bounded by the right side of (28). Thus we have in the limit as $T \to \infty$

$$\begin{aligned} \lambda_{\max}(P) &\geq 1/\lambda_{\min}(\bar{X}_T) \\ &\geq \text{cond}^{-2}(V)(1-\mu^2)\mu^{-2(\lceil \frac{\bar{n}}{|S|} \rceil - 1)} \end{aligned} \tag{29}$$

Substituting $\mu = 1/\eta$ yields the expression (15).

Finally, this analysis for small Q accounts only for input energy costs and not for state regulation costs. It is clear from the structure of the recursion (4) (and from standard comparison lemmas; see, e.g., Chapter 13 in [37]) that for any $Q \succeq 0$ such that $(A, Q^{\frac{1}{2}})$ is detectable the costs can only increase. In particular, if $P^{Q\to 0}$ denotes the solution to (5) for small Q and P^Q the solution for any $Q \succeq 0$ such that $(A, Q^{\frac{1}{2}})$ is detectable, then $P^Q \succeq P^{Q \to 0}$. Thus, the bound remains valid for any such choice of Q.

b) Proof of Corollary 1: To obtain the bound for the first term, consider the controllability Gramian $\bar{X}_T$ relating to the inverse cost matrix for small Q in (27). Let $\bar{X}_{T,\mathcal{B}}$ be the Gramian for $S = \mathcal{B}$. Since $\bar{X}_T \preceq \bar{X}_{T,\mathcal{B}}$, it follows that $\lambda_{\min}(\bar{X}_T) \leq \lambda_{\min}(\bar{X}_{T,\mathcal{B}})$. We then have

$$\begin{aligned} \lambda_{\min}(\bar{X}_{T,\mathcal{B}}) = \lambda_{\min}\left(\sum_{\tau=0}^{T-1} A^{-2\tau}\right) &= \frac{1 - \lambda_{\min}(A^{-1})^{2T}}{1 - \lambda_{\min}(A^{-1})^2} \\ \Rightarrow \lim_{T\to\infty} \lambda_{\min}(\bar{X}_{T,\mathcal{B}}) &= \frac{\lambda_{\max}(A)^2}{\lambda_{\max}(A)^2 - 1}. \end{aligned} \tag{30}$$

The first part then follows since as $T \to \infty$ we have $\lambda_{\max}(P) \geq 1/\lambda_{\min}(\bar{X}_T) \geq 1/\lambda_{\min}(\bar{X}_{T,\mathcal{B}})$. The bound for the second term follows from Theorem 1 with $\eta = \bar{\lambda}_u(A) > 1$ and since the symmetric dynamics matrix admits an orthonormal eigenvector matrix V with $\text{cond}(V) = 1$.

c) Proof of Theorem 2: Our proof follows along the lines of the analogous proof of Theorem 3 in [21], which we include here for completeness even though the proofs are nearly identical. We begin by deriving an upper bound for $\mathbf{tr}(P_{\text{worst}})$, based on the fact that for stable systems the cost is finite even without any actuation. Specifically, with no actuators ($S = \emptyset$) the algebraic Riccati equation (5) reduces to the Lyapunov equation $P^\emptyset = A^\top P^\emptyset A + Q$. Since A is stable, from Lemma 3 there exists a nonsingular similarity transformation T satisfying $\sigma_1(TAT^{-1}) < 1$. Defining $\bar{P} = TP^\emptyset T^\top$, $\bar{Q} = TQT^\top$, and $D = TAT^{-1}$, we have $\bar{P} = D\bar{P}D^\top + \bar{Q}$. Using trace and eigenvalue interlacing properties for sums of symmetric matrices from Lemma 2, there holds $\mathbf{tr}(D\bar{P}D^\top) = \mathbf{tr}(D^\top D\bar{P}) \leq \sigma_1^2(D)\mathbf{tr}(\bar{P})$ so that $\mathbf{tr}(\bar{P}) \leq \frac{\mathbf{tr}(\bar{Q})}{1-\sigma_1^2(D)}$. Similarly, we have $\mathbf{tr}(\bar{P}) = \mathbf{tr}(T^\top TP^\emptyset) \geq \sigma_n^2(T)\mathbf{tr}(P^\emptyset)$ and $\mathbf{tr}(\bar{Q}) = \mathbf{tr}(T^\top TQ) \leq \sigma_1^2(T)\mathbf{tr}(Q)$. Putting it all together yields

$$\begin{aligned} \mathbf{tr}(P_{worst}) &\leq \mathbf{tr}(P^\emptyset) \\ &\leq \frac{\sigma_1^2(T)}{\sigma_n^2(T)} \frac{\mathbf{tr}(Q)}{1-\sigma_1^2(D)} = \alpha_A \mathbf{tr}(Q). \end{aligned} \tag{31}$$

where α_A is the constant defined in (18).

We now provide a lower bound for P_{opt}. For any k-element actuator subset S, there holds

$$\begin{aligned}\mathbf{tr}(P) &\geq \mathbf{tr}(A^\top(Q^{-1}+R(S))^{-1}A+Q)\\ &\geq \lambda_n(AA^\top)\mathbf{tr}(Q^{-1}+R(S))^{-1}+\mathbf{tr}(Q)\\ &= \sigma_n^2(A)\sum_{i=1}^{n}\frac{1}{\lambda_i(Q^{-1}+R(S))}+\mathbf{tr}(Q)\\ &\geq \frac{n\sigma_n^2(A)}{\lambda_1(Q^{-1}+R(S))}+\mathbf{tr}(Q)\\ &\geq \frac{n\sigma_n^2(A)}{\lambda_1(Q^{-1})+\lambda_1(R(S))}+\mathbf{tr}(Q)\\ &\geq \frac{n\sigma_n^2(A)}{\frac{1}{\lambda_1(Q)}+\lambda_1^{max}}+\mathbf{tr}(Q).\end{aligned} \tag{32}$$

The first inequality follows from Lemma 1, and the second and fourth from Lemma 2. Since the bound above holds for any k-element actuator subset, it also holds for the optimal k-element selection. Finally, combining the upper bound for $\mathbf{tr}(P_{worst})$ and the lower bound for $\mathbf{tr}(P_{opt})$ gives (19).

d) Proof of Corollary 2: Since the denominator in the bound (19) is lower bounded by $(1+\lambda_1^{\max}\lambda_n(Q))\mathrm{trace}(Q)$, a looser bound $r(P)\leq\alpha_A$ is obtained that only depends on the system dynamics, and not on the cost matrix Q. In addition, if A is normal, its singular values are equal to the magnitude of its eigenvalues [32], and since A is Schur stable, we have $\sigma_1(A)=|\lambda_1(A)|<1$. Further, the similarity transformation T described in Lemma 3 can be taken to be the identity matrix. Under these conditions, the bound reduces to (20).

e) Proof of Theorem 3: As in Theorem 1, we make a connection between the equilibrium cost matrix for small Q and controllability Gramians for each input associated with in inverse dynamics matrix. As $Q\to 0$, the inverse equilibrium cost matrix in the recursion (11) becomes

$$Z_{t-1}^{-1}=A^{-1}[Z_t^{-1}+(B_SB_S^\top-\gamma^{-2}F_{S_a}F_{S_a}^\top)]A^{-\top}.$$

Defining $X_{T-t}=Z_t^{-1}+(B_SB_S^\top-\gamma^{-2}F_{S_a}F_{S_a}^\top)$, we follow a similar approach to the proof to Theorem 1 to obtain

$$\begin{aligned}X_T &= \underbrace{\sum_{\tau=0}^{T-1}(A^{-1})^\tau BB^\top(A^{-\top})^\tau}_{\overline{X}_T}\\ &\quad-\gamma^{-2}\underbrace{\sum_{\tau=0}^{T-1}(A^{-1})^\tau FF^\top(A^{-\top})^\tau}_{\tilde{X}_T}+\underbrace{(A^{-1})^TQ_T^{-1}(A^{-\top})^T}_{\tilde{Q}}\\ &=\overline{X}_T-\gamma^{-2}\tilde{X}_T+\tilde{Q}.\end{aligned} \tag{33}$$

Note that $\overline{X}_T$ and $\tilde{X}_T$ are T-stage controllability Gramians for the systems (A^{-1},B_S) and (A^{-1},F_{S_a}), respectively. We again apply Theorem 3.1 from [9] to obtain

$$\lambda_{\min}(\overline{X}_T)\leq \mathrm{cond}^2(V)\frac{\mu^{2(\lceil\frac{n}{|S|}\rceil-1)}}{1-\mu^2}, \tag{34}$$

$$\lambda_{\min}(\tilde{X}_T)\leq \mathrm{cond}^2(V)\frac{\mu^{2(\lceil\frac{n}{|S_a|}\rceil-1)}}{1-\mu^2}. \tag{35}$$

for any $\mu\in[\lambda_{\min}(A^{-1}),1)$ and any $T\in[1,\infty)$. Applying Weyl's inequality to the matrix $W_T=\overline{X}_T-\gamma^{-2}\tilde{X}_T$ gives

$$\begin{aligned}\lambda_{\min}(W_T)&\leq\lambda_{\max}(-\gamma^{-2}\tilde{X}_T)+\lambda_{\min}(\overline{X}_T)\\ &=-\gamma^{-2}\lambda_{\min}(\tilde{X}_T)+\lambda_{\min}(\overline{X}_T).\end{aligned} \tag{36}$$

Combining (34)-(36) yields

$$\lambda_{\min}(W_T)\leq\frac{\mathrm{cond}^2(V)}{1-\mu^2}\left[\mu^{2(\lceil\frac{n}{|S_1|}\rceil-1)}-\gamma^{-2}\mu^{2(\lceil\frac{n}{|S_a|}\rceil-1)}\right]. \tag{37}$$

Since A has at least one unstable eigenvalue, then A^{-1} has at least one stable eigenvalue, and the corresponding block of $(A^{-1})^T$ goes to zero as $T\to\infty$. In view of (33), this means that $X_\infty:=\lim_{T\to\infty}\lambda_{\min}(X_T)$ is also bounded by the right side of (37).

Since $Z_0^{-1}=X_T-(B_SB_S^\top-\gamma^{-2}F_{S_a}F_{S_a}^\top)$, by Weyl's inequality we have

$$\begin{aligned}\lambda_{\min}(Z_0^{-1})&\leq\lambda_{\min}(X_T)+\lambda_{\max}(-B_SB_S^\top+\gamma^{-2}F_{S_a}F_{S_a}^\top)\\ &\leq\lambda_{\min}(X_T)-\lambda_{\min}(B_SB_S^\top)+\gamma^{-2}\lambda_{\max}(F_{S_a}F_{S_a}^\top)\\ &\leq\lambda_{\min}(X_T)+\gamma^{-2}\lambda_{\max}(F_{S_a}F_{S_a}^\top)\end{aligned}$$

Thus $\lambda_{\max}(Z_0)\geq 1/(\lambda_{\min}(X_T)+\gamma^{-2}\lambda_{\max}(F_{S_a}F_{S_a}^\top))$. In the limit as $T\to\infty$ we have

$$\lambda_{\max}(Z)\geq\frac{1-\mu^2}{\left[\begin{array}{r}\mathrm{cond}^2(V)\left(\mu^{2(\lceil\frac{n}{|S|}\rceil-1)}-\gamma^{-2}\mu^{2(\lceil\frac{n}{|S_a|}\rceil-1)}\right)\\+\gamma^{-2}\lambda_{\max}(F_{S_a}F_{S_a}^\top)(1-\mu^2)\end{array}\right]}.$$

Setting $\eta=\frac{1}{\mu}$ and defining $\Lambda_F=\lambda_{\max}(F_{S_a}F_{S_a}^\top)$ gives (21), proving the theorem.

f) Proof of Theorem 4: To facilitate a performance bound, in this section we assume that the cost with a set of adversarial inputs with fixed input penalty γ and without any actuators is bounded. Specifically, we assume that the optimal control problem $\max_v\sum_{t=0}^{\infty}(x_t^\top Qx_t-\gamma^2v_t^\top v_t)$ for the system $x_{t+1}=Ax_t+F_{S_a}v_t$ has a finite value. This assumption is equivalent to the existence of a positive definite solution Z to the equation

$$Z=Q+A^\top Z[I-\gamma^{-2}F_{S_a}F_{S_a}^\top Z]^{-1}A \tag{38}$$

that satisfies $\gamma^2I-F_{S_a}^\top ZF_{S_a}>0$. Under this assumption, the optimal adversary state feedback policy is

$$v_t=(\gamma^2I-F_{S_a}^\top ZF_{S_a})^{-1}F_{S_a}^\top ZAx_t, \tag{39}$$

and hence the (stable) closed-loop dynamics matrix is

$$\overline{A}=(I+F_{S_a}(\gamma^2I-F_{S_a}^\top ZF_{S_a})^{-1}F_{S_a}^\top Z)A. \tag{40}$$

It is now possible to utilize the theorem for performance bounds of the stable LQR system proposed in III-B to establish a worst-case network robustness bound for actuator subsets obtained through any selection algorithm. Consider the robustness ratio $r_{\text{robust}}(Z)=\frac{\mathbf{tr}(Z_{\text{worst}})}{\mathbf{tr}(Z_{\text{opt}})}$, where Z_{worst} and Z_{opt} are the solutions to the algebraic Riccati equation (12) corresponding to the optimal and worst k-element selection of actuators for the system (8). The result follows directly from applying Theorem 2 to the system $x_{t+1}=\overline{A}x_t+B_Su_t$.

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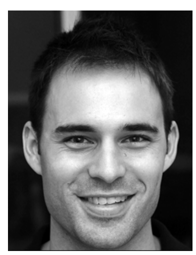

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