Stochastic Dynamic Programming for Wind Farm Power Maximization

Yi Guo, Mario Rotea, Tyler Summers

Abstract—Wind plants can increase annual energy production with advanced control algorithms by coordinating the operating points of individual turbine controllers across the farm. It remains a challenge to achieve performance improvements in practice because of the difficulty of utilizing models that capture pertinent complex aerodynamic phenomena while remaining amenable to control design. We formulate a multi-stage stochastic optimal control problem for wind farm power maximization and show that it can be solved analytically via dynamic programming. In particular, our model incorporates state- and input-dependent multiplicative noise whose distributions capture stochastic wind fluctuations. The optimal control policies and value functions explicitly incorporate the moments of these distributions, establishing a connection between wind flow data and optimal feedback control. We illustrate the results with numerical experiments.

I. INTRODUCTION

Wind energy is an important component of future energy systems to meet growing energy demands. As wind power continues to account for a larger portion of the world-wide energy portfolio, the optimal operation of wind farms offers both challenges and opportunities to further improve performance at the levels of single turbines, wind farms, and power grids. Due to nonlinear aerodynamic interaction through wakes and unpredictable wind variations, future optimal control strategies for wind farms will require sophisticated models to capture and manage stochastic wind fluctuations.

Maximizing the wind power capture has been discussed in the scope of wind turbines [1]–[4] and wind farms [5]–[26]. In Region 2 operation (below-rated wind speed), the wind plant is operated to maximize the power output. In this regime, there are inherent tradeoffs between the wake of upstream turbines and the power extracted from downstream turbines. Due to this aerodynamic coupling, maximizing total power of wind farms cannot be achieved by myopically maximizing the power output for each individual wind turbine in the array [27]. Therefore, depending on layout and wind conditions, it may be essential to have a coordinated control framework for wind farms to determine the optimal control strategy for each wind turbine to improve annual energy production.

Many challenges and related solutions for wind farm power maximization have been highlighted and discussed in [1]. Recent control strategies for optimal operation have been proposed using both model-based [6]–[14], [19], [20], [22], and model-free strategies [17], [21], [24]–[26]. Model-based strategies provide solutions that typically have faster response times than model-free approaches. However, the models used for control design can deviate from actual wind field and turbine characteristics in practice, which can limit the effectiveness of model-based control strategies. The reader is referred to the introduction in [17], and the references therein, for further discussion on model-based and model-free strategies for wind plant power maximization.

In this paper, we focus on wind power maximization in Region 2. The work presented here generalizes the simple actuator disk model (ADM) utilized in [22] to a stochastic version and pose a multi-stage stochastic optimal control problem for wind farm power maximization. The stochastic actuator disk model balances complexity and tractability by incorporating unsteady aerodynamic phenomena into the distributions of random variables in the model. Estimates of the statistics of these distributions can then be exploited in the control algorithm to improve overall efficiency of the farm in the presence of stochastic wind flow.

Our main contributions are as follows:

- We formulate a multi-stage stochastic optimal control problem for wind farm power maximization and show that it can be analytically solved via dynamic programming. In particular, our model generalizes that of [22] by incorporating state- and input-dependent multiplicative noises to capture the uncertain wake effects of wind turbines. The stochastic version of the ADM relaxes a strong assumption of a deterministic ADM, such as steady wind over the rotor disk. In contrast to existing work, the proposed stochastic multi-stage formulation allows us to maximize the wind farm power by explicitly incorporating information about the probability distributions of wind fluctuations into control decisions.
- By solving the proposed multi-stage stochastic optimization, we show that the optimal feedback control policies for the turbines are linear with respect to upstream wind velocity, but in contrast to [22], the optimal gain coefficients depend explicitly on the statistics of the multiplicative noises, which can be estimated from high-fidelity wind flow simulations or experimental data. This provides a direct connection between statistical properties of the unsteady wind flow physics and the optimal feedback control of wind farms. We also show that for the stochastic ADM with both multiplicative and additive noise, the optimal policies are nonlinear.

The framework, while elementary for real-world applica-
tions, illustrates a rigorous process for incorporating flow statistics into the wind farm power maximization problem. The dependence of control solutions on the statistics of the wind fluctuations makes intuitive sense, as one cannot expect a single control algorithm to be optimal under a range of unsteady wind conditions. In future work, we will extend the stochastic approach presented in this paper to more representative, yet tractable, models of the flow physics and loads as done in [18].

II. PROBLEM FORMULATION

Our model is a generalization of the one in [22], which utilizes the actuator disk model (ADM) [28], [29]. Let \( P \) denote the power extracted by an ideal turbine rotor, let \( F \) denote the force done by the wind on the rotor, let \( V_0 \) denote the free stream upwind velocity, let \( V \) denote the wind velocity at the disk, and let \( V_1 \) denote the far wake velocity. The ADM model is then

\[
P = FV, \quad (1a)
\]

\[
F = \rho A (V_0 - V_1) V, \quad (1b)
\]

\[
V = V_0 - u, \quad (1c)
\]

\[
V_1 = V_0 - 2u, \quad (1d)
\]

where \( \rho \) is the air density, \( A \) is the rotor swept area, and \( u \geq 0 \) is the reduction in air velocity between the free stream and the rotor plane, which can be interpreted as a control input. In practice, \( u \) can be controlled by adjusting the angular rotor speed or the collective blade pitch angle.

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Deterministic Model: We consider a one-dimensional cascade of wind turbines, illustrated in Fig. 1. We assume that the wind direction is along the row of turbines and is not varying. The ADM model given in (1) can be written in state-space form by letting \( x_k \) and \( x_{k+1} \) denote the wind velocity upstream and downstream of the \( k \)-th turbine (i.e., \( x_k = V_k \) in (1d), for \( k = 0, 1 \)). The scalar control input for the \( k \)-th turbine is denoted by \( u_k \), which is the controllable wind velocity deficit at the rotor disk, and \( y_k \) is an output to estimate the power extraction of turbine \( k \) (i.e., \( y_k = V_k - u_k \) in (1c)). Then the velocity \( V_{k+1} \) in the far wake of the rotor (1d) and the rotor effect at the disk in velocity (1c) can be written as below in (2a) and (2b). The power extraction of the \( k \)-th wind turbine using ADM model (1) in state-space expression is given in (2c)

\[
x_{k+1} = x_k - 2u_k, \quad (2a)
\]

\[
y_k = x_k - u_k, \quad (2b)
\]

\[
p_k(y_k, u_k) = 2\rho A b_k^2 u_k, \quad (2c)
\]

where the control input is constrained by \( u_k \in [0, \frac{1}{2} x_k] \) so that the wind velocity in the far wake remains positive. To simplify the notation, we eliminate the constant in (2c) and come to the constant-free turbine power function \( \ell(\cdot, u_k) \), which will serve as a stage cost in our subsequent multi-stage optimal control problem

\[
\ell(x_k, u_k) = (x_k - u_k)^2 u_k. \quad (3)
\]

Note that this function is jointly cubic in the state and control input. Further details of this model may be found in [22].

Stochastic Model: The simple model described above captures basic wind farm turbine interactions. But it fails to capture stochastic wind fluctuations that are also relevant to optimizing the total power output. High fidelity computational fluid dynamic models offer extreme detail of flows but are cumbersome to incorporate into high-level operational decision making. Therefore, we consider here a stochastic extension of the deterministic actuator disk model above that can capture more complex phenomena, such as stochastic wind fluctuations, while remaining computationally tractable.

![Fig. 1. A cascade of \( N \) wind turbines; \( k = 0 \) indicates the most upstream location.](image)

![Fig. 2. Stochastic actuator disk model and stream-tube diagram for wind power extraction. The solid and dashed lines indicate the wind field mean and associated stochastic variations, respectively, which relate to the moments of the multiplicative variations parameters \( a_k \) and \( b_k \).](image)

The stochastic actuator disk model is given by

\[
x_{k+1} = a_k x_k + b_k u_k, \quad (5)
\]

where \( a_k \sim \mathcal{P}_{a,k} \) is a state multiplicative random variable and \( b_k \sim \mathcal{P}_{b,k} \) is an input multiplicative random variable. The model is illustrated in Fig. 2. We assume that the random variables \( a_k \) and \( b_k \) are independent for all \( k \) and independent of each other. This model captures stochastic wind fluctuations. In particular, the multiplicative noises \( a_k \) and \( b_k \) provide a simple model for the inherent stochasticity of far wake recovery. We assume that moments up to order three of each of the distributions \( \mathcal{P}_{a,k}, \mathcal{P}_{b,k} \) are known (or can be estimated from high-fidelity simulation or experimental data). For the state mean dynamics to match the deterministic model (2a), we can set \( \mathbb{E}[a_k] = 1, \mathbb{E}[b_k] = -2 \).
\[ \psi_k = -3Q_{k+1}\Sigma_{b,k}\mu_{a,k} - 2 + \sqrt{(3Q_{k+1}\Sigma_{b,k}\mu_{a,k} - 2)^2 - 3(Q_{k+1}\Gamma_{b,k} + 1)(3Q_{k+1}\Sigma_{a,k}\mu_{b,k} + 1)} \div 3(Q_{k+1}\Gamma_{b,k} + 1), \]

\[ Q_k = (1 - \psi_k)^2\psi_k + Q_{k+1}(\Gamma_{a,k} + \Gamma_{b,k}\psi_k^3 + 3\Sigma_{b,k}\mu_{a,k}\psi_k^2 + 3\Sigma_{a,k}\mu_{b,k}\psi_k). \]

where \( G^*_k(x_k) \) represents the optimal (normalized) wind farm power from turbine \( k \) as a function of the state \( x_k \), with initialization \( G^*_N(x_k) = 0 \). We first solve the last tail subproblem at \( k = N - 1 \) with \( G^*_N(x) = 0 \). We have

\[ \frac{\partial\ell(x_{N-1}, u_{N-1})}{\partial u_{N-1}} = (x_{N-1} - u_{N-1})(x_{N-1} - 3u_{N-1}) = 0, \]

for which the policy \( u_{N-1}^* = \frac{1}{3}x_{N-1} \) is the unique maximizer and satisfies the constraint \( u_{N-1} \in [0, 1_2x_{N-1}] \). Substituting this optimal policy back into the value expression yields the optimal power function

\[ G^*_N(x_{N-1}) = 4 \frac{x_{N-1}^3}{27}, \]

Note that this function is a cubic in the state. Accordingly, we parameterize the optimal power functions as \( G^*_k(x_k) = Q_kx_k^3 \) and consider a general step in the backward recursion. To obtain the optimal policy, we define the function inside the maximization operation

\[ G_k(x_k, u_k) := (x_k - u_k)^2u_k + Q_{k+1}E[(a_kx_k + b_ku_k)^3]. \]

Expanding the second term and taking the expectation by utilizing the (raw) moment information from the distributions of \( a_k \) and \( b_k \), and then taking the partial derivative of \( G_k(x_k, u_k) \) with respect to \( u_k \) gives a quadratic polynomial in \( u_k \). As above, one of the roots of this polynomial corresponds to the unique maximizing input, which is a linear function of the state. Carrying out the algebra yields

\[ u_k^* = \pi^*(x_k) = \psi_kx_k, \]

where the gain parameters \( \psi_k \) are given in (4a). Note that the optimal policies all satisfy the constraints on \( u_k \). To obtain a backwards recursion for the value function coefficients \( Q_k \), we substitute \( u_k^* = \psi_kx_k \) back into (10)

\[ G_k^*(x_k, u_k^*) = Q_kx_k^3 \]

\[ = (x_k - u_k^*)^2u_k^* + Q_{k+1}E[(a_kx_k + b_ku_k^*)^3]. \]

(12)

Since \( u_k^* \) is linear in \( x_k \), the optimal value functions are cubic in the state. Matching the coefficients on both sides of (12), we come to (4b). Eq. (8) follows from (12) for \( k = 0, 6 \) and (2c), which concludes the proof.

Remark 1: (Optimal policies and value functions with central moments.) The random variables \( a_k \) and \( b_k \) can also be described by their higher-order central moments, namely their variances \( \sigma_{a,k}^2 \), \( \sigma_{b,k}^2 \) and skewnesses \( \gamma_{a,k} \), \( \gamma_{b,k} \). The optimal linear state feedback control policies can also be written in terms of central moments instead of raw moments by using

\[ \Sigma = \sigma^2 + \mu^2, \quad \Gamma = \sigma^3\gamma + 3\sigma^2\mu + \mu^3. \]

(14)
\[
\pi^*_N(x) = -\Delta_k + \sqrt{\Delta^2_N} - 3(Q_N^{-1}\Gamma_{b,N-2} + 1)\left[ (3Q_N^{-1}\Sigma_{a,N-2}\mu_{b,N-2} + 1)x^2 + 3Q_N^{-1}\Sigma_{c,N-2}\mu_{b,N-2} \right],
\]
where, \( \Delta_N = (3Q_N^{-1}\Sigma_{b,N-2}\mu_{a,N-2} - 2)x \).

**Corollary 1:** Under the assumptions of Theorem 1, we define the efficiency \( \eta_\ell \) of the \( \ell \)-th sub-array\(^1\) by

\[
\eta_\ell := \mathbb{E} \left[ \frac{P_\ell}{\pi^*_\ell} \right],
\]
where \( x_\ell \) is the free stream velocity entering the subarray cascaded turbines from \( \ell \) to \( N - 1 \) and \( P_\ell \) denotes the aggregated power from the \( \ell \)-th subarray of wind turbines. The optimal efficiency \( \eta_\ell^* \) of the \( \ell \)-th sub-array has the form

\[
\eta_\ell^* = 4Q_\ell, \quad \forall \ell \in \{0, \ldots, N - 1\},
\]
which is achieved with the optimal control sequence \( u_\ell^*, \ldots, u_{N-1}^* \), where \( Q_\ell \) is calculated from (4b).

**Proof:** The maximum power produced by the \( N - \ell \) turbines

\[
P_\ell^* = 2\rho A Q_\ell x_\ell^3,
\]
under the optimal control sequence \( u_\ell^*, \ldots, u_{N-1}^* \) with \( Q_\ell \) computed via (4b). We substitute the optimal power (17) into (15) and obtain (16), which concludes the proof. \( \blacksquare \)

Next, we consider a stochastic actuator disk model with both multiplicative and additive noise, which allows a more general description of uncertainty in wind fluctuations. Interestingly, in contrast to classical linear quadratic problems, when additive noise is included the optimal policies are no longer linear in general, and so the optimal value functions are no longer cubic. This highlights a computational limitation with this more general model that makes the approach more difficult to implement in practice.

**Theorem 2:** (Stochastic actuator disk model with additive noise.) Consider the stochastic ADM (5) with additive noise

\[
x_{k+1} = a_k x_k + b_k u_k + c_k,
\]
where \( c_k \sim \mathcal{P}_c \) is a zero-mean additive random variable with second moment \( \Sigma_{c,k} \) and third moment \( \Gamma_{c,k} \). In the penultimate tail subproblem, the optimal policy has the nonlinear form

\[
\pi^*_N(x) = \delta x + \sqrt{\alpha + \beta x^2}
\]
for some constants \( \delta, \alpha, \) and \( \beta \); the exact expression is given in (13). As a result, the corresponding optimal value function at turbine location \( N - 2 \) is non-cubic, and so the remaining optimal policies and value functions are nonlinear and non-cubic, respectively.

**Proof:** Consider again the dynamic programming recursion (9). Since \( G_N^*(x) = 0 \), the last tail subproblem is identical to that in Theorem 1, so that \( G_{N-1}^*(x_{N-1}) = \frac{x_{N-1}^3}{\pi^*_N} \). Consider now the penultimate tail subproblem for \( \ell = N - 2 \)

\[
G_{N-2}(x, u) = (x-u)^2 u + \frac{4}{27} \mathbb{E} \left[ (a_k x + b_k u + c_k)^3 \right],
\]
Taking the expectation of the second term by utilizing the (raw) moments of \( a_k, b_k \) and \( c_k \), and then taking the partial derivative with respect to \( u \) and setting to zero yields a quadratic optimality condition in \( u \). Carrying out some algebra as above, it turns out that the roots of this polynomial are no longer linear in the state, in contrast to the results in Theorem 1. The optimal control policy is thus a nonlinear function of state of the form \( \pi^*_N(x) = \delta x + \sqrt{\alpha + \beta x^2} \) for some constants \( \delta, \alpha, \) and \( \beta \). The exact expression for the maximizing control input derived from the quadratic optimality condition is given in (13). It can also be seen that when the additive noise variance \( \Sigma_{c,k} \) is zero (i.e., the additive noise is absent since it also has zero mean), then \( \alpha = 0 \) and we recover the linear policy of Theorem 1 since \( x \geq 0 \). Finally, these observations also lead to the conclusion that none of the remaining optimal policies and value functions are linear and cubic, respectively, and will in fact become increasingly complicated as the recursion proceeds backward toward the beginning of the array. \( \blacksquare \)

**IV. Numerical Experiments**

To illustrate our results, we consider a cascade with \( N = 10 \) identical turbines to analyze the performance of the optimal gain sequence \( \{\psi^0, \ldots, \psi^9\} \) for the proposed stochastic actuator disk model. As is commonly done in the literature [28], [29], we refer to these gains as *induction factors*. The stochastic model parameters \( a_k \) and \( b_k \) are all independent of each other and spatially homogeneous \((\mu_{a,k} = \mu_a, \mu_{b,k} = \mu_b, \sigma_{a,k} = \sigma_a, \sigma_{b,k} = \sigma_b, \gamma_{a,k} = \gamma_a \) and \( \gamma_{b,k} = \gamma_b, \forall k)^2\).

Fig. 3 illustrates the optimal induction factor sequence \( \{\psi^0, \ldots, \psi^9\} \) and Fig. 4 depicts the optimal efficiency \( \eta_\ell^* \) under different standard deviation values of the input-dependent multiplicative noise \( b_k \). The induction factors in Fig. 3 are normalized by 1/3, which is the value achieving the Betz limit for a single isolated turbine [28]. We set the mean value \( \mu_a \) to 1, and the skewness to zero. Fig. 4 demonstrates that the optimal array efficiency improves with increasing variance on \( b_k \). This result is intuitively reasonable, in the sense that higher variability of the velocity deficits in the far wake may lead to increased power extraction. We speculate that this multiplicative stochastic perturbation on the velocity

\(^1\)The efficiency \( \eta_\ell \) defined here quantifies the energy extraction of sub-array \( \ell \) compared to energy in the wind entering the sub-array. Note that due to aerodynamic wake coupling, it is possible for the optimal efficiency of the sub-array to exceed the efficiency obtained by independently setting individual turbine induction factors to achieve the single-turbine Betz limits.

\(^2\)To have clearer interpretation of our results, we discuss the results in terms of central moments. No additive noise is considered in this section.
induction factors for leading upstream turbines, since in this case the model essentially predicts that additional energy will be injected into the wake further downstream. This indicates that the parameters in the stochastic ADM should be carefully calibrated based on measured data in order to capture appropriate (possibly heterogeneous) spatio-temporal flow variations and obtain reasonable control policies for the array. To appropriately incorporate stochasticity of the wind flow, we set the mean value of $a_k$ to $\mu_a = 0.99$, and vary the standard deviation $\sigma_a$ to describe statistical fluctuations. The key observation is that regardless of the value of $\mu_a$, the proposed approach improves efficiency with increasing variance by exploiting statistical knowledge of wind field fluctuations and incorporating this information into optimal control policies for wind farm power maximization.

Fig. 6. Comparison of optimal efficiency $\eta_t$ for deterministic model ($\mu_a = 0.99, \mu_b = -2$) and stochastic model with various values of state-dependent multiplicative noise standard deviation ($\mu_a = 0.99, \sigma_a > 0, \mu_b = -2, \sigma_b = 0, \gamma_a = 0$ and $\gamma_b = 0$).

The stochastic actuator disk model of a wind farm with cascaded wind turbines captures stochastic wind fluctuations. By definition, the optimal control laws derived from stochastic dynamic programming achieve superior performance to laws derived from a deterministic model of the same complexity, allowing the turbines to recognize and react to the particular wind field characteristics. Data derived directly from measurements or simulations can be incorporated...
directly into the control law to improve the aerodynamic efficiency a wind farm for specific wind fluctuation statistics. It is worth emphasizing that more work is necessary to incorporate tractable noise models that are consistent with the flow physics; this conference paper is a first step in this direction.

V. CONCLUSIONS AND OUTLOOKS

We have formulated a multi-stage stochastic optimal control problem for maximizing the power output of a one dimensional wind farm array and shown that it can be solved analytically via dynamic programming. The optimal control policies depend explicitly on the statistics of multiplicative noise, which can be related to stochastic wind fluctuations.

Our results provide an initial step toward defining a wind farm control strategy that tractably incorporates statistical knowledge of stochastic wind fluctuations. However, there remain several lines of future work that can extend the present results in various ways to more fully understand the possibilities and limits for maximizing annual energy production. Our future work will involve

(a) utilizing more realistic wake models;
(b) estimating necessary statistics from high-fidelity numerical simulations and experimental data;
(c) performance evaluation of the policies on high-fidelity models, which may improve the results in [33];
(d) considering more realistic array geometries;
(e) exploring computationally efficient approximation of nonlinear optimal control strategies; if needed.

REFERENCES