Manipulating Factions Evolved in Signed Networks*

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Abstract— **In this paper the dynamics of signed networks is studied. Initially, we show how factions can be formed in a signed network and demonstrate that the factions formed in a weakly balanced network are consistent with what is expected. Later, we shift our focus to constructing signed networks that result in factions of known structure. In the end, we consider the case in which a group of *influencer* nodes manipulate their local connections in the network in order to steer to nodes to form any arbitrary faction structure. This allows the definition of a novel centrality measure for sets of nodes in a signed network.**

I. Introduction

There has been significant recent interest in studying dynamical processes on large networks consisting of many interacting subsystems. One important research focus is to quantify how combinatorial network properties interact with continuous properties associated with the dynamics. Very recently, dynamics on networks with interactions that can be both positive and negative have been considered. Such networks provide an intriguing contrast with well-studied consensus and synchronization problems; in consensus networks, where weights are typically non-negative, states synchronize to a common value, whereas in signed networks, the presence of both positive and negative weights can cause clusters to emerge that each synchronize to different values. Several interesting mathematical models for real networks feature both positive and negative interactions: in social networks, individual relationships can be friendly or hostile, giving rise to factions and, e.g., clustering in opinion dynamics; in genetic regulatory networks, genes can promote or repress the expression of other genes; in neuronal networks, neurons can excite or inhibit the firing of other neurons.

An important combinatorial property associated with signed graphs is *structural balance*, which traces back to social psychology research in the 1940s by Heider [1]. Harary and Cartwright [2] showed that structurally balanced graphs are those that can be partitioned into two subgraphs such that within each subgraph all edges are positive and between subgraphs all edges are negative. There is also a notion of *weak structural balance*, which allows a partition into more than two such subgraphs [3].

Recent work has considered dynamic processes on both the edges and the vertices in signed networks. Studies of edge dynamics include [4]–[7]. In this work, a real-valued "friendliness level" is associated with each edge and nonlinear differential equations for the evolution of these friendliness levels that almost always converge to structurally balanced states are studied. In [5] it was shown that the final structurally balanced state can be determined from the sign pattern of the dominant eigenvector of the initial state matrix. Non-symmetric graphs and active influence are considered in [6] and [7].

Studies of node dynamics on signed graphs include [8]–[10]. The most commonly considered model is a variation of the standard consensus algorithm that allows weights to be negative. Xia and Cao [8] show that clustering in diffusively coupled networks can arise from three mechanisms, including heterogeneous self-dynamics, delays, and negative couplings. Altafini [9] also considers consensus protocols with negative weights and shows that structural balance plays a role in convergence properties. Both of these works use a non-standard definition of the graph Laplacian that takes absolute values of edge weights. Shi et al. [10] present a stochastic model of dynamics on signed graphs and also make connections to the theory of structural balance. Burger et al. [11] show clustering arising from heterogeneous self-dynamics and saturating but positive coupling. Other ways of clustering can arise in diffusively coupled network dynamics is through state-dependent coupling [12] and pinning control [13].

However, it is still not clear how the clustering structure is determined and how this relates to graphs properties. In this paper, we elucidate several features of dynamic clustering and structural balance. There are four main contributions of our work: we show that (1) the dominant eigenspace determines clustering/faction structure; (2) if the underlying graph is complete (weakly) structurally balanced, dynamics converge to clusters defined by the factions in the underlying graph; (3) one can construct weighted graphs that have a given faction/cluster structure by solving a linear program; (4) a group of "influencer" nodes that manipulate their local edges can result in any desired faction structure.

A novelty in our results come from using the standard definition of the Laplacian with negative coupling. Although the trajectories diverge since the Laplacian is indefinite, we can still quantify clustering of the trajectories and convergence can be seen in an appropriately normalized system.

The structure of the rest of the paper is as follows. In next section, we present some preliminaries and the problem formulation. In Section III, we show that the faction structure of a signed network is determined by its dominant eigenspace. In Section IV, we show how weighted signed networks with

*This work is supported in part by a McKenzie Fellowship and a UoM Early Career Researcher Grant.

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a given faction structure can be constructed by solving a linear program and demonstrate how a group of influencer network can drive the network to form any desired faction structure. Section V gives illustrative numerical examples, and Section VI gives concluding remarks.

II. Preliminaries and Problem Formulation

We start this section by defining what we mean by a weakly balanced graph.

Definition 1. *A signed graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ with nonzero weights for its edges is weakly balanced if there exist $\mathcal{V}_1,\ldots,\mathcal{V}_f$ such that $\mathcal{V}_i\cap\mathcal{V}_j=\emptyset$, $\bigcup_{i=1}^{f}\mathcal{V}_i=\mathcal{V}$, all edges that connect the vertices inside each of $\mathcal{V}_i$ have positive weights and all edges that connect vertices from two different $\mathcal{V}_i$ and $\mathcal{V}_j$ have negative weights.*

For more information on balanced networks see [2].

Furthermore, the Laplacian of a signed graph is defined in the same way as the Laplacian of a graph with positive edge weights:

$$L = A - D \tag{1}$$

where $A_{ij}=A_{ji}$, and $A_{ij}>0$ corresponds to a sympathetic relationship with nodes i and j, and $A_{ij}<0$ is associated with an antagonistic relationship between the pair i and j. $A_{ij}=0$ demonstrates that there is no relationship between the pair and there is no edge between the pair i and j in $\mathcal{G}$. Moreover,

$$D_{ii}=\sum_{j=1}^{n}A_{ij}.$$

Note that L is indefinite and rank deficient with $\mathbf{1}$ in its nullspace. This property of the Laplacian matrix as defined above has persuaded many researchers to change the definition of Laplacian for signed graphs, e.g. see [9], [14]. However, in this paper we argue that it is not necessary and still much insight can be obtained from studying networks with this definition of the graph Laplacian.

Let $x_i(t)$ be the state of node i. We assume that the following equation governs the states of the nodes in the network:

$$\dot{x}(t) = Lx(t). \tag{2}$$

We have the following definitions.

Definition 2. *For a given network of n nodes with the underlying graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ we say nodes i and j belong to the same faction if $\|x_i(t)-x_j(t)\|\to 0$ as t goes to infinity, where $x_i(t)$ is the state of node i at time t.*

We address the following three problems in this paper. The first problem deals with how the states of nodes evolve in a signed graph. The second question considers the problem of constructing a graph where nodes in the graph form a faction of given structure. Finally, the third problem addresses the question of the possibility of transforming a graph to another one with a desired faction structure through manipulating its edges by a subset of nodes.

Problem 1. *For a given network of n nodes with an underlying signed graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, it is desired to calculate the number of factions, f, and characterise the members of each faction when the node states are governed by (2).*

Problem 2. *Given f sets $\mathcal{V}_1,\ldots,\mathcal{V}_f$, determine how a signed graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ can be constructed such that under (2) the nodes form f factions $\mathcal{V}_1,\ldots,\mathcal{V}_f$.*

Problem 3. *For a given network of n nodes with an underlying signed graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, the Laplacian L, a set of* Influencer Nodes, *$\mathcal{I}\subset\mathcal{V}$ and f sets $\mathcal{V}_1,\ldots,\mathcal{V}_f$ determine under what conditions it is possible to find a perturbation matrix Δ with nonzero entries at i-th rows and columns where $i\in\mathcal{I}$ such that the nodes in a network with the interconnection $L+\Delta$ form f factions $\mathcal{V}_1,\ldots,\mathcal{V}_f$.*

III. Emergence of Factions in Networks

We first propose the solution to Problem 1. Since, L is generically an indefinite matrix for signed graphs states of the nodes will "blow up". However, the trajectories form groups that are consistent with factions as defined in Definition 2. We address Problem 1 in the next theorem.

Theorem 1. *Consider the system described by (2). Let $v_1,\ldots,v_p$ be the eigenvectors associated with the dominant eigenvalue $\lambda_1(L)$, which is of multiplicity p. If $\nu_i=\nu_i$ for all vectors ν in the linear span of $v_1,\ldots,v_p$, then $\|x_i(t)-x_j(t)\|$ converges to 0 for any initial condition x_0 such that $\bar{V}x_0\neq 0$, where $\bar{V}$ is the matrix with columns comprising the normalised eigenvector of L associated with $\lambda_1(L)$.*

Proof: Let $\lambda_i(\cdot)$ be the i-th eigenvalue of its $n\times n$ matrix argument and $\lambda_n(\cdot)\le\cdots\le\lambda_1(\cdot)$. As L is a real symmetric matrix there exist a unitary matrix V such that

$$L = V\Lambda V^{\top},$$

where $\Lambda=\mathrm{diag}(\lambda_1(L),\ldots,\lambda_n(L))$ has the eigenvalues of L as its diagonal entries and the i-th column of V is the normalised eigenvector of L associated with λ_i. Hence,

$$x(t) = Ve^{\Lambda t}V^{\top}x(0). \tag{3}$$

Equivalently,

$$x(t) = e^{\lambda_1(L)t}(\bar{V}\bar{V}^T + M(t))x_0$$

where $M(t)\to 0$ exponentially fast with increasing t. Let

$$\nu(t) = \frac{x(t)}{\|x(t)\|}.$$

Then,

$$\nu(t) = \frac{e^{\lambda_1(L)t}(\bar{V}\bar{V}^T + M(t))x_0}{e^{\lambda_1(L)t}\|(\bar{V}\bar{V}^T + M(t))x_0\|}$$

which approaches

$$\nu^{\star} = \frac{\bar{V}\bar{V}^T x_0}{\|\bar{V}\bar{V}^T x_0\|}. \tag{4}$$

Since $\nu^\star$ is in the linear span of $v_1, \ldots, v_p$, it follows by hypothesis that $\nu_i^\star = \nu_j^\star$ and hence that $\|x_i(t) - x_j(t)\|$ converges to 0.

□

Since L is generically an indefinite matrix for signed graphs, each $x_i(t)$ in (2) can go to infinity. However, from the observations made thus far, $x_i(t)$ and $x_j(t)$ converge to the same value for all i and j belonging to the same faction as described in Definition 2.

Theorem 1 establishes that the factions are determined by the structure of the dominant eigenvectors of L and the space that they span. Now we shift our focus to study the dominant eigenspace of a weakly balanced graph as described in Definition 1.

Remark: Theorem 1 in fact holds for any symmetric autonomous linear dynamical system, not just systems with Laplacian dynamics. This can be seen easily by noting that the arguments in the proof are identical in that case.

For a weakly balanced network we have the following

Theorem 2. *Consider a network of n nodes with an underlying signed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with edge weights of either +1 or -1. Moreover, suppose there exist $\mathcal{V}_1, \ldots, \mathcal{V}_f$ such that $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$, $\bigcup_{i=1}^{f} \mathcal{V}_i = \mathcal{V}$, and the states of nodes are governed by (2) for with arbitrary initial condition. Then,*

(i) $\lambda_1(L)$ has multiplicity $p = f - 1$; and

(ii) As $t \to \infty$ the vertex sets $\mathcal{V}_1, \ldots, \mathcal{V}_f$ characterise the f factions in the graph.

Proof: Note that

$$L = \widetilde{L} - 2\hat{L} \tag{5}$$

where $\widetilde{L}$ is the Laplacian of the complete graph on $\mathcal{V}$ with all edge weights equal to 1 and $\hat{L} = \mathrm{diag}(\overline{L}_1, \ldots, \overline{L}_f)$ with $\overline{L}_i$ being the Laplacian of the complete graph induced by $\mathcal{V}_i$ with all edge weights equal to 1, $i = 1, \ldots, f$. From Weyl's matrix inequality we have

$$\lambda_n(\widetilde{L}) + \lambda_1(\hat{L}) \leq \lambda_1(L) \leq \lambda_1(\widetilde{L}) + \lambda_1(\hat{L}) \tag{6}$$

It can be observed that $\lambda_1(\hat{L}) = 0$, $\lambda_1(\widetilde{L}) = n$, and $\lambda_n(\widetilde{L}) = 0$. We obtain

$$0 \leq \lambda_1(L) \leq n. \tag{7}$$

Let $\mathbf{1}_i$ be a vector of all ones of length n_i, and $v \triangleq [\alpha_1 \mathbf{1}_1^\top, \ldots, \alpha_f \mathbf{1}_f^\top]^\top$ where α_i are arbitrary real scalars. First, observe that v is an eigenvector of L when $\alpha_1 = \cdots = \alpha_f$ and corresponds to a zero eigenvalue. Now, consider the case where not all α_i are equal. We postulate that the a v with such a structure is an eigenvector of L with its corresponding eigenvalue equal to to n. We have

$$\begin{aligned} Lv &= (\widetilde{L} - 2\hat{L})v \\ &= \widetilde{L}v - 2\hat{L}v \\ &= \widetilde{L}v. \end{aligned} \tag{8}$$

For v to be an eigenvector of L, the following equation should be satisfied

$$\widetilde{L}v = nv. \tag{9}$$

For (9) to be satisfied the following should hold

$$\sum_{i=1}^{f} n_i \alpha_i = 0. \tag{10}$$

Then any v such that (10) holds is an eigenvector of L with corresponding eigenvalue of n. From (7) we know that the largest eigenvalue of L is at most n and as is obtained in the light of the above arguments. Moreover, the space spanned by all such vectors that satisfy (10) is an $f - 1$ dimensional space. So, the multiplicity of the dominant eigenvalue is $f - 1$. In light of the Theorem 1 and the structure of the dominant eigenspace of L, the formation of different factions consistent with $\mathcal{V}_1, \ldots, \mathcal{V}_f$ is apparent. □

We continue by making an observation on the relationship between the evolution of $x(t)/\|x(t)\|$ and the gradient of the Rayleigh quotient of L, $R(L, y)$. Recall that,

$$R(L, y) = \frac{y^\top L y}{y^\top y}.$$

Moreover, consider the evolution of the trajectories of $u(t)$ where $u(t) \triangleq \dfrac{x(t)}{\|x(t)\|}$. We have

$$\begin{aligned} \dot{u}(t) &= \frac{\dot{x}(t)\|x(t)\| - \dfrac{x(t)^\top \dot{x}(t)}{\|x(t)\|}}{\|x(t)\|^2} \\ &= \frac{Lx(t)\|x(t)\| - \dfrac{x(t)^\top L x(t)}{\|x(t)\|}}{\|x(t)\|^2} \end{aligned}$$

$$\dot{u}(t) = Lu(t) - (u^\top(t) L u(t)) u(t). \tag{11}$$

On the other hand, for the gradient of the Rayleigh quotient of L we have

$$\frac{\partial R(L, y)}{\partial y} = \frac{2}{y^\top y}\left(Ly - \frac{y^\top L y}{y^\top y} y\right). \tag{12}$$

Note that if y is constrained to have unit norm, the right hand side of (11) will have the same direction as (12).

We end this section by relating the factions defined for $x(t)$ and $u(t)$ establishes the relationship between the factions defined under (2) and (11).

Proposition 1. *For a given network of n nodes with the underlying graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ if nodes i and j belong to the same faction then $\|u_i(t) - u_j(t)\| \to 0$ and $\|x_i(t) - x_j(t)\| \to 0$, where $u_i(t)$ is the state of node i at time t as t goes to infinity, and vice versa.*

IV. Constructing and Manipulating Graphs for a Given Set of Factions

In this section we initially address Problem 2 where the objective is to construct a graph from a given set of factions. Let $v^\star$ be a vector that defines a set of factions in the following way: the entries of $v^\star$ that correspond to nodes in the same faction are equal and entries corresponding to nodes not in the same faction are not equal. Moreover, the entries of $v^\star$ should sum to zero since $\mathbf{1}_n^\top v^\star = 0$.

Now consider the following linear program with variables $\lambda \in \mathbf{R}$ and $L \in \mathbf{R}^{n\times n}$ and data $v^\star$

$$\begin{aligned} &\underset{\lambda, L}{\text{maximize}} \quad \lambda \\ &\text{subject to} \quad L \in \mathcal{L}, \quad Lv^\star = \lambda v^\star, \quad \|vec(L)\|_\infty \le 1 \end{aligned} \tag{13}$$

where $vec(L)$ is a column vector consisting of the entries of L and $\mathcal{L}$ is the subspace of admissible symmetric Laplacian matrices and is defined

$$\begin{aligned} \mathcal{L} = \{L: \ & L \in \mathbb{R}^{n\times n}, \ L = L^\top, L\mathbf{1} = 0, \\ & L_{ij} = 0, \ \forall (i,j) \in \mathcal{E}_0, \ L_{lk} = 0, \ \forall (l,k) \in \mathcal{E}_1\}. \end{aligned} \tag{14}$$

Moreover, the sets $\mathcal{E}_0$ and $\mathcal{E}_1$ are the sets of all the edges that the constructed graph should exclude and include respectively. These sets can be used to enforce desired structure on the resulting graph. Note that the constraint $\|vec(L)\|_\infty \le 1$ ensures that the problem is bounded above. The idea is to automatically compute a matrix for which λ and $v^\star$ is the dominant eigenpair. This will be formalised later in this section.

In the following theorem we characterise the situations that the graph obtained from solving (13) has $v^\star$ as its dominant eigenvector.

Theorem 3. *Let $\lambda^\star$ and $L^\star$ be obtained from solving the optimisation problem (13). Then $\lambda^\star$ and $v^\star$ are the dominant eigenpair of $L^\star$.*

Proof: We prove the theorem by way of contradiction. Assume $\psi > \lambda^\star$ is the largest eigenvalue of $L^\star$. Then because of the fact that there are $\frac{n^2+n}{2}$ decision variables and at most $|\mathcal{E}_0| + |\mathcal{E}_1| = \frac{n^2-n}{2}$, there exists a $\Psi \neq L^\star$ such that

$$\begin{aligned} &\Psi\mathbf{1} = 0, \quad \Psi = \Psi^\top, \quad \Psi v^\star = \psi v^\star \\ &\Psi_{ij} = \Psi_{ji} = 0, \ \forall (i,j) \in \mathcal{E}_0, \ \Psi_{lk} = \Psi_{kl} = 1, \ \forall (l,k) \in \mathcal{E}_1 \end{aligned}$$

Thus there exists another feasible solution that yields a larger value and it is a contradiction. □

The abovementioned theorem enables us to proceed to address Problem 3. Note that the value of the perturbation matrix depends on the choices of the desired factions and $\mathcal{I}$. Moreover, as stated earlier the structure of the factions can be captured by a vector $v^\star$. The following linear program is proposed to find $\Delta(\mathcal{I}, v^\star)$:

$$\begin{aligned} [\lambda^\star, \Delta^\star(\mathcal{I}, v^\star)] = \ &\underset{\lambda, \Delta}{\text{argmax}} \quad \lambda \\ &\text{subject to} \quad (L+\Delta)\mathbf{1} = 0, \quad (L+\Delta)v^\star = \lambda v^\star, \\ &\qquad \Delta = \Delta^\top, \|vec(L+\Delta)\|_\infty \le 1 \\ &\qquad \Delta_{ij} = 0, \ \forall (i,j) \in \{(i,j) : \{i,j\} \cap \mathcal{I} = \emptyset\} \end{aligned} \tag{15}$$

First, we note that if there is only one influencer node, the problem (15) do not have a nonzero solution for $\Delta(\mathcal{I}, v^\star)$. To see this note that the equation $\Delta(\mathcal{I}, v^\star) = 0$ does not have a nonzero solution for the case where $|\mathcal{I}| = 1$. For the case where there are two influencer nodes one cannot make any statements, however, for the case where $|\mathcal{I}| \ge 3$ one can prove that any faction structure is attainable. To this aim, as before, we need to show $L^\star(\mathcal{I}, v^\star) = L + \Delta^\star(\mathcal{I}, v^\star)$ has the dominant eigenpair $\lambda^\star$ and $v^\star$ where $\Delta^\star(\mathcal{I}, v^\star)$ and $\lambda^\star$ are obtained from solving (15). We formalise it in the next theorem.

Theorem 4. *Let $\lambda^\star$ and $\Delta^\star(\mathcal{I}, v^\star)$ be obtained from solving the optimisation problem (15). Moreover, assume $|\mathcal{I}| \ge 3$. Then $\lambda^\star$ and $v^\star$ are the dominant eigenpair of $L^\star(\mathcal{I}, v^\star) = L + \Delta^\star(\mathcal{I}, v^\star)$.*

Proof: First, note that (15) can be recast as

$$\begin{aligned} [\lambda^\star, L^\star(\mathcal{I}, v^\star)] = \ &\underset{\lambda, L_\Delta}{\text{argmax}} \quad \lambda \\ &\text{subject to} \quad L_\Delta\mathbf{1} = 0, \quad L_\Delta = L_\Delta^\top, \\ &\qquad L_\Delta v^\star = \lambda v^\star, \quad L_{\Delta ij} = L_{ij}, \ i,j \in \mathcal{V} \setminus \mathcal{I} \\ &\qquad \|vec(L_\Delta)\|_\infty \le 1 \end{aligned} \tag{16}$$

Similar to the proof of Theorem 3, assume $\psi > \lambda^\star$ is the largest eigenvalue of the solution to (16), $L^\star(\mathcal{I}, v^\star)$. Since there are $\frac{2n|\mathcal{I}| + |\mathcal{I}| - |\mathcal{I}|^2}{2}$ decision variables and $2n$ linear constraints then for $|\mathcal{I}| \ge 3$ there exists a $\Psi \neq L^\star(\mathcal{I}, v^\star)$ that satisfies

$$\begin{aligned} \Psi\mathbf{1} = 0, \quad \Psi = \Psi^\top, \quad \Psi v^\star = \lambda v^\star \\ \Psi_{ij} = L_{ij}, \ i,j \in \mathcal{V} \setminus \mathcal{I}, \end{aligned}$$

which results in a contradiction. □

The existence of a nonzero $\Delta(\mathcal{I}, v^\star)$ enables the implementation of a feedback control signal to steer the nodes to form any desirable faction structure depending on the choice of $v^\star$. In other words, by changing the Laplacian of the graph, (2) becomes

$$\dot{x}(t) = Lx(t) + u_\mathcal{I}(t), \tag{17}$$

where $u_\mathcal{I} = \Delta(\mathcal{I}, v^\star)x(t)$.

We conclude this section by commenting on the magnitude of $\Delta(\mathcal{I}, v^\star)$ and its application as a *centrality measure* for a signed network.

Network centrality measures are real-valued functions that assign a relative "importance" to each vertex within the graph. Examples include degree, betweenness, closeness, and eigenvector centrality. Along these lines, one can assign importance to a subset of nodes in a network as well. To this aim, in the context of signed graphs, the magnitude of the perturbation required to achieve a desired faction structure, i.e. $\Delta(\mathcal{I}, v^\star)$ defines a class of network centrality measures. In particular, this input magnitude assigns an importance or influence value to different sets of nodes and indicates how easily that subset of nodes can perturb the network into the desired faction structure. For a given desired state, this magnitude can be used to rank the influence of all subsets of nodes in the network by computing the value for influencer nodes set and sorting the result; the smallest magnitude corresponds to the most influential set. The required perturbation magnitude and the ranking depends of course on the desired factions and the number of the influencer nodes. We thus define the *Factional Influence Index* for signed graphs, parameterised by the desired factions structure, $v^\star$, and the number of influencer nodes, $N_{\mathcal{I}}$, as follows.

Definition 3 (Factional Influence Index)**.** *Given a signed graph with n vertices and an associated Laplacian matrix L as defined in (1). Let v^* correspond to a desired faction structure. The Factional Influence Index of set $\mathcal{I}_i \subset \mathcal{V}$ and $|\mathcal{I}_i| = N_{\mathcal{I}}$ given v^* is the norm of the perturbation by the influencer nodes in $\mathcal{I}_i$ required to achieve v^*:*

$$\varphi_{v^*}(\mathcal{I}_i) = \|\Delta(\mathcal{I}_i, v^\star)\|. \tag{18}$$

V. Illustrative Examples

In this section we initially study the evolution of the states of the nodes in a network with the a weakly balanced underlying graph of 25 nodes and 8 factions as defined in Definition 1. The value of $x_i(t)/\|x(t)\|$ are depicted in Fig. 1 where states of nodes within the same faction converge to the same value.

In the second example, we consider a network of 20 nodes under an initial weakly balanced interconnection. It is further assumed that $\mathcal{I} = \{1, 2, 3\}$. At times 20 and 40 the influencer nodes change their local interconnections as described in Section IV to reflect a new faction structure. The value of $x_i(t)/\|x(t)\|$ for all $i \in \mathcal{V}$ and the factions formed in each time interval are depicted in Fig. 2.

To conform with the principles of reproducible research the scripts to generate the results in this paper are available at [15].

VI. Concluding Remarks

In this paper we consider the problem of evolution of states of a set of nodes in a signed network.

We initially demonstrated how the states of the nodes evolve in a signed network. Moreover, It was demonstrated how nodes form factions and how the structure of these factions depends on the dominant eigenspace of the signed Laplacian of the network. Furthermore, the problem of constructing networks with factional properties was considered. Later, it was shown how a subset of influencer nodes in the network can steer the rest of the network to any desired faction structure by modifying their local interconnections only. This observation allowed the definition of a novel centrality measure for subsets of nodes in a signed network. Some numerical examples were provided as well. Formulating the problem of finding the control input $u_{\mathcal{I}}$ as a linear quadratic optimal control problem is a future research direction.

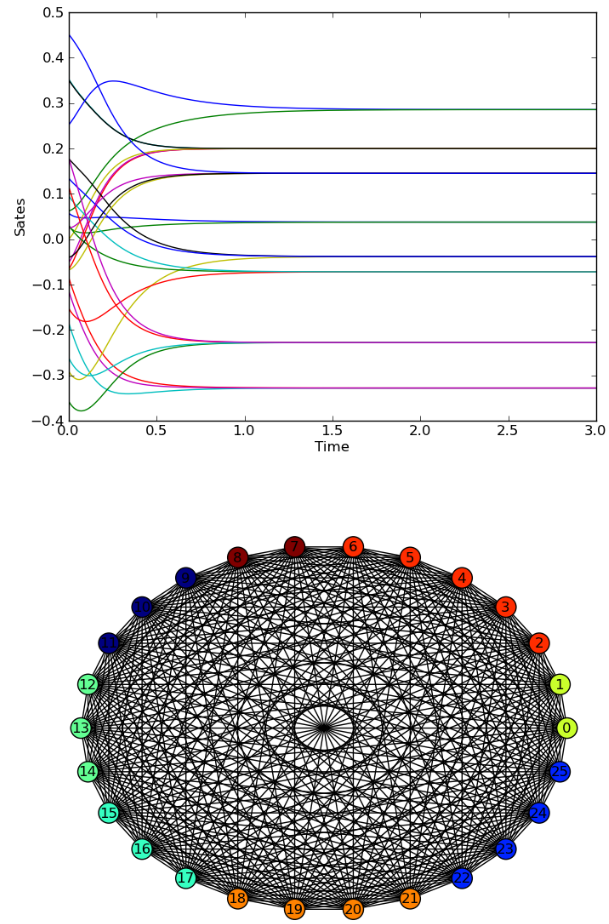

Fig. 1. Eight Factions and 25 nodes where nodes of the same colour belong to the same faction.

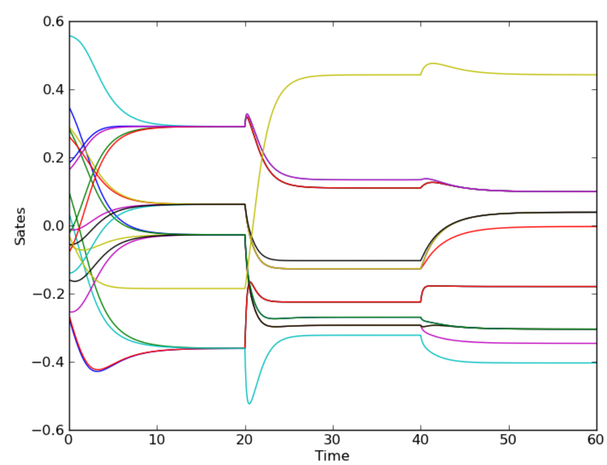

Fig. 2. A network of 20 nodes and 3 influencer nodes where manipulate their interconnection to drive the evolution of factions to different desired states.

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