Actuator placement in networks using optimal control performance metrics

Tyler Summers

***Abstract*— Quantifying controllability in large dynamical networks and designing network structures with good controllability properties has generated significant recent interest. We consider actuator placement problems in dynamical networks and show that the mappings from actuator subsets to four fundamental optimal control metrics are in general neither supermodular nor submodular set functions via a simple counterexample. We also find a set of restrictive conditions under which these mappings are modular set functions. Although this implies that simple greedy algorithms do not in general produce actuator placements with guaranteed near optimal closed-loop control performance, we find in computational experiments that greedy algorithms can exceed performance and far exceed scalability of convex relaxation heuristics with general purpose semidefinite programming solvers.**

I. Introduction

Quantifying controllability in large dynamical networks, including power grids, transportation networks, and various biological networks, has generated significant recent interest. Although the notion of controllability in dynamical systems has been around for decades [1], there are renewed efforts to elaborate on various aspects of controllability in structured networks.

Many metrics can be used to quantify different aspects of controllability and relate it to network structure properties. Once an appropriate metric is defined, one can consider the problem of selecting or placing actuator sets that improve network controllability. Recent prominent work has considered classical binary controllability metrics based on Kalman rank [2], [3], [4], [5], [6], [7], [8]. Since these can be extremely crude and misleading controllability quantifications, other work has focused on non-binary metrics, e.g., using the controllability Gramians to obtain input energy-related quantifications and optimal control costs that capture essential control notions of feedback control performance and robustness.

Actuator selection problems for optimizing network controllability are combinatorial and for large networks require heuristics. There are two broad heuristic approaches: convex relaxation and combinatorial greedy algorithms. Convex relaxations for sensor and actuator selection using optimal control metrics are considered in [9], [10], [11]. Combinatorial methods are considered using Gramian metrics in [13], [14], [15], [16]. A key focus in analysis of combinatorial greedy algorithms is to determine if set functions mapping actuator subsets to a controllability metric are submodular or supermodular. When this is true, greedy algorithms are guaranteed to produce near optimal actuator subsets. Several classes of Gramian metrics are shown to be submodular in [13], [14], [15], [16]. Other similar problems that feature Gramians, submodularity or supermodularity, and greedy algorithms are considered in [17], [18], [19], [20], [21], [22]. Many of the results and methods for actuator selection and controllability have analogous counterparts for sensor selection and observability.

Here we consider four fundamental optimal and dynamic game metrics that quantify feedback control performance and robustness, and we study the use of greedy algorithms for actuator selection in dynamical networks. The main contributions are as follows. We show that in general none of the four fundamental metrics are supermodular functions via a simple counterexample. This parallels a recent result demonstrating lack of supermodularity in Kalman filtering-based estimation metrics [23]. We also identify special conditions on the problem data under which all of the metrics are modular. Finally, we investigate the empirical performance of simple greedy algorithms in a class of random networks and show that the greedy algorithm can outperform a convex relaxation while achieving far faster computation times.

The paper is organized as follows. Section II provides the problem formulation and preliminaries. Section III gives a counterexample that shows that the four metrics are in general not supermodular, and identifies special conditions on the problem data under which the metrics are modular. Section IV presents the results of computational experiments. Section V concludes and describes future research directions.

II. Problem formulation and preliminaries

In this section, we formulate four actuator selection problems based on optimal values and equilibrium values of fundamental optimal control and dynamic game problems. These metrics capture feedback control performance and robustness. Specifically, we consider deterministic and stochastic linear quadratic optimal control problems and dynamic games. Properties and heuristic solution algorithms will be discussed in Section III.

A. Deterministic optimal control

Consider the linear dynamical system

$$x_{t+1} = Ax_t + B_S u_t, \quad t = 0, ..., T, \tag{1}$$

where $x_t \in \mathbf{R}^n$ is the system state at time t, $u_t \in \mathbf{R}^{(m+|S|)}$ is the input at time t, A is the dynamics matrix. Let $V = \{b_1, ..., b_M\}$ be a finite set of n-dimensional column vectors associated with possible locations for actuators that could be placed in the system. For any $S \subset V$, the input matrix B_S comprises a (possibly empty) input matrix $B_0 \in \mathbf{R}^{n\times m}$

corresponding to existing actuators and the columns indexed by S, i.e., $B_S = [B_0, b_{s_1}, ..., b_{s_{|S|}}] \in \mathbf{R}^{n\times(m+|S|)}$.

We would like to select a subset of actuators that maximizes a performance metric for the system. Here, we consider the optimal open-loop linear quadratic regulator objective of an input sequence $\mathbf{u} = [u_0^T, ..., u_{T-1}^T]^T$. The cost function is

$$V^*_{LQR}(S, x_0) = \min_{\mathbf{u}} \sum_{t=0}^{T-1} (x_t^T Q x_t + u_t^T R_S u_t) + x_T^T Q x_T,$$

where $Q \succeq 0$ is a state cost matrix and $R_S \succ 0$ is an input cost matrix, which is a principal submatrix of a total input cost matrix $R \in \mathbf{R}^{M\times M}$ consisting of the rows and columns indexed by S. The optimal open-loop cost can be computed by solving a least squares problem. In particular, we have

$$\begin{aligned} V^*_{LQR}(S, x_0) &= x_0^T G^T (I + H\mathbf{B}_S\mathbf{B}_S^T H^T)^{-1} G x_0 \\ &= x_0^T P_0 x_0, \end{aligned} \tag{2}$$

where

$$H = \mathrm{diag}(Q^{\frac{1}{2}}) \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ I & 0 & 0 & \cdots & 0 \\ A & I & 0 & \cdots & 0 \\ A^2 & A & I & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ A^{T-1} & A^{T-2} & \cdots & A & I \end{bmatrix},$$

$$G = \mathrm{diag}(Q^{\frac{1}{2}}) \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^T \end{bmatrix}, \quad \mathbf{B}_S = \mathrm{diag}(B_S R_S^{-\frac{1}{2}}).$$

Via dynamic programming, the optimal cost matrix P_0 can be alternatively computed from the backward Riccati recursion

$$P_{t-1} = Q + A^T P_t A - A^T P_t B_S (R_S + B_S^T P_t B_S)^{-1} B_S^T P_t A, \tag{3}$$

for $t = T, ..., 1$ with $P_T = Q$. It is well known that for deterministic systems, the optimal open-loop control sequence and optimal closed-loop feedback control policy generate the same state trajectories and have the same cost.

The optimal cost function (2) depends on the initial state. To obtain a scalar performance metric, we can assume that the initial state x_0 is a zero-mean random variable with finite covariance X_0 and consider expected performance. We obtain

$$\begin{aligned} J^*_{LQR}(S) &= \mathbf{E}_{x_0} V^*(S, x_0) \\ &= \mathbf{tr}[G^T (I + H\mathbf{B}_S\mathbf{B}_S^T H^T)^{-1} G X_0] \\ &= \mathbf{tr}[P_0 X_0]. \end{aligned} \tag{4}$$

Problem 1: The mapping $J^*_{LQR} : 2^V \to \mathbf{R}$ shown above is a set function that maps actuator subsets to the optimal LQR cost. To select a k-element subset of actuators to minimize this cost, we can pose the following set function optimization problem

$$\min_{S\subset V,\ |S|=k} J^*_{LQR}(S). \tag{5}$$

For infinite horizon problems, the set function J^*_{LQR} produces a finite value whenever the system is stabilizable via the actuator subset S; otherwise $J^*_{LQR}(S) = \infty$.[1]

B. Stochastic optimal control

Consider now the stochastic linear dynamical system

$$x_{t+1} = Ax_t + B_S u_t + w_t, \quad t = 0, ..., T, \tag{6}$$

where $\{w_t\}$ is an identically and independently distributed Gaussian process with $w_t \sim \mathcal{N}(0, W)$. Here one optimizes an expected cost over the set Π of causal, measurable state feedback *policies* $\pi : \mathbf{R}^n \to \mathbf{R}^m$

$$V^*_{LQG}(S, x_0) = \min_{\pi\in\Pi} \mathbf{E}_w \sum_{t=0}^{T-1} (x_t^T Q x_t + u_t^T R_S u_t) + x_T^T Q x_T.$$

It can be shown via dynamic programming that the optimal policy is linear state feedback and that the optimal cost function is quadratic and given by

$$V^*_{LQG}(S, x_0) = x_0^T P_0 x_0 + \sum_{t=1}^{T} \mathbf{tr} P_t W, \tag{7}$$

where the P_t for $t = 0, ..., T$ are generated by (3), the same recursion as in the deterministic problem. Assuming again that the initial condition is a zero-mean random variable with finite covariance X_0, we have the scalar metric

$$J^*_{LQG}(S) = \mathbf{tr} P_0 X_0 + \sum_{t=1}^{T} \mathbf{tr} P_t W. \tag{8}$$

Problem 2: The mapping $J^*_{LQG} : 2^V \to \mathbf{R}$ is a set function that maps actuator subsets to the optimal *closed-loop* LQG cost under the optimal feedback policy. Our second problem is to select a k-element subset of actuators to minimize this cost, which can be posed as the following set function optimization problem

$$\min_{S\subset V,\ |S|=k} J^*_{LQG}(S). \tag{9}$$

C. Deterministic dynamic game

Consider the linear dynamical system

$$x_{t+1} = Ax_t + B_S u_t + F v_t, \quad t = 0, ..., T, \tag{10}$$

where $v_t \in \mathbf{R}^p$ is the input of a strategic attacker. Here we consider a two-player zero-sum dynamic game (ZSDG) between the system defender, who controls u, and the attacker, who controls v. In zero-sum dynamic games, the

[1]We will discuss later an alternative performance quantification for actuator subsets that cannot stabilize the system.

appropriate equilibrium concept is the saddle point, whose value is given by

$$V^*_{ZSDG}(S, x_0) = \min_{\mathbf{u}} \max_{\mathbf{v}} \sum_{t=0}^{T-1} (x_t^T Q x_t + u_t^T R_S u_t - \gamma^2 v_t^T v_t) + x_T^T Q x_T. \tag{11}$$

This problem is closely related to $\mathcal{H}_\infty$ optimal control, which is of fundamental importance for robust control of systems with model uncertainties; see, e.g., [24], [25]. As in deterministic optimal control, the saddle point value in open-loop strategies can be computed by solving least squares problems associated with both the defender and attacker. The optimal value is

$$\begin{aligned} V^*_{ZSDG}(S, x_0) &= x_0^T G^T (I + H\mathbf{B}_S\mathbf{B}_S^T H^T - \gamma^{-2} H\mathbf{F}\mathbf{F}^T H^T)^{-1} G x_0 \\ &= x_0^T \mathcal{P}_0 x_0, \end{aligned} \tag{12}$$

where $\mathbf{F} = \text{diag}(F)$. The optimal cost matrix P_0 can be alternatively computed via a generalized backward Riccati recursion

$$\mathcal{P}_{t-1} = Q + A^T \mathcal{P}_t [I + (B_S B_S^T - \gamma^{-2} F F^T)\mathcal{P}_t]^{-1} A, \tag{13}$$

for $t = T, ..., 1$ with $\mathcal{P}_T = Q$. It is also well known that the optimal open-loop input sequences and closed-loop feedback strategies yield the same state trajectories and associated saddle point equilibrium value [26], [24].

Assuming again that the initial state is a random variable with finite covariance X_0, we obtain

$$\begin{aligned} J^*_{ZSDG}(S) &= \mathbf{E}_{x_0} V^*(S, x_0) \\ &= \mathbf{tr}[G^T (I + H\mathbf{B}_S\mathbf{B}_S^T H^T - \gamma^{-2} H\mathbf{F}\mathbf{F}^T H^T)^{-1} G X_0]. \end{aligned} \tag{14}$$

Problem 3: The mapping $J^*_{ZSDG} : 2^V \to \mathbf{R}$ is a set function that maps actuator subsets to the saddle point equilibrium value of the dynamic game. Our third problem is to select a k-element subset of actuators to minimize this value, which again can be posed as the following set function optimization problem

$$\min_{S \subset V,\ |S|=k} J^*_{ZSDG}(S). \tag{15}$$

D. Stochastic dynamic game

Consider the linear dynamical system

$$x_{t+1} = Ax_t + B_S u_t + F v_t + w_t, \quad t = 0, ..., T, \tag{16}$$

where $\{w_t\}$ is an identically and independently distributed Gaussian process with $w_t \sim \mathcal{N}(0, W)$. We now consider a stochastic zero-sum dynamic game (SZSDG). In particular, the saddle point value in *feedback strategies*

$$J^*_{SZSDG}(S) = \mathbf{E}_{x_0} \min_{\pi \in \Pi} \max_{\mu \in \Pi} \mathbf{E}_w \sum_{t=0}^{T-1} (x_t^T Q x_t + u_t^T R u_t - \gamma^2 v_t^T v_t^T) + x_T^T Q x_T \tag{17}$$

is given by

$$J^*_{SZSDG}(S) = \mathbf{tr}\mathcal{P}_0 X_0 + \sum_{t=1}^{T} \mathbf{tr}\mathcal{P}_t W. \tag{18}$$

Problem 4: The mapping $J^*_{SZSDG} : 2^V \to \mathbf{R}$ is a set function that maps actuator subsets to the *closed-loop* saddle point equilibrium value under the optimal feedback strategies for both players. Our fourth problem is to select a k-element subset of actuators to minimize this value, which can again be posed as the following set function optimization problem

$$\min_{S \subset V,\ |S|=k} J^*_{SZSDG}(S). \tag{19}$$

E. Set functions and submodularity

The actuator placement problems described above are formulated as cardinality constrained set function optimization problems. These problems are combinatorial and finite, and so can be solved simply by brute force enumeration. However, this approach quickly becomes intractable even for moderately sized problems. Our setting of large networked dynamical systems forces a different approach.

Rather than attempting to find a global optimum, we focus instead on structural properties of set functions that allow simple and computationally scalable algorithms to achieve provably good results. One such property, called *submodularity*, plays a similar role in combinatorial optimization as convexity and concavity play in continuous optimization [27], [28].

Definition 1: A set function $f : 2^V \to \mathbf{R}$ is called *submodular* if for all subsets $A \subseteq B \subseteq V$ and all elements $s \notin B$, it holds that

$$f(A \cup \{s\}) - f(A) \geq f(B \cup \{s\}) - f(B), \tag{20}$$

or equivalently, if for all subsets $A, B \subseteq V$, it holds that

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B). \tag{21}$$

A set function is called *supermodular* if the reversed inequalities in (20) and (21) hold and is called *modular* if (20) and (21) hold with equality.

Intuitively, submodularity is a diminishing returns property where adding an element to a smaller set gives a larger gain than adding it to a larger set. Maximization of submodular functions (equivalently, minimization of supermodular functions) is NP-hard, but a simple greedy heuristic can be used to obtain a solution that is provably close to the optimal solution [29]. The greedy algorithm for set function minimization is shown in Algorithm 1. Modular functions can be globally optimized by evaluating the set function for each element and sorting, as described in [14]. Several problems in systems and control that feature greedy algorithms and sub- or supermodularity have been recently explored [17], [18], [13], [14], [15], [20], [22], [16].

Algorithm 1 A greedy heuristic for set function optimization.

$S \leftarrow \emptyset$
while $|S| \leq k$ **do**
  $e^\star = \underset{e \in V \setminus S}{\text{argmax}} \quad f(S \cup \{e\})$
  $S \leftarrow S \cup \{e^\star\}$
end while
$S^\star \leftarrow S$

III. Main results

In this section, we first show via a simple counterexample that in general the four set functions are neither supermodular nor submodular. As a consequence, simple greedy algorithms do not necessarily produce actuator subsets that provide provably near optimal closed-loop performance and robustness for optimal control and dynamic games. We then identify a set of restrictive conditions under which all of the four set functions described in the previous section based on optimal control and dynamic game performance metrics are modular. Finally, we describe an alternative metric for infinite horizon problems with unstable systems.

A. *The optimal control and dynamic game metrics are neither submodular nor supermodular*

Consider the expression for the optimal open-loop LQR cost in (4). At first glance, it seems at least plausible that there might be a supermodular diminishing returns property to the optimal cost as actuators are added. When an actuator is added, a positive semidefinite term is added to expression inside the inverse, and so adding actuators will eventually cause saturation in certain directions. It is a matrix analog of the scalar function $\frac{1}{1+|S|}$, which decreases increasingly slowly as $|S|$ increases. Indeed, one can easily verify that for many problem instances and many actuator subsets, the supermodularity inequality in Definition 1 is satisfied (suggesting that the metric is not submodular, but perhaps supermodular). The same observations can be made for the remaining metrics.

Consider though the following counterexample with

$$A = \begin{bmatrix} 0.75 & -1 \\ 0.25 & -1 \end{bmatrix}, \; V = \left\{ b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \\ Q = I, \quad R = I, \quad X_0 = I, \quad T = 2. \tag{22}$$

There are four actuator subsets of V: $2^V = \{\emptyset, \{b_1\}, \{b_2\}, V\}$. Let $S_1 = \{b_1\}$, $S_2 = \{b_2\}$. Note that $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = V$. We can compute directly that

$$\begin{aligned} J^*_{LQR}(S_1) &= 4.31, \quad J^*_{LQR}(S_1 \cap S_2) &= 5.35 \\ J^*_{LQR}(S_2) &= 4.74, \quad J^*_{LQR}(S_1 \cup S_2) &= 3.38 \end{aligned} \tag{23}$$

and so

$$\begin{aligned} J^*_{LQR}(S_1) + J^*_{LQR}(S_2) = 9.05 \geq \\ J^*_{LQR}(S_1 \cap S_2) + J^*_{LQR}(S_1 \cup S_2) = 8.73. \end{aligned} \tag{24}$$

This violates the inequality for supermodularity in Definition 1 and shows that the set function J^*_{LQR} is not supermodular.

The same counterexample shows that J^*_{LQG} is not supermodular in general. The first term in (8) is the same as the corresponding deterministic cost, and there is a positive definite covariance matrix W for the disturbance that makes the second term small enough for the supermodularity inequality to be violated.

Similarly, the same counterexample also shows that J^*_{ZSDG} and J^*_{SZSDG} are not supermodular in general. The cost of the deterministic dynamic game converges to the cost of the deterministic dynamic game as the attacker input penalty $\gamma \to \infty$. Thus, for any attacker input matrix F there is a γ large enough for the supermodularity inequality for J^*_{ZSDG} to be violated. There is then also a disturbance covariance matrix W for the disturbance that makes the second term in (18) small enough for the supermodularity inequality for J^*_{SZSDG} to be violated.

Although the metrics are in general neither submodular nor supermodular, this does not preclude the use of greedy algorithms as a heuristic for selecting actuator subsets. The greedy algorithm's simplicity yields serious computational advantages, and it can produce actuator subsets that significantly improve feedback control performance and robustness despite the lack of theoretical guarantees.

B. *Special cases*

We have just seen that the four metrics are neither submodular nor supermodular in general. The next question is whether there are any specific conditions on the problem data under which the metrics do have a supermodularity property. We now present a set of conditions under which all of the metrics are actually modular. The conditions are quite restrictive, but the modularity means that globally optimal actuator subsets can be obtained. We have the following result.

Theorem 1: Suppose the horizon $T = 1$, the cost matrices $Q = I$, $R = I$, and the set defining possible actuator locations is $V = \{e_1, ..., e_n\}$, the set of standard unit basis vectors in $\mathbf{R}^n$, meaning that each possible input signal affects a single state variable. Then the set function $J^*_{LQR} : 2^V \to \mathbf{R}$ described in Section II-A is modular.

Proof: First, note that if $Q = I$, $R = I$ and $V = \{e_1, ..., e_n\}$ then for any $S \subset V$ we have $R + B_S^T Q B_S = 2I$. The optimal cost matrix for $T = 1$ from the Riccati recursion (3) for $S \subset V$ reduces to

$$\begin{aligned} P_0 &= I + A^T A - \frac{1}{2} A^T B_S B_S^T A \\ &= I + A^T A - \frac{1}{2} \sum_{i \in S} A^T e_i e_i^T A. \end{aligned}$$

Now for any $S \subset V$ and $s \in V \setminus S$, it follows that

$$J^*_{LQR}(S \cup \{s\}) - J^*_{LQR}(S) = -\mathbf{tr}(A^T e_s e_s^T A X_0),$$

and so by Definition 1, J^*_{LQR} is modular. ■

The same result holds for the stochastic case.

Theorem 2: Suppose the horizon $T = 1$, $Q = I$, $R = I$, and the set defining possible actuator locations is $V = \{e_1, ..., e_n\}$, the set of standard unit basis vectors in $\mathbf{R}^n$. Then the set function $J^*_{LQG} : 2^V \to \mathbf{R}$ described in Section II-B is modular.

Proof: The optimal closed-loop LQG cost $J^*_{LQG}(S)$ for $S \subset V$ is given by (8). The first term is the same as the open-loop cost for the corresponding deterministic system, which is modular by Theorem 1. For $T = 1$, the second term reduces to $\mathbf{tr}QW$, a constant independent of S. This constant cancels in Definition 1 for modularity, so J^*_{LQG} is also modular. ■

A similar result also holds for deterministic and stochastic dynamic games with the same arguments when $F = I$ and $\gamma > 1$.

Theorem 3: Suppose the horizon $T = 1$, $Q = I$, $R = I$, $F = I$, $\gamma > 1$, and the set defining possible actuator locations is $V = \{e_1, ..., e_n\}$, the set of standard unit basis vectors in $\mathbf{R}^n$. Then the set functions $J^*_{ZSDG} : 2^V \to \mathbf{R}$ and $J^*_{SZSDG} : 2^V \to \mathbf{R}$ described in Sections II-C and II-D are both modular.

Proof: Under the assumptions, (13) reduces to

$$\mathcal{P}_0 = I + \frac{\gamma^2}{\gamma^2 - 1} A^T A - \frac{\gamma^4}{(\gamma^2 - 1)(2\gamma^2 - 1)} A^T B_S B_S^T A.$$

For any $S \subset V$ and $s \in V \setminus S$, it follows that

$$J^*_{ZSDG}(S \cup \{s\}) - J^*_{ZSDG}(S) = - \frac{\gamma^4}{(\gamma^2 - 1)(2\gamma^2 - 1)} \mathbf{tr}(A^T e_s e_s^T A X_0),$$

and so by Definition 1, J^*_{ZSDG} is modular. Further, for $T = 1$ the second term in (18) is $\mathbf{tr}QW$, a constant independent of S, so J^*_{SZSDG} is also modular. ■

C. Algorithms and quantifying actuator quality for unstable systems and infinite horizons

For finite horizon problems, Algorithm 1 can be directly applied with any of the metrics. Infinite horizon problems raise the issue of instability. For large networks with unstable dynamics, it may be possible that no single actuator provides closed-loop stability, and so it is not clear how to proceed with the greedy algorithm. As measured by the optimal cost, each actuator is infinitely bad. However, it is possible to get a quantitative measure of actuator quality by looking instead at recursions for the cost inverse.

Consider the deterministic optimal control problem. Inverting the Riccati recursion (3) yields

$$P_{t-1}^{-1} = [Q + A^T (P_t - P_t B_S (R_S + B_S^T P_t B_S)^{-1} B_S^T P_t) A]^{-1}. \tag{25}$$

Applying the Woodbury matrix identity to the expression in parenthesis gives

$$P_{t-1}^{-1} = [Q + A^T (P_t^{-1} + B_S R^{-1} B_S^T)^{-1} A]^{-1} \tag{26}$$

Applying the Woodbury matrix identity again to the expression in brackets gives

$$P_{t-1}^{-1} = Q^{-1} - Q^{-1} A^T (P_t^{-1} + B_S R^{-1} B_S^T + A Q^{-1} A^T)^{-1} A Q^{-1} \tag{27}$$

The inverse cost matrix P_0^{-1} becomes rank deficient in directions in which an actuator (or subset of actuators) fails to stabilize the system, causing P_0 to be infinite in these directions. In other directions, one obtains valuable quantitative information about the effectiveness of a particular actuator subset for controlling the system even when it fails to stabilize. Until a stabilizing actuator subset is found, the greedy algorithm can be run to maximize $\mathbf{tr}(P_0^{-1} X_0^{-1})$ instead of minimizing $\mathbf{tr}(P_0 X_0)$.

Similar expressions can be obtained for the stochastic optimal control and dynamic game problems.

IV. Numerical experiments

In this section we present numerical experiments to compare the performance of the greedy algorithm with a convex relaxation technique.

We consider a class of randomly generated dynamics matrices where the entries of A are independently drawn from a standard normal distribution. We then scale A so that its spectral radius is 1 so that the system is marginally stable. The remaining problem data are $n = 25$, $V = \{e_1, ..., e_n\}$, $Q = I$, $R = I$, $X_0 = I$, and $T = 20$. We generated 100 instances of the dynamics matrix and ran the greedy algorithm to select a set of 10 actuators to minimize J^*_{LQR} the deterministic LQR optimal cost. The mean over the 100 samples was 52.4, and computation time for a single instance was about a third of a second on a 1.7 GHz Intel Core i7 processor. For comparison, we implemented a convex relaxation algorithm, which was a discrete-time, finite-horizon version of the actuator selection method in [9]. We ran this method[2] for 100 instances using the general purpose semidefinite programming solver SDPT3 [30]. The mean performance was 54.5, and the computation time for a single instance was about 20 seconds. The greedy algorithm provided better performance on average than the convex relaxation while being nearly two orders of magnitude faster to compute for this problem class and size.

More extensive computational experiments are needed to fully compare the performance of convex relaxations and greedy algorithms for each of the metrics, and this will be pursued in future work. Specialized methods can be developed to improve computation times for these types of convex relaxations [11], and the greedy algorithm is also trivially parallelizable and can scale more easily to large networks. Despite the lack of supermodularity and associated theoretical guarantees, greedy algorithms can match and exceed performance of convex relaxation techniques and

[2]In the convex relaxation algorithm, the number of added actuators is fixed indirectly by varying a cost parameter multiplying a sparsity-inducing penalty function. A bisection algorithm was implemented on the cost parameter to ensure that 10 actuators were selected.

possess simplicity that can give significant computational advantages.

V. Summary, Conclusions, and Outlook

We formulated actuator placement problems for optimizing controllability of dynamical networks using four fundamental optimal control and dynamic game metrics that quantify network controllability in terms of feedback control performance and robustness. Unlike other controllability metrics involving the Gramian, these turn out in general to be neither submodular nor supermodular. Consequently, greedy algorithms do not necessarily yield provably good actuator subsets. However, this does not preclude the use of greedy algorithms as a heuristic, and we find in computational experiments that greedy algorithms can exceed the performance of convex relaxations and scale to much larger networks.

We did find a set of restrictive conditions under which all four of the metrics are modular. An immediate open problem is to identify if possible less restrictive conditions on problem data under which the metrics are supermodular. Other future work will include extensive computational experiments for various random networks and networks from application domains. Additionally, the metrics we considered here quantify performance and robustness of feedback control with *centralized* information architectures; for large scale systems it will be important to consider metrics that capture limitations imposed by structural information constraints on the feedback information architecture.

References

[1] R. Kalman, "On the general theory of control systems," *IRE Transactions on Automatic Control*, vol. 4, no. 3, pp. 110–110, 1959.

[2] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, "Controllability of complex networks," *Nature*, vol. 473, no. 7346, pp. 167–173, 2011.

[3] I. Rajapakse, M. Groudine, and M. Mesbahi, "Dynamics and control of state-dependent networks for probing genomic organization," *Proceedings of the National Academy of Sciences*, vol. 108, no. 42, pp. 17 257–17 262, 2011.

[4] N. Cowan, E. Chastain, D. Vilhena, J. Freudenberg, and C. Bergstrom, "Nodal dynamics, not degree distributions, determine the structural controllability of complex networks," *PLOS ONE*, vol. 7, no. 6, p. e38398, 2012.

[5] T. Nepusz and T. Vicsek, "Controlling edge dynamics in complex networks," *Nature Physics*, vol. 8, no. 7, pp. 568–573, 2012.

[6] W.-X. Wang, X. Ni, Y.-C. Lai, and C. Grebogi, "Optimizing controllability of complex networks by minimum structural perturbations," *Physical Review E*, vol. 85, no. 2, p. 026115, 2012.

[7] A. Olshevsky, "Minimal controllability problems," *IEEE Transactions on Control of Network Systems*, vol. 1, no. 3, pp. 249–258, 2014.

[8] S. Pequito, S. Kar, and A. Aguiar, "A framework for structural input/output and control configuration selection in large-scale systems," *IEEE Transactions on Automatic Control*, vol. 61, no. 2, pp. 303–318, 2016.

[9] B. Polyak, M. Khlebnikov, and P. Shcherbakov, "An lmi approach to structured sparse feedback design in linear control systems," in *European Control Conference*. IEEE, 2013, pp. 833–838.

[10] U. Munz, M. Pfister, and P. Wolfrum, "Sensor and actuator placement for linear systems based on and optimization," *IEEE Transactions on Automatic Control*, vol. 59, no. 11, pp. 2984–2989, 2014.

[11] N. K. Dhingra, M. R. Jovanovic, and Z.-Q. Luo, "An ADMM algorithm for optimal sensor and actuator selection," in *IEEE Conference on Decision and Control*. IEEE, 2014, pp. 4039–4044.

[12] A. Hassibi, J. How, and S. Boyd, "Low-authority controller design via convex optimization," *AIAA Journal of Guidance, Control, and Dynamics*, vol. 22, no. 6, pp. 862–872, 1999.

[13] T. Summers, F. Cortesi, and J. Lygeros, "On submodularity and controllability in complex dynamical networks," *IEEE Transactions on Control of Network Systems*, vol. 3, no. 1, pp. 1–11, 2016.

[14] T. Summers and J. Lygeros, "Optimal sensor and actuator placement in complex dynamical networks," in *IFAC World Congress, Cape Town, South Africa*, 2014, pp. 3784–3789.

[15] F. Cortesi, T. Summers, and J. Lygeros, "Submodularity of energy related controllability metrics," in *IEEE Conference on Decision and Control, Los Angeles, CA*, 2014, pp. 2883–2888.

[16] V. Tzoumas, M. A. Rahimian, G. Pappas, and A. Jadbabaie, "Minimal actuator placement with bounds on control effort," *to appear, IEEE Transactions on Control of Network Systems*, 2016.

[17] L. Bushnell, A. Clark, and R. Poovendran, "A supermodular optimization framework for leader selection under link noise in linear multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 59, no. 2, pp. 283–296, 2014.

[18] A. Clark, B. Alomair, L. Bushnell, and R. Poovendran, "Minimizing convergence error in multi-agent systems via leader selection: A supermodular optimization approach," vol. 59, no. 6, pp. 1480–1494, 2014.

[19] F. Pasqualetti, S. Zampieri, and F. Bullo, "Controllability metrics, limitations and algorithms for complex networks," *Control of Network Systems, IEEE Transactions on*, vol. 1, no. 1, pp. 40–52, 2014.

[20] T. Summers, I. Shames, J. Lygeros, and F. Dorfler, "Topology design for optimal network coherence," in *European Control Conference*. IEEE, 2015, pp. 575–580.

[21] G. Yan, G. Tsekenis, B. Barzel, J.-J. Slotine, Y.-Y. Liu, and A.-L. Barabási, "Spectrum of controlling and observing complex networks," *Nature Physics*, vol. 11, pp. 779–786, 2015.

[22] I. Shames and T. Summers, "Rigid network design via submodular set function optimization," *IEEE Transactions on Network Science and Engineering*, vol. 2, no. 3, pp. 84–96, 2015.

[23] H. Zhang, R. Ayoub, and S. Sundaram, "Sensor selection for optimal filtering of linear dynamical systems: Complexity and approximation," in *Proc. of 54th IEEE Conference on Decision and Control*, 2015.

[24] T. Başar and P. Bernhard, *H-infinity optimal control and related minimax design problems: a dynamic game approach*. Springer Science & Business Media, 1995.

[25] J. C. Doyle, B. A. Francis, and A. R. Tannenbaum, *Feedback control theory*. Courier Corporation, 1990.

[26] T. Basar and G. J. Olsder, *Dynamic noncooperative game theory*. Siam, 1999, vol. 23.

[27] L. Lovász, "Submodular functions and convexity," *Mathematical programming: the state of the art*, pp. 235–257, 1983.

[28] A. Krause and D. Golovin, "Submodular function maximization," *Tractability: Practical Approaches to Hard Problems*, vol. 3, 2012.

[29] G. Nemhauser, L. Wolsey, and M. Fisher, "An analysis of approximations for maximizing submodular set functions—I," *Mathematical Programming*, vol. 14, no. 1, pp. 265–294, 1978.

[30] K.-C. Toh, M. J. Todd, and R. H. Tütüncü, "SDPT3: a MATLAB software package for semidefinite programming, version 1.3," *Optimization methods and software*, vol. 11, no. 1-4, pp. 545–581, 1999.